A quenched local limit theorem for stochastic flows

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Abstract

We consider a particle undergoing Brownian motion in Euclidean space of any dimension, forced by a Gaussian random velocity field that is white in time and smooth in space. We show that conditional on the velocity field, the quenched density of the particle after a long time can be approximated pointwise by the product of a deterministic Gaussian density and a spacetime-stationary random field U. If the velocity field is additionally assumed to be incompressible, then $U \equiv 1$ almost surely and we obtain a local central limit theorem.

1 Introduction

Let $(\Omega, \mathcal{H}, \mathbf{P})$ be a probability space. Fix a spatial dimension $d \in \mathbf{N}$. Let $V = (V_1, \ldots, V_d)$ be a Wiener process on $L^2(\mathbf{R}^d; \mathbf{R}^d)$ that is spatially-smooth, with covariance function formally given by

$$\mathbf{E} V_i(\mathrm{d}t, x) V_j(\mathrm{d}s, y) = \delta(t-s) R_{ij}(x-y) \mathrm{d}s \mathrm{d}t$$

for some covariance function $R \in \mathcal{C}_\mathrm{c}^\infty(\mathbf{R}^d; \mathbf{R}^{d\times d})$. (That is, R is smooth with compact support.) Let $\{\mathcal{F}_t\}$ be the usual filtration associated to V (generated by $\{V(s)\ :\ s \le t\}$) and let $\mathcal{F} = \bigvee_{t<\infty} \mathcal{F}_t$. Let $B = (B_1, \ldots, B_d)$ be a Brownian motion taking values in $\mathbf{R}^d$ (independent of $\mathcal{F}$) with quadratic variation

$$\langle B_i, B_j \rangle(t) = \nu \delta_{ij} t \tag{1.1}$$

for some $\nu > 0$. Let $\{\mathcal{G}_t\}$ be the usual filtration associated to B, and let $\mathcal{G} = \bigvee_{t<\infty} \mathcal{G}_t$. Let $\mathcal{H}_t = \mathcal{F}_t \vee \mathcal{G}_t$. We assume that the σ-algebra $\mathcal{H}$ is given by $\mathcal{H} = \mathcal{F} \vee \mathcal{G}$.

We are interested in the stochastic differential equation

$$\mathrm{d}X(t) = V(\mathrm{d}t, X(t)) + \mathrm{d}B(t); \tag{1.2}$$
$$X(0) = 0, \tag{1.3}$$

which models a passive scalar in a environment that decorrelates rapidly in time. We will interpret (1.2) in the manner of [28, Section 3.4]; that is, as equivalent to the *Itô* integral equation

$$X(t) = B(t) + \int_0^t V(\mathrm{d}s, X(s)),$$

where $\{X(t)\}$ is assumed to be a continuous $\mathbf{R}^d$-valued process adapted to $\{\mathcal{H}_t\}$.

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This problem has a unique solution by [28, Theorem 3.4.1] (using our assumption on the smoothness of R). The process $\{X(t)\}$ is a continuous martingale with quadratic variation process given by

$$\langle X_i, X_j\rangle(t) = (\nu\delta_{ij} + R_{ij}(0))t, \tag{1.4}$$

by [28, Theorem 3.2.4]. Thus, the annealed law of $\{X(t)\}$ is actually a d-dimensional Brownian motion with covariance matrix $(\nu I_d + R(0))t$ at time t. Here we used I_d to denote the $d\times d$ identity matrix. We will think of the forcing V as a random velocity field and the forcing B as a molecular diffusion, so ν is the "molecular diffusivity". Our interest will be in the quenched (with respect to the molecular diffusion) law of X given by

$$\mu_t = \mathrm{Law}[X(t) \mid \mathcal{F}]. \tag{1.5}$$

We will show in Section 2.1 that, for $t > 0$, μ_t has a density with respect to Lebesgue measure on $\mathbf{R}^d$ that exists as a random field $\big(u(t,x)\big)_{t>0,x\in\mathbf{R}^d}$ (as a consequence of the molecular diffusion):

$$\mu_t(\mathrm{d}x) = u(t,x)\mathrm{d}x, \qquad t > 0, x \in \mathbf{R}^d. \tag{1.6}$$

Thus, $u(t,\cdot)$ is a density function that feels the randomness of the velocity field. Let G_t be the solution to the PDE

$$\begin{aligned} \partial_t G_t(x) &= \frac{1}{2}\mathrm{tr}[(\nu I_d + R(0))\nabla^2 G_t(x)];\\ G_0 &= \delta_0. \end{aligned} \tag{1.7}$$

Thus G_t is a Gaussian density centered at the origin with covariance matrix $(\nu I_d + R(0))t$, and so G_t is the density of annealed law of $X(t)$. The goal of this paper is to study the relationship between the quenched law $u(t,\cdot)$ and the annealed law $G_t(\cdot)$, and to understand how the randomness from the environment affects the *local* behavior of the passive scalar. Here is the main theorem:

Theorem 1.1. *There is a spacetime-stationary random field U, positive almost surely with $\mathbf{E}U \equiv 1$, and, for every $\varepsilon > 0$, a constant $C = C(R,\nu,\varepsilon) < \infty$ so that*

$$\sup_{x\in\mathbf{R}^d} \mathbf{E}|u(t,x) - G_t(x)U(t,x)|^2 \le Ct^{-d}\left(t^{-1/3}\log t\,\mathbb{1}_{d=1} + t^{-2/(2+d)+\varepsilon}\mathbb{1}_{d\ge 2}\right) \tag{1.8}$$

for all $t \ge C$. In particular, for any $c < \frac{1}{3}\mathbb{1}_{d=1} + \frac{2}{2+d}\mathbb{1}_{d\ge2}$, we have

$$\lim_{t\to\infty} \sup\left\{\mathbf{E}\left|\frac{u(t,x)}{G_t(x)} - U(t,x)\right|^2 \;:\; (\nu I_d + R(0))^{-1}x\cdot x \le ct\log t\right\} = 0. \tag{1.9}$$

We in fact have $U \equiv 1$ almost surely if and only if $\sum_{i=1}^d \frac{\partial R_{ij}}{\partial x_i} \equiv 0$ for each j (which holds if and only if V is incompressible almost surely). In general, we have

$$|\mathbf{E}(U(t,x_1) - 1)(U(t,x_2) - 1)| \le C(1 + |x_1 - x_2|)^{-d}. \tag{1.10}$$

Define $X_\varepsilon(t) = \varepsilon X(\frac{t}{\varepsilon^2})$, so the quenched density of $X_\varepsilon(t)$ is

$$u_\varepsilon(t,x) = \varepsilon^{-d}u(\tfrac{t}{\varepsilon^2}, \tfrac{x}{\varepsilon}).$$

By (1.8), we have for any $t > 0, x \in \mathbf{R}^d$, that

$$\mathbf{E}|u_\varepsilon(t,x) - G_t(x)U(\tfrac{t}{\varepsilon^2}, \tfrac{x}{\varepsilon})|^2 \to 0, \text{ as } \varepsilon \to 0.$$

In other words, the quenched density of the diffusively rescaled process is approximately the Gaussian density multiplying a stationary random field, which can be viewed as the "corrector" in stochastic homogenization. This corrector is the constant 1 in the incompressible case, so we obtain a local central limit theorem.

Alternatively, Theorem 1.1 can be seen as a continuous-space, continuous-time version of the local limit theorems for random walk in a random environment proved in [12, 17]. (See also the survey [13, §1.4.3] regarding the result of [12].) Since the result concerns the long-time behavior of the system, one does not expect a substantial difference between the discrete and continuous settings. However, the local temporal roughness of the driving force introduces substantial complications in establishing the required estimates, as we discuss in Section 1.1 below. Moreover, Theorem 1.1 is meaningful in the entire diffusive bulk region (i.e. $|x| \lesssim t^{1/2}$), while [12, Theorem 2] only holds for $|x| \ll t^{1/3}$. Similar results were shown in [34, 14] for certain exactly-solvable models, with the one-point distribution of the correction field U characterize explicitly. It is also worth mentioning that for reversible random walks/diffusions in random environments, e.g., the random conductance model, one can actually prove the local central limit theorem. Using our notation this says that $u_\varepsilon(t,x) \approx G_t(x)$ for $\varepsilon \ll 1$, similar to our result when V is incompressible. We refer the reader to [1, 2] and the references therein.

The stationary random field U is a spacetime stationary solution to the Fokker-Planck equation (2.7) with $\mathbf{E}U = 1$, which is closely related to the invariant measure of the process of "environment seen from the particle," a crucial object in the study of random walk/diffusion in random environment. This connection was also made in [17] for random walks in a balanced random environment. We refer to Remark 3.3 for more discussion.

Our interest in the quenched density $u(t,x)$ is motivated in part by the recent work on the moderate- and large-deviations regime of diffusion in a time-dependent random environment, which decorrelates rapidly; see the discussions [4, 5, 6, 30] in both the physics and mathematics literature. In the diffusive regime $x \sim \sqrt{t}$, which is what we consider here, it is well-known that the diffusion scales to a Brownian motion, see e.g. the discussion on similar models in [9, 33, 21] and [24] for a monograph on the subject. (In our special setting of white-in-time noise, the annealed law of $X(t)$ is actually exactly the Brownian motion.) To study the error, one can consider quantities of the form

$$\mathbf{E}[f(\varepsilon X(\tfrac{t}{\varepsilon^2})) \mid \mathcal{F}] - \int_{\mathbf{R}^d} f(x)G_t(x)\mathrm{d}x = \int_{\mathbf{R}^d} f(x)[u_\varepsilon(t,x) - G_t(x)]\mathrm{d}x,$$

where $f : \mathbf{R}^d \to \mathbf{R}$ is an arbitrary smooth function. The Edwards-Wilkinson type fluctuation is proved in [3, 10, 35], i.e., after a proper rescaling, $\{u_\varepsilon(t,x) - G_t(x)\}_{t>0, x\in\mathbf{R}^d}$ converges in law and weakly in space to a Gaussian field that solves a stochastic heat equation with an additive Gaussian noise. Compared to our result, the difference is that we consider the fluctuation $u_\varepsilon(t,x) - G_t(x)$ for any fixed (t,x), rather than performing a spatial averaging under which the local fluctuations average out so that one needs to consider the next order error to observe random fluctuations. One can also look at super-diffusive regimes. In the moderate-deviations regime of $x \sim t^{3/4}$, the KPZ equation arises [5] (see a similar result in a weak noise regime [16]), and the large-deviations regime $x \sim t$ is associated with the KPZ fixed point and the Tracy-Widom type distribution was derived in [4].

In [30], the relation between the diffusion in time-dependent random environments and the KPZ universality class was explored. For $\log u(t,x)$, the Edwards–Wilkinson universality was actually conjectured to prevail in the diffusive regime, and it was also pointed out that the expected normal statistics seems to be different from the one studied in [3]. Our result of $u(t,x) \approx G_t(x)U(t,x)$ in $t \gg 1$ shows that the random fluctuation is governed by the stationary random field U, but we do not observe log-normal fluctuations of $U(t,x)$. Instead, as it will become clear later in the proof, $U(t,x)$ is a deterministic functional of the local random environment near (t,x), so there is actually no averaging taking place. It is very similar to the case of a directed polymer in a random environment in dimension $d \geq 3$ at high temperature, where it is well-known that the polymer path is diffusive and the partition function is approximately a deterministic functional of the random environment near the endpoint.

We approach the problem from a more analytic perspective. We will show in Proposition 2.1 below that u satisfies the stochastic PDE

$$\mathrm{d}u(t) = \frac{1}{2}\operatorname{tr}[(\nu I_d + R(0))\nabla^2 u(t)]\mathrm{d}t - \nabla\cdot[u(t)V(\mathrm{d}t)], \qquad t > 0; \tag{1.11}$$

$$u(0) = \delta_0, \tag{1.12}$$

which can be seen as a Fokker–Planck equation with random coefficients. Here and throughout the paper, we use ∇^2 to mean the Hessian operator, *not* the Laplacian. Then the field U in Theorem 1.1 is in fact a spacetime-stationary solution to (1.11), starting from constant initial data $u(0,x) \equiv 1$. Thus, Theorem 1.1 is quite similar to the "homogenization-type" theorems of [19, 15] proved for the stochastic heat equation with weak noise in $d \geq 3$, in that it shows how to approximate the solution to a stochastic PDE with a compactly-supported initial condition by a deterministic evolution multiplied by a random spacetime-stationary solution. (A similar result was proved for directed polymers in $d \geq 3$ in [11].)

In the case when the forcing is assumed to be incompressible (i.e. $\nabla \cdot V \equiv 0$ almost surely), the SPDE (1.11) has been extensively studied in the turbulence community as the "rapid decorrelation in time model" or "Kraichnan model." See [32] and the references therein. incompressibility and indeed an important case is when $d = 1$, in which nontrivial incompressibility is impossible.

1.1 Proof strategy

As pointed out above, our result is quite similar in form to the results on the stochastic heat equation in $d \geq 3$. If we ignore convergence issues and formally write the mild solution formula to (1.11)

$$\begin{aligned} u(t) &= G_t * u(0) - \int_0^t \int G_{t-s} * \nabla \cdot [u(s)V(\mathrm{d}s)] \\ &= G_t * u(0) - \int_0^t \nabla G_{t-s} * [u(s)V(\mathrm{d}s)], \end{aligned} \tag{1.13}$$

then we immediately see the similarity between (1.11) and the stochastic heat equation, with the only difference coming from the use of ∇G_{t-s} instead of G_{t-s} in the stochastic integral term. This extra gradient is the reason our result holds in $d \geq 1$, rather than the requirement of $d \geq 3$ for the stochastic heat equation. To see it more clearly, one can look at the first order "chaos", which is the first random term obtained by iterating the mild formulation: for SHE, we obtain $\int_0^t G_{t-s} * V(\mathrm{d}s)$, which converges to a stationary Gaussian field in large time, only in $d \geq 3$; for the Fokker-Planck equation, the convergence of $\int_0^t \nabla G_{t-s} * V(\mathrm{d}s)$ to a stationary Gaussian field holds in any dimension. The extra gradient also means that making (1.13) rigorous seems quite nontrivial, due to the worse singularity of $\nabla G_t(x)$ near $(t,x) = (0,0)$. (Some progress in developing such a theory was made in [23] for a special class of V.) Thus, we do not use the formulation (1.13) in the present work, and instead use another approach to make sense of the SPDE (1.11).

While it is not difficult to formally derive (1.11) as the Fokker–Planck equation associated with the passive scalar evolution (1.2), solution theories for the stochastic PDE (1.11) are rather intricate; see the discussion in [23, pp. 2–3]. We will use a solution theory due to Kunita [26] (similar to the approach described in [28, §6.2]) that uses stochastic flows to make sense of the stochastic PDE. We note that we require a somewhat stronger solution theory than simply deriving the problem (1.11)–(1.12) solved by the density, because, as indicated above, we will also need to construct spacetime-stationary solutions to (1.11), with the initial data $u(0,x) \equiv 1$. We recall the results we will need in Section 2.1. This approach requires R to be (qualitatively) several times differentiable, which we have assumed in our work. Alleviating this restriction was part of the goal of [23], but results in this direction are not yet strong enough for our purposes.

To justify the approximation

$$u(t,x) \approx G_t(x)U(t,x), \qquad t \gg 1, \tag{1.14}$$

and thus prove Theorem 1.1, our strategy is similar to that of [18] for the 2D nonlinear stochastic heat equation. Namely, we first approximate (1.11) by the equation for which the noise has been turned off in the time interval $[0,q]$, for some properly chosen q so that $1 \ll t-q \ll t$. Then we show that the latter solution can be approximated locally in space by a stationary solution. Basically, the evolution of (1.11) in the time interval $[0,q]$, which is almost of length t, generates the factor $G_t(x)$ in (1.14), while the evolution in the remaining interval $[q,t]$, which is macroscopically small but microscopically large, "feels" the random environment and produces the factor $U(t,x)$ in (1.14). A difference is that [18] works with a stochastic heat equation in $d=2$, where spacetime-stationary solutions do not exist. Thus, as we have stated before, phenomenologically the situation is more similar to that considered in [19, 15], although in those works a different approach based on the Feynman–Kac formula was used in the proofs.

Proving the mentioned bounds in [18] was done using the mild solution formula, the analogue of (1.13). A discrete chaos expansion was also the key technique used for the proof in [12]. As we have stated, we do not (at present) have a mild solution theory for the SPDE (1.11). Thus we work in a more analytic way, using the PDE satisfied by the two-point correlation function of the solution to (1.11) in Section 2.2. This PDE has been used before in the case of the Kraichnan model (i.e. when the forcing is assumed incompressible); see for example [31]. Then we use tools from the theory of parabolic PDE (in particular [22, 20]) to prove the required bounds on the correlations. We establish these bounds in Section 2.2.1. Then we apply them in Section 3 to prove the existence of the spacetime-stationary solution U and in Section 4 to complete the proof of Theorem 1.1.

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2 Setup and preliminaries

Throughout the paper, the letter C will denote a positive constant depending on R and ν, and only on other parameters if specified explicitly. We will allow C to change from line to line if necessary.

We wish to derive a stochastic PDE satisfied by μ_t, but before we do this we will generalize (1.2)–(1.3) to the setting of *stochastic flows* (see [28, Chapter 4]). Let $\varphi_{s,t}(x)$ ($s,t \in \mathbf{R}$, $x \in \mathbf{R}^d$) be the family of random diffeomorphisms solving the family of SDEs

$$\mathrm{d}_t\varphi_{s,t}(x) = V(\mathrm{d}t, \varphi_{s,t}(x)) + \mathrm{d}B(t); \tag{2.1}$$

$$\varphi_{s,s}(x) = x, \tag{2.2}$$

by which we mean solving the stochastic Itô integral equations

$$\varphi_{s,t}(x) = x + \int_s^t [V(\mathrm{d}r, \varphi_{s,r}(x)) + \mathrm{d}B(r)], \qquad t \geq s. \tag{2.3}$$

This means that the solution to (1.2)–(1.3) will be given by $X(t) = \varphi_{0,t}(0)$. Such a solution exists and is unique by [28, Theorem 4.5.1].

2.1 The stochastic PDE

Now for a Borel measure μ_0 on $\mathbf{R}^d$, which we assume to live in some weighted Sobolev space (of negative regularity) with at most polynomial growth at infinity, let $\tilde{\mu}_t$ be the pushforward measure of μ_0 by $\varphi_{0,t}$, so for any $A \subset \mathbf{R}^d$, we have

$$\tilde{\mu}_t(A) = \mu_0(\varphi_{0,t}^{-1}(A)). \tag{2.4}$$

Thus, $\tilde{\mu}_t$ is an $\mathcal{H}_t$-measurable random measure.

The definition (2.4) is similar to [27, (2.14)] and [26, (2.4)], which define the composition of a tempered distribution and a stochastic flow. We emphasize, however, that the composition of a tempered distribution with a diffeomorphism is *not* a generalization of the pushforward of a measure by a diffeomorphism, as the former construction involves a factor of the Jacobian determinant of the diffeomorphism. That is, our definition (2.4) is in fact the same as defining

$$\tilde{\mu}_t = \left(\frac{\mu_0}{\det D\varphi_{0,t}}\right) \circ \varphi_{0,t}^{-1}, \tag{2.5}$$

where the $\circ$ denotes composition of distributions, in the sense that

$$\langle \tilde{\mu}_t, f\rangle = \int \frac{1}{\det D\varphi_{0,t}(x)} (\det D\varphi_{0,t}(x)) f(\varphi_{0,t}(x))\,\mathrm{d}\mu_0(x) = \int f(\varphi_{0,t}(x))\,\mathrm{d}\mu_0(x),$$

which agrees with (2.4). The determinants involved in the last two formulas are positive, so there is no need to take an absolute value.

Now we define

$$\mu_t = \mathbf{E}[\tilde{\mu}_t \mid \mathcal{F}], \tag{2.6}$$

so (1.5) represents the special case when $\mu_0 = \delta_0$. Conditional expectations of the form (2.6) were constructed and studied in [26]. By [26, Theorem 3.2] (which relies on the partial Malliavin calculus developed in [7, 29]), for all $t > 0$ the measure μ_t has a (spatially) smooth density $u(t)$ with respect to the Lebesgue measure almost surely. This property comes from the ellipticity implied by (1.1) of the molecular diffusion. The following proposition shows that (2.6) solves the Fokker-Planck in an appropriate sense:

Proposition 2.1. *The function u, considered as a time-indexed family of tempered distributions on $\mathbf{R}^d$, is the unique solution of the Itô stochastic PDE*

$$\mathrm{d}u(t) = \frac{1}{2}\operatorname{tr}[(\nu I_d + R(0))\nabla^2 u(t)]\mathrm{d}t - \nabla\cdot[u(t)V(\mathrm{d}t)], \qquad t > 0; \tag{2.7}$$

$$\lim_{t\downarrow 0} u(t) = \mu(0) \tag{2.8}$$

in the "generalized solution" sense analogous to [26, (3.3)]: for almost every realization of the random environment, we have for all Schwartz functions $h : \mathbf{R}^d \to \mathbf{R}$ that

$$\langle u(t), h\rangle = \langle \mu(0), h\rangle + \int_0^t \frac{1}{2}\langle u(s), \operatorname{tr}[(\nu I_d + R(0))\nabla^2 h]\rangle\,\mathrm{d}s + \int_0^t \langle u(s), \nabla h\cdot V(\mathrm{d}s)\rangle, \qquad t > 0. \tag{2.9}$$

Remark 2.2. In the sequel, we will use the standard abuse of notation and write (2.8) as $u(0) = \mu(0)$, even if $\mu(0)$ does not have a density.

Proof. We will derive (2.7) by applying [26, Theorem 3.1]. In order to use this theorem, we must show how our problem fits into the framework of [26]. This is done via the following list of correspondences, in which the left-side quantities (also written in sans-serif type to avoid confusion with the notation used in the present paper) are the notations of [26] and the right-side quantities are our notations:

$$\mathsf{W}^i = -V_i, \qquad 1 \le i \le d; \tag{2.10}$$

$$\mathsf{W}^{d+1} = -\nabla \cdot V; \tag{2.11}$$

$$\mathsf{B}(x,t) = -B(t) + \frac{1}{2}(\nabla \cdot R)(0)t; \tag{2.12}$$

$$\mathsf{L} = \frac{1}{2}\nu\Delta + \frac{1}{2}(\nabla \cdot R)(0)\cdot\nabla, \text{ i.e. } \mathsf{a}^{ij} \equiv \nu\delta_{ij},\ \mathsf{b}^i \equiv \frac{1}{2}\sum_{j=1}^{d}\frac{\partial R_{ij}}{\partial x_j}(0),\ \mathsf{d} \equiv 0; \tag{2.13}$$

$$\mathsf{X} = \mu(0); \tag{2.14}$$

$$\mathsf{G} \equiv 0. \tag{2.15}$$

Here and henceforth, by $(\nabla \cdot R)(0)$ we denote the vector with the ith component $\sum_{j=1}^{d}\frac{\partial R_{ij}}{\partial x_j}(0)$.

As [26] works with Stratonovich rather than Itô integrals, we rewrite (2.1) in the Stratonovich form. Using the Itô–Stratonovich correction given in [28, Theorem 3.2.5], we have

$$\begin{aligned}\varphi_{s,t}(x) &= x + \int_s^t V(\mathrm{d}r, \varphi_{s,r}(x)) + B(t) - B(s)\\ &= x + \int_s^t V(\circ\mathrm{d}r, \varphi_{s,r}(x)) + B(t) - B(s) - \frac{1}{2}(\nabla \cdot R)(0)(t-s),\end{aligned} \tag{2.16}$$

or equivalently

$$\mathrm{d}_t\varphi_{s,t}(x) = V(\circ\mathrm{d}t, \varphi_{s,t}(x)) + B(\mathrm{d}t) - \frac{1}{2}(\nabla \cdot R)(0)\mathrm{d}t.$$

Using the correspondences (2.10)–(2.13), we see that (2.16) matches [26, (3.4)], with $\psi = \varphi$.

Now using the differentiation rule [28, (3.3.21)] we have that

$$\mathrm{d}_t[D\varphi_{s,t}(x)] = \mathrm{d}(DV)(\circ\mathrm{d}t, \varphi_{s,t}(x)) \cdot D\varphi_{s,t}(x).$$

Therefore, by the Jacobi formula and the chain rule for Stratonovich integrals, we have

$$\begin{aligned}\mathrm{d}_t[\det D\varphi_{s,t}(x)] &= \mathrm{tr}[(\mathrm{adj}\, D\varphi_{s,t}(x))DV(\circ\mathrm{d}t, \varphi_{s,t}(x)) \cdot D\varphi_{s,t}(x)]\\ &= \det D\varphi_{s,t}(x)(\nabla \cdot V)(\circ\mathrm{d}t, \varphi_{s,t}(x)),\end{aligned}$$

where adj denotes the classical adjoint (adjugate) matrix. This implies that

$$\mathrm{d}_t[\log\det D\varphi_{s,t}(x)] = (\nabla \cdot V)(\circ\mathrm{d}t, \varphi_{s,t}(x)),$$

so

$$\det D\varphi_{s,t}(x) = \exp\left\{\int_0^t (\nabla \cdot V)(\circ\mathrm{d}r, \varphi_{s,r}(x))\right\}, \tag{2.17}$$

Therefore, recalling (2.11) and (2.13), we have

$$\gamma_{s,t}(x) = \frac{1}{\det D\varphi_{s,t}(x)}, \qquad \mathsf{X}(t) = \frac{\mu(0)}{\det D\varphi_{0,t}}, \tag{2.18}$$

with the left sides in the notation of [26, (3.5)–(3.6)] and the right sides in our notation.

Now we see that [26, Theorem 3.1] applies, and it tells us that

$$\begin{aligned}\mu(t) = \mu(0) + \int_0^t \left[\frac{1}{2}\nu\Delta\mu(s) + \frac{1}{2}(\nabla\cdot R)(0)\cdot\nabla\mu(s)\right] \mathrm{d}s - \int_0^t \nabla\mu(s)\cdot V(\circ \mathrm{d}s) \\ - \int_0^t \mu(s)(\nabla\cdot V)(\circ\mathrm{d}s).\end{aligned} \tag{2.19}$$

Since $u(t)$ is the density of $\mu(t)$, the same equation holds for u.

To complete the proof, it remains to convert (2.19) into an Itô integral equation by subtracting the appropriate correction term. This computation is carried out on in [28, p. 302]. Again using a sans-serif font for the notation there, we have $\mathsf{F}^{\mathsf{i}} = -V_i$ for $i = 1,\dots,d$ and $\mathsf{F}^{\mathsf{d+1}} = -\nabla\cdot V$. Thus we have the "local characteristic"

$$\mathsf{A}^{\mathsf{ij}}(x,y,t) = \begin{cases} R_{ij}(x-y), & 1\le i,j\le d; \\ -\sum_{k=1}^d \frac{\partial R_{ik}}{\partial x_k}(x-y) = -(\nabla\cdot R)_i(x-y), & 1\le i\le d,\ j=d+1; \\ \sum_{k=1}^d \frac{\partial R_{kj}}{\partial x_k}(x-y), & i=d+1,\ 1\le j\le d; \\ -\sum_{k,\ell=1}^d \frac{\partial^2 R_{k\ell}}{\partial x_k\partial x_\ell}(x-y), & i=j=d+1. \end{cases} \tag{2.20}$$

We also have the auxiliary functions

$$\begin{aligned}\mathsf{C}^j(x,t) &= \sum_{i=1}^d \frac{\partial \mathsf{A}^{ij}}{\partial \mathsf{y}^i}(x,y,t)|_{y=x} = -\sum_{i=1}^d \frac{\partial R_{ij}}{\partial x_i}(0) = (\nabla\cdot R)_j(0) \\ \mathsf{D}(x,t) &= \sum_{i=1}^d \frac{\partial \mathsf{A}^{i,d+1}}{\partial y^i}(x,y,t)|_{y=x} = \sum_{k,\ell=1}^d \frac{\partial R_{k\ell}}{\partial x_k\partial x_\ell}(0).\end{aligned}$$

Thus, [28, p. 302, (3)] becomes in our setting

$$\begin{aligned}\tilde{\mathsf{L}}u &= \frac{1}{2}\sum_{i,j=1}^d R_{ij}(0)\frac{\partial^2 u}{\partial x_i\partial x_j} + \sum_{i=1}^d \left(-(\nabla\cdot R)_i + \frac{1}{2}(\nabla\cdot R)_i\right)\frac{\partial u}{\partial x_i} \\ &= \frac{1}{2}\sum_{i,j=1}^d R_{ij}(0)\frac{\partial^2 u}{\partial x_i\partial x_j} - \frac{1}{2}(\nabla\cdot R)\cdot\nabla u,\end{aligned}$$

and so by [28, p. 302, (4)], we have

$$\begin{aligned}\mu(t) &= \mu(0) + \int_0^t \left[\frac{1}{2}\nu\Delta\mu(s) + \frac{1}{2}(\nabla\cdot R)(0)\cdot\nabla u(s) + \tilde{\mathsf{L}}u(s)\right]\mathrm{d}s - \int_0^t \nabla\mu(t)\cdot V(\mathrm{d}s) \\ &\quad - \int_0^t \mu(s)(\nabla\cdot V)(\mathrm{d}s) \\ &= \mu(0) + \int_0^t \frac{1}{2}\operatorname{tr}[(\nu I_d + R(0))\nabla^2\mu(s)]\,\mathrm{d}s - \int_0^t \nabla\mu(t)\cdot V(\mathrm{d}s) - \int_0^t \mu(s)(\nabla\cdot V)(\mathrm{d}s).\end{aligned} \tag{2.21}$$

Thus u satisfies the Itô SPDE (2.7). □

2.2 The second-moment PDE

As described in the introduction, we now want to write a PDE for the second moments of $u(t)$. To this end, we first consider

$$u_2(t,x,y) = u(t,x)u(t,y). \tag{2.22}$$

Since $u(t,\cdot)$ is the quenched density of $X(t)$, we know that if $u(0)$ is a delta, then $u_2(t,x,y)$ is the joint quenched density of $(X(t),Y(t))$ with

$$\begin{aligned} X(t) &= X(0) + B_1(t) + \int_0^t V(\mathrm{d}s, X(s)), \\ Y(t) &= Y(0) + B_2(t) + \int_0^t V(\mathrm{d}s, Y(s)), \end{aligned} \tag{2.23}$$

where B_1, B_2 are independent Brownian motions that are also independent from V. Thus, u_2 encodes the correlation of the two passive scalars in the same random environment. From the Itô formula and the SPDE (2.7) (or, in the case when $u(0)$ is a delta, by redoing the computation in Proposition 2.1 but for a flow on $\mathbf{R}^{2d}$ where the first and last d coordinates are forced by the same instance of V but two independent Brownian motions B_1 and B_2), we see that u_2 satisfies the SPDE

$$\begin{aligned} \mathrm{d}u_2(t,x,y) = \frac{1}{2}\operatorname{tr}[(\nu I_d + R(0))^{\otimes 2}\nabla^2 u_2](t,x,y)\mathrm{d}t - u(t,x)\nabla\cdot[u(t,y)V(\mathrm{d}t,y)] - u(t,y)\nabla\cdot[u(t,x)V(\mathrm{d}t,x)] \\ + \sum_{i,j=1}^d \frac{\partial^2}{\partial x_i \partial y_j}(u_2(t,x,y)R_{ij}(x-y))\mathrm{d}t, \end{aligned} \tag{2.24}$$

again in the sense of [26, (3.3)]. If we define

$$Q_t(x,y) = \mathbf{E}u_2(t,x,y),$$

then Q_t lives in polynomially-weighted Sobolev space by [26, Lemma 3.1]. By definition, Q_t is the annealed density of $(X(t),Y(t))$ defined in (2.23). Now we take expectations in (2.24). Rigorously, this could be done by using [26, Theorem 3.1] again, but this time taking conditional expectation with respect to the null filtration. In this way, we see that Q_t, considered as a tempered distribution, is the unique solution to the PDE

$$\partial_t Q_t(x,y) = \frac{1}{2}\operatorname{tr}[\nabla^2[(\nu I_d + R(0))^{\otimes 2} Q_t]](x,y) + \sum_{i,j=1}^d \frac{\partial^2}{\partial x_i \partial y_j}(Q_t(x,y)R_{ij}(x-y)), \qquad t > 0; \tag{2.25}$$

$$Q_0(x,y) = u_0(x)u_0(y) \tag{2.26}$$

in the "generalized" sense of [26, (2.1)] (which means that the corresponding integral equation holds when Q_t is integrated against a Schwartz test function).

Now we make change of variables

$$x \mapsto w + z/2, \qquad y \mapsto w - z/2, \tag{2.27}$$

and put

$$S_t(w,z) = Q_t(w+z/2, w-z/2).$$

With $X(t),Y(t)$ defined in (2.23), we further define the center of mass and the relative distance by

$$W(t) = (X(t)+Y(t))/2, \qquad Z(t) = X(t) - Y(t), \tag{2.28}$$

so $S_t(w,z)$ is the annealed density of $(W(t),Z(t))$. Define the matrix $A(z)$ by

$$\begin{aligned} A(z) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}(z) &= \frac{1}{2}\begin{pmatrix} \frac{1}{2}I_d & \frac{1}{2}I_d \\ I_d & -I_d \end{pmatrix}\left[\nu I_{2d} + \begin{pmatrix} R(0) & R(z)^{\mathrm{T}} \\ R(z) & R(0) \end{pmatrix}\right]\begin{pmatrix} \frac{1}{2}I_d & I_d \\ \frac{1}{2}I_d & -I_d \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} \frac{1}{2}[\nu I_d + R(0)] + \frac{1}{4}\left[R(z) + R(z)^{\mathrm{T}}\right] & \frac{1}{2}\left[R(z)^{\mathrm{T}} - R(z)\right] \\ \frac{1}{2}\left[R(z) - R(z)^{\mathrm{T}}\right] & 2\nu I_d + 2R(0) - \left[R(z) + R(z)^{\mathrm{T}}\right] \end{pmatrix}. \end{aligned} \tag{2.29}$$

Then from (2.25) we obtain

$$\partial_t S_t(w,z) = \mathcal{L}^* S_t(w,z) = \operatorname{tr}\left[\nabla^2 (A S_t)(w,z)\right]; \tag{2.30}$$
$$S_0(w,z) = u_0(w+z/2)u_0(w-z/2), \tag{2.31}$$

where we have defined the differential operator

$$\mathcal{L} f(w,z) = \operatorname{tr}[A(z)\nabla^2 f(w,z)], \tag{2.32}$$

and $\mathcal{L}^*$ its adjoint

$$\mathcal{L}^* f(w,z) = \operatorname{tr}[\nabla^2 (Af)(w,z)],$$

where we use the notation, if $A = (a_{ij})$,

$$\operatorname{tr}[\nabla^2(Af)] = \sum_{i,j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j}[a_{ij} f].$$

We emphasize that (2.30)–(2.31) is simply a deterministic change of variables from (2.25)–(2.26). Alternatively, one could start from (2.23) to write down the equation satisfied by $(W(t), Z(t))$, then derive the PDE satisfied by its annealed density, which is (2.30).

If $X : \mathbf{R}^d \to \mathbf{R}^d$ is a stationary Gaussian random field with correlation function $\mathbf{E} X(z)X(0)^{\mathrm{T}} = R(z)$, then we have

$$\mathbf{E}\begin{pmatrix} X(z) \\ X(0) \end{pmatrix}\begin{pmatrix} X(z)^{\mathrm{T}} & X(0)^{\mathrm{T}} \end{pmatrix} = \begin{pmatrix} R(0) & R(z) \\ R(-z) & R(0) \end{pmatrix} = \begin{pmatrix} R(0) & R(z) \\ R(z)^{\mathrm{T}} & R(0) \end{pmatrix},$$

so the matrix on the right is nonnegative-definite, and thus from (2.29) we conclude that $A(z)$ is positive-definite uniformly over all $z \in \mathbf{R}^d$. By the assumption $R \in \mathcal{C}_{\mathrm{c}}^\infty$, $A(z)$ is also smooth in z.

Now by the theory of parabolic PDEs (which relies on the ellipticity of A; see e.g. [22, §1.6]), we know that the PDE (2.30) has a fundamental solution. Thus, (2.30)–(2.31) has a classical solution given by integration of the initial measure against the fundamental solution. Since it is clear that this classical solution is also a tempered distribution and satisfies (2.30)–(2.31) in the "generalized" sense of Kunita [26], for which there is a uniqueness statement, the function S_t in fact is given by integration of the initial condition (2.31) against the fundamental solution. In the sequel, we mean this solution when we talk about "the" solution to (2.30)–(2.31). (Any other solution must have extremely fast growth as $|x| \to \infty$.)

2.2.1 Bounds on the fundamental solution

For notational convenience, we will often write $\omega = (w,z)$. Let Γ_t be the fundamental solution for (2.30), so that the solution to (2.30) satisfies

$$S_t(\omega) = \int \Gamma_{t-s}(\omega;\omega') S_s(\omega')\,\mathrm{d}\omega'$$

for $s < t$ and $\omega \in \mathbf{R}^{2d}$. We note that Γ_t is the fundamental solution for the non-divergence form parabolic PDE

$$\partial_t g = \mathcal{L} g, \tag{2.33}$$

with its arguments swapped, i.e.,

$$g_t(\omega) = \int \Gamma_{t-s}(\omega';\omega) g_s(\omega')\,\mathrm{d}\omega'.$$

In this section we will prove some bounds on Γ_t using tools from the theory of parabolic PDE.

Recall from (2.29) that

$$A_{22}(z) = \nu I_d + R(0) - \frac{1}{2}\left[R(z) + R(z)^{\mathrm{T}}\right]. \tag{2.34}$$

We first need the following proposition, which will also be useful later.

Proposition 2.3. *There is a unique function $\chi \in C^\infty(\mathbf{R}^d; \mathbf{R})$ and a constant $C < \infty$ so that*

$$\operatorname{tr}[\nabla^2(A_{22}\chi)] \equiv 0, \tag{2.35}$$

$$\chi - 1 \in L^p(\mathbf{R}^d) \qquad \text{for any } p > 1, \tag{2.36}$$

$$C^{-1} \le \inf_{\mathbf{R}^d} \chi \le \sup_{\mathbf{R}^d} \chi \le C, \tag{2.37}$$

and

$$|\chi(x) - 1| \le C|x|^{-d} \text{ for all } x \text{ with } |x| \ge 1. \tag{2.38}$$

Remark 2.4. In the case of $R(z) = f(z)I_d$ for some scalar function $f \in C_c^\infty(\mathbf{R}^d; \mathbf{R})$ (which is always the case in $d = 1$), we can take

$$\chi(z) = \frac{\nu + f(0)}{\nu + f(0) - f(z)}$$

which evidently satisfies (2.35)–(2.37). In fact, it satisfies (2.36) with $p = 1$ as well.

Remark 2.5. In the case when $\sum_{i=1}^d \frac{\partial R_{ij}}{\partial x_i} \equiv 0$ for each j (i.e. when V is incompressible almost surely), it is clear from (2.34) that $\chi \equiv 1$.

Remark 2.6. From (2.34) and the fact that R is compactly supported, one can view A_{22} as a perturbation of the constant matrix $\nu I_d + R(0)$. Since (2.35) is the equation for the invariant measure of the process $Z(t)$ defined in (2.28), Proposition 2.3 is essentially to quantify the fact that the invariant measure is a perturbation of the Lebesgue measure.

Proof of Proposition 2.3. By Remark 2.4, we can assume that $d \ge 2$, so we can use the results of [20]. Since A is uniformly positive definite, Theorem 1.1 of [20] implies that there is a unique, up to a scalar multiple normalization, $\chi : \mathbf{R}^d \to \mathbf{R}_{\ge 0}$ satisfying (2.35) in a weak sense. Using the assumption that R is smooth, [8, Theorem 1.4.6] ensures that χ is smooth as well. Therefore, χ in fact satisfies (2.35) in a classical sense.

Now we need to prove (2.36) and (2.37). Our approach is based on the proof of [20, Theorem 1.5], the difference being that we make stronger assumptions and obtain stronger results. For the purpose of this proof only, we make a deterministic, linear change of coordinates so that we can assume that $\nu I_d + R(0) = I_d$. This does not affect the conclusions of the proposition (up to the choice of constants). This means that $A_{22}(z) = I_d + E(z)$, where E is compactly-supported, say on $B_M(0)$ for some $M > 0$. Throughout the proof, to simplify the notation we write $\sum_{i,j} = \sum_{i,j=1}^d$ and $\mathcal{A} = A_{22}$. Now define

$$f_x(r) = \int_{B_r(x)} \chi(z)\,\mathrm{d}z.$$

Then we claim that

$$r\int_{B_r(x)} \chi(z) \operatorname{tr}\mathcal{A}(z)\,\mathrm{d}z = \int_{\partial B_r(x)} \chi(z)\mathcal{A}(z)(z-x)\cdot(z-x)\,\mathrm{d}\mathcal{H}^{d-1}(z), \tag{2.39}$$

where $\mathrm{d}\mathcal{H}^{d-1}$ is the surface measure. To show (2.39), we write

$$\begin{aligned}
r\int_{B_r(x)}\chi(z)\,\mathrm{tr}\,\mathcal{A}(z)\,\mathrm{d}z &= r\sum_{i,j}\int_{B_r(x)}\chi(z)\mathcal{A}_{ij}(z)\delta_{ij}\mathrm{d}z\\
&= \frac{r}{2}\sum_{i,j}\int_{B_r(x)}\chi(z)\mathcal{A}_{ij}(z)\partial_{z_iz_j}(|z-x|^2-r^2)\mathrm{d}z\\
&= \frac{r}{2}\sum_{i,j}\int_{B_r(x)}\partial_{z_i}\Big(\chi(z)\mathcal{A}_{ij}(z)\partial_{z_j}(|z-x|^2-r^2)\Big)\mathrm{d}z\\
&\qquad -\frac{r}{2}\sum_{i,j}\int_{B_r(x)}\partial_{z_i}\Big(\chi(z)\mathcal{A}_{ij}(z)\Big)\partial_{z_j}(|z-x|^2-r^2)\mathrm{d}z\\
&=: I_1-I_2.
\end{aligned}$$

For I_1, we apply the divergence theorem to see that it is equal to the r.h.s. of (2.39). For I_2, by the fact that $\mathrm{tr}[\nabla^2(\mathcal{A}\chi)]\equiv 0$, we have

$$I_2=\frac{r}{2}\sum_{i,j}\int_{B_r(x)}\partial_{z_iz_j}\Big(\chi(z)\mathcal{A}_{ij}(z)(|z-x|^2-r^2)\Big)\mathrm{d}z=0$$

where the last identity comes from another application of divergence theorem. So (2.39) is proved.

Now we have

$$\begin{aligned}
f_x'(r) &= \int_{\partial B_r(x)}\chi(z)\,\mathrm{d}\mathcal{H}^{d-1}(z)=\frac{1}{r^2}\int_{\partial B_r(x)}|z-x|^2\chi(z)\,\mathrm{d}\mathcal{H}^{d-1}(z)\\
&= \frac{1}{r^2}\int_{\partial B_r(x)}\chi(z)\mathcal{A}(z)(z-x)\cdot(z-x)\,\mathrm{d}\mathcal{H}^{d-1}(z)-D_1(r,x)\\
&= \frac{1}{r}\int_{B_r(x)}\chi(z)\,\mathrm{tr}\,\mathcal{A}(z)\,\mathrm{d}z-D_1(r,x)=\frac{d}{r}\int_{B_r(x)}\chi(z)\,\mathrm{d}z+D_2(r,x)-D_1(r,x)\\
&= \frac{d}{r}f_x(r)+D_2(r,x)-D_1(r,x).
\end{aligned}\tag{2.40}$$

where we used the fact that $\mathcal{A}(z)=I_d+E(z)$ and we defined

$$\begin{aligned}
D_1(r,x) &= \frac{1}{r^2}\int_{\partial B_r(x)}\chi(z)E(z)(z-x)\cdot(z-x)\,\mathrm{d}\mathcal{H}^{d-1}(z)\\
&= \frac{1}{r^2}\int_{\partial B_r(x)\cap B_M(0)}\chi(z)E(z)(z-x)\cdot(z-x)\,\mathrm{d}\mathcal{H}^{d-1}(z).
\end{aligned}$$

and

$$D_2(r,x)=\frac{1}{r}\int_{B_r(x)\cap B_M(0)}\chi(z)\,\mathrm{tr}\,E(z)\,\mathrm{d}z.$$

We note that there is a constant D, independent of x and r, so that

$$|D_1|,r|D_2(r,x)|\le D,\qquad r>0, x\in\mathbf{R}^d.\tag{2.41}$$

First we consider the case $x=0$. We note that, whenever $r\ge M$, we have $D_1(r,0)=0$ and $D_2(r,0)=\frac{M}{r}D_2(M,0)$. Therefore, we have for $r\ge M$ that

$$f_0'(r)=\frac{d}{r}\left[f_x(r)+d^{-1}MD_2(M,0)\right],$$

so, solving the ODE, we obtain for $r \geq M$ that

$$f_0(r) = \kappa r^d - d^{-1} M D_2(M,0) \tag{2.42}$$

for some constant κ. We fix the normalization of χ so that κ is the volume of the unit ball in $\mathbf{R}^d$. In other words, for $r \gg 1$, we have

$$\int_{B_r(0)} \chi(z)\,\mathrm{d}z = f_0(r) \sim \int_{B_r(0)} \mathbb{1}(z)\,\mathrm{d}z.$$

Now we consider general x. From (2.40) we have

$$f_x'(r) = \frac{d}{r} f_x(r) + D_2(r,x) - D_1(r,x). \tag{2.43}$$

We note that $D_1(r,x) = 0$ and $D_2(r,x) = \frac{|x|+M}{r} D_2(|x|+M,x)$ whenever $r \geq |x|+M$. Therefore, we have

$$f_x'(r) = \frac{d}{r}\left[f_x(r) + d^{-1}(|x|+M)D_2(|x|+M,x)\right], \qquad r \geq |x|+M,$$

and thus

$$f_x(r) = \kappa_x r^d - d^{-1}(|x|+M)D_2(|x|+M,x), \qquad r \geq |x|+M, \tag{2.44}$$

for some constant κ_x. But since $f_0(r-|x|) \leq f_x(r) \leq f_0(r+|x|)$ for all $r \geq |x|$, comparing (2.42) and (2.44) and taking r large we see that in fact we must have $\kappa_x = \kappa$ for all x, and thus

$$f_x(r) = \kappa r^d - d^{-1}(|x|+M)D_2(|x|+M,x), \qquad r \geq |x|+M. \tag{2.45}$$

Using (2.41) in (2.45), we have that

$$|f_x(r) - \kappa r^d| \leq d^{-1} D, \qquad r \geq |x|+M. \tag{2.46}$$

Now assume that $|x| \geq M+1$. We have from (2.43) and (2.41) that, as long as $r \geq |x|-M$, we have $r \geq 1$ and hence

$$\left|\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{-d} f_x(r)\right)\right| \leq D r^{-d}[1+1/r] \leq 2Dr^{-d},$$

so

$$\left|(|x|+M)^{-d} f_x(|x|+|M|) - (|x|-M|)^{-d} f_x(|x|-M)\right| \leq 4DM(|x|-M)^{-d}. \tag{2.47}$$

Since χ is harmonic in $B_{|x|-M}(x)$ (recall that $\mathcal{A}(z) = I_d$ for $|z| \geq M$), we have

$$(|x|-M|)^{-d} f_x(|x|-M) = \kappa \chi(x). \tag{2.48}$$

Using (2.46) and (2.48) in (2.47), we have

$$\kappa\,|1-\chi(x)| \leq d^{-1} D(|x|+M)^{-d} + 4DM(|x|-M)^{-d}. \tag{2.49}$$

Now since χ is smooth, (2.49) implies (2.36). Also, (2.49) implies (2.37) and (2.38) for large $|x|$. But for x in any bounded domain, (2.37) holds by the smoothness of χ and E and the strong maximum principle. This completes the proof. □

Lemma 2.7. *There exists a constant $C = C(\nu, R) < \infty$ so that, for all $t > 0$ and all $\omega, \omega' \in \mathbf{R}^{2d}$, we have*

$$\Gamma_t(\omega;\omega') \leq Ct^{-d} \exp\left\{-C^{-1}t^{-1}|\omega-\omega'|^2\right\}. \tag{2.50}$$

Proof. Recall that we write $\omega = (w, z)$ and $\omega' = (w', z')$. Let

$$\tilde{\chi}(\omega) = \tilde{\chi}(w, z) = \frac{\chi(z)}{\int_{|\omega'|^2 \le 1} \chi(z')\,\mathrm{d}w'\mathrm{d}z'},$$

with χ as in Proposition 2.3, which satisfies $\mathcal{L}^*\tilde{\chi} = 0$ by (2.35), and also satisfies $\int_{|\omega|^2 \le 1} \tilde{\chi}(\omega)\,\mathrm{d}\omega = 1$. By (2.37), there is a constant $C = C(\nu, R) < \infty$ so that for all $t > 0$ and all $\omega \in \mathbf{R}^{2d}$, we have

$$\tilde{\chi}(\omega) \le C$$

and

$$\int_{B_t(\omega)} \tilde{\chi}(\omega')\,\mathrm{d}\omega' \ge C^{-1} t^{2d}.$$

Using these bounds in the result of [20, Theorem 1.2] (noting that our $\tilde{\chi}$ is denoted there by W), we have another constant $C = C(\nu, R)$ so that, for all $t > 0$ and all $\omega, \omega' \in \mathbf{R}^{2d}$, the estimate (2.50) holds. Note that [20] is written in terms of the nondivergence form PDE (2.33), but the fundamental solutions are related by simply swapping the arguments and so the same bound holds for Γ_t. The proof is complete. □

Remark 2.8. Note that [20] assumes that the dimension d is at least 2, but the proof of the upper bound in [20, Theorem 1.2] given there works also for $d = 1$. Actually, the proof is in fact simpler as it follows just from the Krylov–Safonov Harnack inequality [20, Theorem 3.1] and the construction of a subsolution [20, Lemma 3.1] as in the derivation leading to [20, (3.8)], using the explicit construction of the invariant measure given in Remark 2.4 and the fact that in $d = 1$, what [20] calls a "normalized adjoint solution" is in fact just a solution to the original nondivergence-form equation.

Lemma 2.9. *There exists a constant $C = C(\nu, R) < \infty$ so that, for all $t \ge 1$ and all $\omega, \omega' \in \mathbf{R}^{2d}$, we have*

$$\left|\nabla^2_{\omega'}\Gamma_t(\omega;\omega')\right| \le C t^{-d} \exp\left\{-C^{-1}t^{-1}|\omega - \omega'|^2\right\}. \tag{2.51}$$

Proof. By the Chapman–Kolmogorov equation we have

$$\Gamma_t(\omega;\omega') = \int \Gamma_{t-1/2}(\omega;\omega'')\Gamma_{1/2}(\omega'';\omega')\,\mathrm{d}\omega''.$$

Thus we have

$$|\nabla^2_{\omega'}\Gamma_t(\omega;\omega')| \le \int |\Gamma_{t-1/2}(\omega;\omega'')| \cdot |\nabla^2_{\omega'}\Gamma_{1/2}(\omega'';\omega')|\,\mathrm{d}\omega''. \tag{2.52}$$

By [22, Theorem 9.6.7 on p. 261] (which again concerns the fundamental solution for the adjoint problem (2.33), but that corresponds to our fundamental solution by swapping the arguments), using the assumed smoothness of A, we have a constant $C < \infty$ so that

$$|\nabla^2_{\omega'}\Gamma_{1/2}(\omega'';\omega')| \le C \exp\{-C^{-1}|\omega'' - \omega|^2\}.$$

Using this bound along with Lemma 2.7 in (2.52) we obtain (2.51). □

3 The stationary solution

Let $u^{[M]}$ solve (2.7) but with constant initial condition 1 at time $-M$, i.e.,

$$\begin{aligned} \mathrm{d}u^{[M]}(t) &= \frac{1}{2}\mathrm{tr}[(\nu I_d + R(0))\nabla^2 u^{[M]}(t)]\mathrm{d}t - \nabla \cdot [u^{[M]}(t)V(\mathrm{d}t)], \qquad t > -M \\ u^{[M]}(-M) &\equiv 1. \end{aligned} \tag{3.1}$$

The main result of this section is the following proposition.

Proposition 3.1. *For any $t \in \mathbf{R}$, $x \in \mathbf{R}^d$, the sequence $(u_t^{[M]}(x))_{M>-t}$ is a Cauchy sequence in $L^2(\Omega)$. In particular, for any $\varepsilon > 0$ (or $\varepsilon = 0$ if $d = 1$) there exists a constant $C = C(\varepsilon, \nu, R) < \infty$ so that for any $M_1, M_2 > -t$,*

$$\mathbf{E}\left|u^{[M_1]}(t,x) - u^{[M_2]}(t,x)\right|^2 \le C[(t+M_1)^{-(d-\varepsilon)/2} + (t+M_2)^{-(d-\varepsilon)/2}]. \tag{3.2}$$

Proof. Define

$$S_t^{[M]}(z) = \mathbf{E}u^{[M]}(t,0)u^{[M]}(t,z),$$

which is equal to $\mathbf{E}u^{[M]}(t,w-z/2)u^{[M]}(t,w+z/2)$ for all $w \in \mathbf{R}^d$, due to the fact that the noise is spatially translation-invariant and that the initial data is constant. Thus $S_t^{[M]}$ satisfies the PDE

$$\partial_t S_t^{[M]}(z) = \overline{\mathcal{L}}^* S_t^{[M]}(z) = \frac{1}{2}\operatorname{tr}[\nabla^2(A_{22}S_t^{[M]})(z)], \tag{3.3}$$

$$S_{-M}^{[M]}(z) = 1, \tag{3.4}$$

where we have defined $\overline{\mathcal{L}}f(z) = \frac{1}{2}\operatorname{tr}[A_{22}(z)\nabla^2 f(z)]$. The problem (3.3)–(3.4) is obtained from (2.30)–(2.31) by using the space-translation-invariance and the fact that the initial data is a constant. The PDE (3.3) has fundamental solution $\overline{\Gamma}$ given by

$$\overline{\Gamma}_t(z;z') = \int \Gamma(y,z;y',z')\,\mathrm{d}y,$$

which in fact is independent of y'. Integrating (2.50) over y, we have a constant $C = C(\nu, R) < \infty$ so that

$$\overline{\Gamma}_t(z;z') \le Ct^{-d/2}\exp\left\{-C^{-1}t^{-1}|z-z'|^2\right\}. \tag{3.5}$$

Now we recall the function χ from Proposition 2.3. We note that

$$\int \overline{\Gamma}_t(z;z')\chi(z')\,\mathrm{d}z' = \chi(z) \tag{3.6}$$

by (2.35). Thus we have

$$\begin{aligned} S_t^{[M]}(z) = \int \overline{\Gamma}_{t+M}(z;z')\,\mathrm{d}z' &= \int \overline{\Gamma}_{t+M}(z;z')(\chi(z') - [\chi(z')-1])\,\mathrm{d}z' \\ &= \chi(z) - \int \overline{\Gamma}_{t+M}(z;z')[\chi(z')-1]\,\mathrm{d}z'. \end{aligned} \tag{3.7}$$

By Hölder's inquality, for $1/p + 1/q = 1$ we have

$$\left|\int \overline{\Gamma}_{t+M}(z;z')[\chi(z')-1]\,\mathrm{d}z'\right| \le \|\overline{\Gamma}_{t+M}(z;\cdot)\|_{L^q(\mathbf{R}^d)}\|\chi - 1\|_{L^p(\mathbf{R}^d)}. \tag{3.8}$$

By (3.5), we have

$$\begin{aligned} \|\overline{\Gamma}_t(z;\cdot)\|_{L^q(\mathbf{R}^d)} &\le \left(\int Ct^{-dq/2}\exp\left\{-qC^{-1}t^{-1}|z-z'|^2\right\}\mathrm{d}z'\right)^{1/q} \\ &= \left(\int Ct^{-d(q-1)/2}\exp\left\{-qC^{-1}|z'|^2\right\}\mathrm{d}z'\right)^{1/q} \le Ct^{-\frac{d}{2p}}. \end{aligned} \tag{3.9}$$

Thus, using (2.36) and (3.9) in (3.8) and then substituting into (3.7), we have for any $\varepsilon > 0$ (choosing $p = d/(d-\varepsilon)$), there exists a constant C so that

$$|S_t^{[M]}(z) - \chi(z)| \le C(t+M)^{-(d-\varepsilon)/2}. \tag{3.10}$$

When $d=1$, by Remark 2.4 we can take $p=1$, $q=\infty$, and thus $\varepsilon=0$.

Now define, for $M_1<M_2$,

$$S_t^{[M_1,M_2]}(z)=\mathbf{E}(u^{[M_1]}(t,0)-u^{[M_2]}(t,0))(u^{[M_1]}(t,z)-u^{[M_2]}(t,z)),\qquad t>-M_1, z\in\mathbf{R}^d.$$

Then $S_t^{[M_1,M_2]}$ again satisfies the PDE (3.3), since $u^{[M_1]}-u^{[M_2]}$ also satisfies (2.7) by linearity. On the other hand, we have the corresponding initial condition

$$\begin{aligned}S_{-M_1}^{[M_1,M_2]}(z)&=\mathbf{E}(u^{[M_1]}(-M_1,0)-u^{[M_2]}(-M_1,0))(u^{[M_1]}(-M_1,z)-u^{[M_2]}(-M_1,z))\\&=\mathbf{E}(1-u^{[M_2]}(-M_1,0))(1-u^{[M_2]}(-M_1,z))\\&=\mathbf{E}u^{[M_2]}(-M_1,0)u^{[M_2]}(-M_1,z)-1\\&=S_{-M_1}^{[M_2]}(z)-1.\end{aligned}$$

Here we used the fact that $\mathbf{E}u^{[M_2]}\equiv 1$. From this and the linearity of (3.3) we further conclude that

$$S_t^{[M_1,M_2]}(z)=S_t^{[M_2]}(z)-S_t^{[M_1]}(z),\qquad t>-M_1, z\in\mathbf{R}^d. \tag{3.11}$$

Combining (3.10), (3.11), and the triangle inequality, we have

$$\mathbf{E}\left|u^{[M_1]}(t,x)-u^{[M_2]}(t,x)\right|^2=S_t^{[M_1,M_2]}(0)\le C[(t+M_1)^{-(d-\varepsilon)/2}+(t+M_2)^{-(d-\varepsilon)/2}],$$

which is (3.2). ◻

Corollary 3.2. *There is a random function $U:\mathbf{R}\times\mathbf{R}^d\to\mathbf{R}$ so that, for every $t\in\mathbf{R}$ and $x\in\mathbf{R}^d$, we have for any $\varepsilon>0$ (or $\varepsilon=0$ if $d=1$) that*

$$\lim_{M\to\infty}(t+M)^{(d-\varepsilon)/2}\mathbf{E}|u^{[M]}(t,x)-U(t,x)|^2=0.$$

Moreover, U is stationary under time and space translations, $\mathbf{E}U\equiv 1$, and it is a solution to the SPDE (2.7). Finally, for $x_1,x_2\in\mathbf{R}$, we have

$$\mathbf{E}U(t,x_1)U(t,x_2)=\chi(x_1-x_2), \tag{3.12}$$

with χ as in Proposition 2.3.

Proof. We can construct U as the limit of the $u^{[M]}$ in an appropriate spatially- and temporally-weighted L^2 space using Proposition 3.1 and Fubini's theorem. The limit preserves the expectation, so $\mathbf{E}U\equiv 1$. Spatial and temporal stationarity, and the fact that solving the SPDE (2.7) (i.e. solving the integral equation (2.9)) passes to the limit, is clear. Finally, (3.12) follows directly from the convergence of $u^{[M]}(t,x)\to U(t,x)$ in $L^2(\Omega)$ and equation (3.10). ◻

Remark 3.3. The spacetime stationary random field U solves equation (2.7), which is related to the Fokker-Planck equation for the process of "environment seen from the particle." If we use $U(0,0)$ as the Radon-Nikodym derivative to tilt the probability measure $\mathbf{P}$, then the new measure is actually the invariant measure for the "environment seen from the particle." For a model of random walk in balanced random environment, [17] proved a similar result as Theorem 1.1. The $U(t,x)$ constructed above corresponds to the $\rho_\omega(x,t)$ defined in [17, Page 3], and the SPDE (2.7) corresponds to [17, Equation (3)]. For a Markovian velocity field with a large spectral gap, the invariant measure was constructed in [25], in parallel to our construction of the spacetime stationary solution to equation (2.7), although the velocity field here is white in time which corresponds to an infinite spectral gap.

Corollary 3.4. *For any sequence $((t_k,M_k))_{k\ge 0}$ so that $t_k+M_k\to\infty$ as $k\to\infty$, we have for any $\varepsilon>0$ (or $\varepsilon=0$ if $d=1$) that*

$$\lim_{k\to\infty}(t_k+M_k)^{(d-\varepsilon)/2}\mathbf{E}|u^{[M_k]}(t_k,x)-U(t_k,x)|^2=0. \tag{3.13}$$

Proof. This is clear from the translation-invariance and Corollary 3.2. ◻

4 Proof of Theorem 1.1

Let $u(t)$ solve (2.7) with $u(0) = \delta_0$, a Dirac delta measure at zero. For any $q \le t$, let $\mathcal{F}_{q,t}$ be the σ-algebra generated by $V(s) - V(r)$ for $q \le r \le s \le t$. Given $q > 0$, define

$$\tilde{u}_q(t) = \mathbf{E}[u(t) \mid \mathcal{F}_{q,t}], \qquad t \ge q.$$

Then $t \mapsto \tilde{u}_q(t)$ satisfies (2.7) for $t > q$, and we have the initial condition

$$\tilde{u}_q(q) = G_q,$$

where we recall that G solves (1.7). An important step in the proof of Theorem 1.1 is the following proposition.

Proposition 4.1. *For any $\beta \in (0,1)$, there exists a constant $C = C(R, \nu, \beta) < \infty$ so that, for all $t \ge C$,*

$$\sup_{x \in \mathbf{R}^d} \mathbf{E}|u(t,x) - \tilde{u}_{t-t^\beta}(t,x)|^2 \le C t^{-(d \wedge (d/2+1)) - \beta d/2} \log t. \tag{4.1}$$

Proof. Step 1: taking second moments. Define

$$\overline{A} = \begin{pmatrix} \frac{1}{4}[\nu I_d + R(0)] & \\ & \nu I_d + R(0) \end{pmatrix}, \qquad \tilde{A}(z) = \begin{pmatrix} \frac{1}{8}\left[R(z) + R(z)^{\mathrm{T}}\right] & \frac{1}{4}\left[R(z) - R(z)^{\mathrm{T}}\right] \\ \frac{1}{4}\left[R(z)^{\mathrm{T}} - R(z)\right] & -\frac{1}{2}\left[R(z) + R(z)^{\mathrm{T}}\right] \end{pmatrix},$$

so we can decompose (2.29) as

$$A(z) = \overline{A} + \tilde{A}(z). \tag{4.2}$$

Let H_t be the solution to the PDE

$$\begin{aligned} \partial_t H_t(\omega) &= \operatorname{tr}[\overline{A} \nabla^2 H_t(\omega)]; \\ H_0 &= \delta_0. \end{aligned}$$

This means that $H_t(w,z) = G_{t/2}(w) G_{2t}(z)$. For any $q > 0$, if we define

$$S_t(y,z) = \mathbf{E} u(t, y + z/2) u(t, y - z/2), \qquad \tilde{S}_{q,t}(y,z) = \mathbf{E} \tilde{u}_q(t, y + z/2) \tilde{u}_q(t, y - z/2),$$

then S_t satisfies (2.30) with initial condition

$$S_0(y,z) = \delta_0(y)\delta_0(z) \tag{4.3}$$

and $t \mapsto \tilde{S}_{q,t}$ satisfies (2.30) for $t > q$ with initial condition

$$\tilde{S}_{q,q} = H_q. \tag{4.4}$$

In particular, we have

$$S_t(\omega) = \Gamma_t(\omega; 0). \tag{4.5}$$

Then define

$$\overline{S}_{q,t}(y,z) = \mathbf{E}(u(t, y + z/2) - \tilde{u}_q(t, y + z/2))(u(t, y - z/2) - \tilde{u}_q(t, y - z/2)).$$

Again $t \mapsto \overline{S}_{q,t}$ satisfies (2.30) in $t > q$. The initial condition is

$$\begin{aligned} \overline{S}_{q,q}(y,z) &= \mathbf{E}\left(u(q, y + z/2) - \tilde{u}_q(q, y + z/2)\right)\left(u(q, y - z/2) - \tilde{u}_q(q, y - z/2)\right) \\ &= \mathbf{E}\left(u(q, y + z/2) - G_q(y + z/2)\right)\left(u(q, y - z/2) - G_q(y - z/2)\right) \\ &= \mathbf{E} u(q, y + z/2) u(q, y - z/2) - G_q(y + z/2) G_q(y - z/2) \\ &= S_q(y,z) - \tilde{S}_{q,q}(y,z), \end{aligned} \tag{4.6}$$

where we used the fact that $\mathbf{E}u(t) = G_t$. Therefore, by the linearity of (2.30) we in fact have

$$\overline{S}_{q,t}(y,z) = S_t(y,z) - \tilde{S}_{q,t}(y,z)$$

for all $t > q$.

Step 2: proving the bound. Similar to the proof of Proposition 3.1, our goal is to prove an upper bound on $\overline{S}_{q,t}(y,0)$. Since $t \mapsto \overline{S}_{q,t}$ satisfies (2.30) we have the identity

$$\overline{S}_{q,t}(\omega) = \int \Gamma_{t-q}(\omega;\omega')\overline{S}_{q,q}(\omega')\,\mathrm{d}\omega'. \tag{4.7}$$

By the Duhamel principle applied to the PDE (2.30), using the decomposition (4.2), we have

$$S_q(\omega') = H_q(\omega') + \int_0^q \int H_{q-s}(\omega'-\omega'')\,\mathrm{tr}\left[\nabla^2[\tilde{A}S_s](\omega'')\right]\mathrm{d}\omega''\,\mathrm{d}s.$$

Subtracting (4.4) and recalling (4.6), we obtain

$$\overline{S}_{q,q}(\omega') = \int_0^q \int H_{q-s}(\omega'-\omega'')\,\mathrm{tr}\left[\nabla^2[\tilde{A}S_s](\omega'')\right]\mathrm{d}\omega''\,\mathrm{d}s.$$

Now plugging this into (4.7) and using Fubini's theorem, we obtain

$$\overline{S}_{q,t}(\omega) = \int_0^q \int K_{q-s,t-q}(\omega;\omega')\,\mathrm{tr}\left[\nabla^2[\tilde{A}S_s](\omega')\right]\mathrm{d}\omega'\,\mathrm{d}s, \tag{4.8}$$

where we have defined

$$K_{r_1,r_2}(\omega;\omega') = \int \Gamma_{r_2}(\omega;\omega'')H_{r_1}(\omega''-\omega')\,\mathrm{d}\omega''. \tag{4.9}$$

Integrating by parts in (4.8), we have

$$\overline{S}_{q,t}(\omega) = \int_0^q \int S_s(\omega')\,\mathrm{tr}\left[\tilde{A}(\omega')\nabla^2_{\omega'}K_{q-s,t-q}(\omega;\omega')\right]\mathrm{d}\omega'\,\mathrm{d}s. \tag{4.10}$$

Using Lemma 4.2 below, and also another application of Lemma 2.7 (and (4.5)) to bound $S_s(\omega')$, in (4.10), we obtain, for $t \geq q+1$,

$$\begin{aligned}\left|\overline{S}_{q,t}(\omega)\right| &\leq C\int_0^q [(q-s)^{-1}\wedge 1](t-s)^{-d}s^{-d}\int |\tilde{A}(z')|\exp\left\{-\frac{|\omega'-\omega|^2}{C(t-s)} - \frac{|\omega'|^2}{Cs}\right\}\mathrm{d}\omega'\,\mathrm{d}s\\ &= C\left(\int_0^1 + \int_1^q\right)[(q-s)^{-1}\wedge 1](t-s)^{-d}s^{-d}\int |\tilde{A}(z')|\exp\left\{-\frac{|\omega'-\omega|^2}{C(t-s)} - \frac{|\omega'|^2}{Cs}\right\}\mathrm{d}\omega'\,\mathrm{d}s\\ &:= I_1 + I_2.\end{aligned} \tag{4.11}$$

To control the above integral, we consider the region of $s \in (0,1)$ and $s \in (1,q)$ separately. For the integration in $s \in (0,1)$, by the fact that $\tilde{A}$ is uniformly bounded, we integrate in ω' to derive

$$I_1 \leq Ct^{-d}\int_0^1 [(q-s)^{-1}\wedge 1]\mathrm{d}s \leq Ct^{-d}(q-1)^{-1}. \tag{4.12}$$

For the integration in $s \in (1,q)$, to control the inner integral, we write

$$\begin{aligned}&\int |\tilde{A}(z')|\exp\left\{-\frac{|\omega'-\omega|^2}{C(t-s)} - \frac{|\omega'|^2}{Cs}\right\}\mathrm{d}\omega'\\ &= \left(\int |\tilde{A}(z')|\exp\left\{-\frac{|z'-z|^2}{C(t-s)} - \frac{|z'|^2}{Cs}\right\}\mathrm{d}z'\right)\left(\int \exp\left\{-\frac{|y'-y|^2}{C(t-s)} - \frac{|y'|^2}{Cs}\right\}\mathrm{d}y'\right)\\ &\leq C\|\tilde{A}\|_{L^1(\mathbf{R}^d;\mathbf{R}^{2d\times 2d})}\left(\frac{s(t-s)}{t}\right)^{d/2}\end{aligned}$$

for a new constant C, still depending only on R and ν. Using this bound in (4.11), we obtain

$$\begin{aligned} I_2 &\le Ct^{-d/2}\int_1^q [(q-s)^{-1}\wedge 1](t-s)^{-d/2}s^{-d/2}\,\mathrm{d}s \\ &\le Ct^{-d/2}(t-q)^{-d/2}\int_1^q [(q-s)^{-1}\wedge 1]s^{-d/2}\,\mathrm{d}s. \end{aligned} \tag{4.13}$$

Now we estimate the last integral in two parts. First we have

$$\int_{q/2}^q [(q-s)^{-1}\wedge 1]s^{-d/2}\,\mathrm{d}s \le (q/2)^{-d/2}\int_{q/2}^q [(q-s)^{-1}\wedge 1]\,\mathrm{d}s = (q/2)^{-d/2}[1+\log(q/2)].$$

Second, we have

$$\int_1^{q/2}(q-s)^{-1}s^{-d/2}\,\mathrm{d}s \le 2q^{-1}\int_1^{q/2} s^{-d/2}\,\mathrm{d}s \le Cq^{-1}\left(\sqrt{q}\mathbb{1}_{d=1}+\log q\,\mathbb{1}_{d=2}+\mathbb{1}_{d\ge 3}\right) \le Cq^{-((d/2)\wedge 1)}\log q.$$

Using the last two inequalities in (4.13) and taking $q = t - t^\beta$ we obtain, for a C now depending also on β,

$$I_2 \le Ct^{-(d\wedge(d/2+1))-\beta d/2}\log t.$$

Combining this with (4.12), we obtain (4.1). $\square$

Now we must prove the lemma we used in the previous proof.

Lemma 4.2. *Recall the definition (4.9) of K_{r_1,r_2}. There is a constant $C = C(\nu, R) < \infty$ so that, for all $r_1 > 0$ and $r_2 \ge 1$, we have*

$$\left|\nabla^2_{\omega'}K_{r_1,r_2}(\omega;\omega')\right| \le C(r_1^{-1}\wedge 1)(r_1+r_2)^{-d}\exp\left\{-\frac{|\omega''-\omega|^2}{C(r_1+r_2)}\right\}. \tag{4.14}$$

Proof. Differentiating (4.9) and using Lemma 2.7, we have

$$\begin{aligned} \left|\nabla^2_{\omega'}K_{r_1,r_2}(\omega;\omega')\right| &\le \int \Gamma_{r_2}(\omega;\omega'')\left|\nabla^2_{\omega'}H_{r_1}(\omega''-\omega')\right|\mathrm{d}\omega'' \\ &\le Cr_2^{-d}r_1^{-d-1}\int \exp\left\{-C^{-1}r_2^{-1}|\omega-\omega''|^2 - C^{-1}r_1^{-1}|\omega''-\omega'|^2\right\}\mathrm{d}\omega'' \\ &\le Cr_1^{-1}(r_1+r_2)^{-d}\exp\left\{-C^{-1}(r_1+r_2)^{-1}|\omega-\omega'|^2\right\}, \end{aligned} \tag{4.15}$$

where we allowed the constant C to change from line to line. Alternatively, we can use integration by parts and Lemma 2.9 to derive that

$$\begin{aligned} \left|\nabla^2_{\omega'}K_{r_1,r_2}(\omega;\omega')\right| &\le \int \left|\nabla^2_{\omega''}\Gamma_{r_2}(\omega;\omega'')\right|H_{r_1}(\omega''-\omega')\,\mathrm{d}\omega'' \\ &\le Cr_2^{-d}r_1^{-d}\int \exp\left\{-C^{-1}r_2^{-1}|\omega-\omega''|^2 - C^{-1}r_1^{-1}|\omega''-\omega'|^2\right\}\mathrm{d}\omega'' \\ &\le C(r_1+r_2)^{-d}\exp\left\{-C^{-1}(r_1+r_2)^{-1}|\omega-\omega'|^2\right\}, \end{aligned} \tag{4.16}$$

where again C changed from line to line. Together, (4.15) and (4.16) imply (4.14). $\square$

Now we want to show that, when $1 \ll t-q \ll t$, the field $\tilde{u}_q(t)$ is well-approximated by the stationary solution $U(t)$ multiplied by G_t. Let $\underline{u}_q(t)$ solve (2.7) in $t > q$, with initial condition

$$\underline{u}_q(q) \equiv 1,$$

so by Corollary 3.4 we have

$$\lim_{t-q\to\infty} \sup_{x\in\mathbf{R}^d} \mathbf{E}|\underline{u}_{q,t}(x) - U_t(x)|^2 = 0. \tag{4.17}$$

Proposition 4.3. *There exists a constant C so that, for any $x \in \mathbf{R}$ and $t > q$, we have*

$$\mathbf{E}|\tilde{u}_q(t,x) - G_q(x)\underline{u}_q(t,x)|^2 \le Cq^{-d-1}(t-q), \tag{4.18}$$

and in particular, for any $\beta \in (0,1)$, there exists a constant $C = C(R,\nu,\beta)$ such that for all $t > 1$

$$\sup_{x\in\mathbf{R}^d} \mathbf{E}\left|\tilde{u}_{t-t^\beta}(t,x) - G_{t-t^\beta}(x)\underline{u}_{t-t^\beta}(t,x)\right|^2 \le Ct^{-d-1+\beta}. \tag{4.19}$$

Proof. Fix q, x. First recall that $\tilde{u}_q, \underline{u}_q$ both solve (2.7) in $t > q$. Let

$$\underline{S}_{q,t,x}(y,z) = \mathbf{E}(\tilde{u}_q(t,y+z/2) - G_q(x)\underline{u}_q(t,y+z/2))(\tilde{u}_q(t,y-z/2) - G_q(x)\underline{u}_q(t,y-z/2)).$$

Then as a function of (t,y,z), we have $\underline{S}_{q,t,x}$ solves (2.30) with initial condition

$$\underline{S}_{q,q,x}(y,z) = \left(G_q(y+z/2) - G_q(x)\right)\left(G_q(y-z/2) - G_q(x)\right).$$

Therefore, we have

$$\underline{S}_{q,t,x}(y,z) = \int \left(G_q(y'+z'/2) - G_q(x)\right)\left(G_q(y'-z'/2) - G_q(x)\right)\Gamma_{t-q}(y,z;y',z')\,\mathrm{d}y'\,\mathrm{d}z',$$

and so

$$\begin{aligned}
\underline{S}_{q,t,x}(x,0) &= \int \left|G_q(y'+z'/2) - G_q(x)\right|\left|G_q(y'-z'/2) - G_q(x)\right|\Gamma_{t-q}(x,0;y',z')\,\mathrm{d}y'\,\mathrm{d}z' \\
&\le C(t-q)^{-d}\int \left|G_q(y'+z'/2) - G_q(x)\right|\left|G_q(y'-z'/2) - G_q(x)\right|\mathrm{e}^{-\frac{|y'-x|^2+|z'|^2}{C(t-q)}}\,\mathrm{d}y'\,\mathrm{d}z' \\
&\le C(t-q)^{-d}\|\nabla G_q\|_\infty^2 \int |y'+z'/2|\cdot|y'-z'/2|\mathrm{e}^{-\frac{|y'|^2+|z'|^2}{C(t-q)}}\,\mathrm{d}y'\,\mathrm{d}z' \\
&\le Cq^{-d-1}(t-q)\int |y'+z'/2|\cdot|y'-z'/2|\mathrm{e}^{-\frac{|y'|^2+|z'|^2}{C}}\,\mathrm{d}y'\,\mathrm{d}z',
\end{aligned}$$

where in the first inequality we used Lemma 2.7. This completes the proof of (4.18), and (4.19) follows immediately. ◻

Now we can prove our main theorem.

Proof of Theorem 1.1. Fix $\beta \in (0,1)$ and let $q = t - t^\beta$. We use the triangle inequality to write

$$\begin{aligned}
\mathbf{E}|u(t,x) - G_t(x)U(t,x)|^2 &\le C\mathbf{E}|u(t,x) - \tilde{u}_q(t,x)|^2 + C\mathbf{E}|\tilde{u}_q(t,x) - G_q(x)\underline{u}_q(t,x)|^2 \\
&\quad + C|G_q(x) - G_t(x)|^2\mathbf{E}\underline{u}_q(t,x)^2 + CG_t(x)^2\mathbf{E}|\underline{u}_q(t,x) - U(t,x)|^2.
\end{aligned} \tag{4.20}$$

We note that

$$\sup_{x\in\mathbf{R}^d}|G_q(x)-G_t(x)|\le(t-q)\sup_{x\in\mathbf{R}^d,s\in[q,t]}|\partial_s G_s(x)|\le Ct^{\beta-d/2-1}. \tag{4.21}$$

Applying (4.1), (4.19), (4.21) (along with the fact that $\mathbf{E}\underline{u}_q(t,x)^2$ is uniformly bounded by Corollary 3.4), and (3.13), respectively, to the four terms on the right side of (4.20), we obtain for every $\varepsilon>0$ (or $\varepsilon=0$ if $d=1$), there is a constant $C=C(R,\nu,\beta,\varepsilon)<\infty$ so that

$$\mathbf{E}|u(t,x)-G_t(x)U(t,x)|^2\le C\left(t^{-(d\wedge(d/2+1))-\beta d/2}\log t+t^{-d-1+\beta}+t^{2\beta-d-2}+t^{-d-\beta d/2+\beta\varepsilon}\right),$$

Then we take

$$\beta=\tfrac{2}{3}\mathbb{1}_{d=1}+\tfrac{d}{d+2}\mathbb{1}_{d\ge2}$$

to further derive that

$$\mathbf{E}|u(t,x)-G_t(x)U(t,x)|^2\le Ct^{-d}\left(t^{-1/3}\mathbb{1}_{d=1}+t^{-2/(2+d)+\varepsilon d/(2+d)}\mathbb{1}_{d\ge2}\right)\log t.$$

Changing ε yields (1.8), and (1.9) is then a consequence of the formula for the Gaussian density.

If $\sum_{i=1}^d\frac{\partial R_{ij}}{\partial x_i}\equiv0$ for each j, then by Remark 2.5, (3.12) and the fact that $\mathbf{E}U\equiv1$, we have $U\equiv1$ almost surely. On the other hand, if V is not incompressible, then it is clear that the constant 1 does not solve (2.7), and so U cannot be a.s. identically equal to 1 by Corollary 3.2. Finally, (1.10) follows from (3.12) and (2.38). This completes the proof of the theorem. □

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