HIGH TEMPERATURE BEHAVIORS OF THE DIRECTED POLYMER ON A CYLINDER

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ABSTRACT. In this paper, we study the free energy of directed polymers on a cylinder of radius L with the inverse temperature β. Assuming the random environment is given by a Gaussian process that is white in time and smooth in space, with an arbitrary compactly supported spatial covariance function, we obtain precise scaling behaviors of the limiting free energy for high temperatures $\beta \ll 1$, followed by large $L \gg 1$, in all dimensions. Our approach is based on a perturbative expansion of the PDE hierarchy satisfied by the multipoint correlation function of the polymer endpoint distribution. We also study the case where the random environment is given by the $1+1$ spacetime white noise, and derive an explicit expression of the limiting free energy.

KEYWORDS: Directed polymer, Lyapunov exponent, PDE hierarchy.

1. INTRODUCTION

1.1. **Main result.** The random polymer model studied in this paper is associated with the following stochastic heat equation (SHE) on the d-dimensional torus $\mathbb{T}_L^d$ of size $L > 0$, i.e.,

$$\partial_t u = \frac{1}{2}\Delta u + \beta u V(t,x), \qquad t > 0,\ x \in \mathbb{T}_L^d. \tag{1.1}$$

The d dimensional torus is the product of d copies of $\mathbb{T}_L$, understood as the interval $[-\frac{L}{2}, \frac{L}{2}]$ with identified endpoints. The random potential V is a Gaussian noise that is white in time and smooth in the spatial variable, and we assume

$$\mathbb{E}\left[V(t,x)V(s,y)\right] = \delta(t-s)R(x-y), \quad (t,x),\,(s,y) \in \mathbb{R} \times \mathbb{T}_L^d. \tag{1.2}$$

Throughout the paper we assume $R(\cdot)$ belongs to $C_0^\infty(\mathbb{R}^d)$ - the space of smooth and compactly supported functions. It is a fixed non-negative function that does not depend on the parameter L. We consider the case when L is so large that the support of $R(\cdot)$ is contained within $\mathbb{T}_L^d$, and it is normalized so that $\int_{\mathbb{R}^d} R(z)dz = 1$. The parameter $\beta > 0$, referred to as the inverse of temperature, controls the strength of the noise.

Suppose $u(0,x)=\delta(x)$, then $Z_t=\int_{\mathbb{T}_L^d}u(t,x)dx$ is the partition function of a directed polymer model, as can be seen from the Feynman-Kac representation

$$Z_t=\mathbb{E}_B\exp\left\{\beta\int_0^t V(s,B_s)ds-\frac{1}{2}\beta^2R(0)t\right\}, \tag{1.3}$$

where $\mathbb{E}_B$ is the expectation with respect to the standard Brownian motion B on $\mathbb{T}_L^d$ starting from the origin and independent of the noise $V(t,x)$. It is well-known, see e.g. [23, Theorem 2.5], that the following limit exists

$$\gamma_L(\beta)=\lim_{t\to\infty}\frac{1}{t}\log Z_t=\lim_{t\to\infty}\frac{1}{t}\mathbb{E}\log Z_t, \tag{1.4}$$

and is the thermodynamic limit of the free energy of the directed polymer. The $\gamma_L(\beta)$ will depend on the particular choice of the spatial covariance function $R(\cdot)$. Here we are interested in extracting the universal behaviors in the high temperature regime of $\beta\to 0$, followed by $L\to\infty$. Define the Fourier transform of $R(\cdot)$ by

$$\hat{R}(\xi)=\int_{\mathbb{R}^d}R(x)e^{-i2\pi\xi\cdot x}dx \quad \text{for any } \xi\in\mathbb{R}^d. \tag{1.5}$$

Since $R(\cdot)$ is a covariance function, we have $\hat{R}(\xi)\ge 0$ for all ξ. The Fourier coefficients of the L-periodic version of R, with the size L of the torus satisfying $\operatorname{supp}R(\cdot)\subset[-L/2,L/2]^d$, are given by $\hat{R}\left(\frac{n}{L}\right)$, $n\in\mathbb{Z}^d$.

Here is the main result of the paper:

Theorem 1.1. *Fix any $L>0$. Then,*

$$\gamma_L(\beta)=\gamma_L^{(2)}\beta^2+\gamma_L^{(4)}\beta^4+O(\beta^6), \quad \text{as } \beta\ll 1, \tag{1.6}$$

with

$$\gamma_L^{(2)}=-\frac{1}{2L^d}, \qquad \gamma_L^{(4)}=-\frac{1}{8\pi^2L^{2d-2}}\sum_{0\ne n\in\mathbb{Z}^d}\frac{1}{|n|^2}\hat{R}^2\left(\frac{n}{L}\right).$$

In addition,

$$\begin{aligned}
&\lim_{L\to\infty}\gamma_L^{(4)}=-\frac{1}{24}, && d=1,\\
&\lim_{L\to\infty}\frac{L^2}{\log L}\gamma_L^{(4)}=-\frac{1}{4\pi}, && d=2,\\
&\lim_{L\to\infty}L^d\gamma_L^{(4)}=-\frac{1}{8\pi^2}\int_{\mathbb{R}^d}|\xi|^{-2}\hat{R}^2(\xi)d\xi, && d\ge 3.
\end{aligned} \tag{1.7}$$

1.2. **Context.** The study of directed polymers in random environments is an active area in probability and statistical physics. The interests are in the transversal displacements of the polymer endpoint, the fluctuations of the free energy, the localization behaviors of the sample paths etc. We refer to the monograph [15] for a general introduction to the subject. The partition function of the directed polymer is naturally connected to the heat equation with a random potential, through the Feynman-Kac representation as (1.3). After the Hopf-Cole transformation, it is related to the Kardar-Parisi-Zhang

(KPZ) equation, which is a default model for interface growth subjected to random perturbations, see the reviews [16, 36] on the recent developments on the 1+1 KPZ universality class.

Besides studying the directed polymers on the free space where the sample paths are spread out without any constraint, there have been many recent developments on understanding how the underlying geometry or the boundary conditions affect the large scale behaviors of the polymer measure and the associated SHE and KPZ problem, see e.g. [18, 35, 20, 17, 13, 4, 5] and the references therein. In this paper, we consider the polymers confined to a cylinder and study the high temperature behaviors of the limiting free energy. Our study is partly motivated by the results in [30, 7, 33, 34], where the same problem has been considered in the whole space. If we denote the limiting free energy by $\gamma_\infty(\beta)$ in this case, it has been shown in the aforementioned works, for a large class of discrete models and as $\beta \to 0$,

$$\frac{1}{\beta^4}\gamma_\infty(\beta) \to -\frac{1}{24}, \qquad \text{in } d = 1, \tag{1.8}$$

and

$$\beta^2 \log \gamma_\infty(\beta) \to -\pi, \qquad \text{in } d = 2. \tag{1.9}$$

The limiting constants $-\frac{1}{24}$ and $-\pi$ are universal as they do not depend on the specific distributions of the underlying random environment. Compare to the expansion in (1.6), there is some similarity in $d = 1$. It is worth emphasizing that the free energy defined in (1.4) is actually the difference between the quenched and annealed free energies considered in those works. This is only a matter of convention: if we define the partition function by $\mathbb{E}_B e^{\beta\int_0^t V(s,B_s)ds}$, then the quenched free energy is $t^{-1}\log \mathbb{E}_B e^{\beta\int_0^t V(s,B_s)ds}$, and the annealed free energy is $\frac{1}{2}\beta^2 R(0)$, so their difference is precisely $t^{-1}\log Z_t$ with Z_t defined as in (1.3). It is well-known that the free energy is associated to the localization properties of the polymer paths, and is related to the overlap fraction of two replicas, see e.g. the discussion in [15, Chapter 5 and 6]. Therefore, the study of $\gamma_L(\beta)$ for small β sheds light on the localization properties of the polymer paths in high temperature regimes.

Another motivation comes from the replica method used to compute the free energy. In [12, 11], the authors considered the same problem of directed polymers on a cylinder. For the environment of a $1+1$ spacetime white noise, using the Bethe ansatz method, they derived the expansions of the ground state energy $E(n,\beta,L)$ of the Delta Bose gas in $d = 1$

$$\mathcal{H}_n = \frac{1}{2}\sum_{i=1}^n \nabla_i^2 + \beta^2 \sum_{1\le i<j\le n} \delta(x_i - x_j).$$

In [12, Equation (49)], it says that

$$E(n,\beta,L) = -\Big(\frac{\beta^2}{2L} + \frac{\beta^4}{24}\Big)n + c_2 n^2 + c_3 n^3 + \dots,$$

for some explicit $c_2, c_3, \ldots$. If the replica method works here, then the coefficient of the $O(n)$ term, which is $-\left(\frac{\beta^2}{2L} + \frac{\beta^4}{24}\right)$, should give us $\gamma_L(\beta)$ in the case of $R(\cdot) = \delta(\cdot)$. We will show in Section 4 below, in this particular case, $\gamma_L(\beta)$ can be written as an explicit integral, see (4.2). Performing a small β expansion leads to

$$\frac{1}{\beta^6}\left(\gamma_L(\beta) + \frac{\beta^2}{2L} + \frac{\beta^4}{24}\right) \to \gamma^{(6)} \neq 0, \qquad \text{as } \beta \to 0,$$

see (4.4). This shows that the high order terms are missing in the replica method calculation. Nevertheless for the problem on the whole space, the replica method actually leads to the correct answer [26, 14, 9, 2, 38, 39, 21]. At the end of [12], the authors mentioned that "another interesting extension of the present work would be to consider more general correlations of the noise" and "one could try to extend the approach to higher dimension as the relation between the directed polymer problem and the quantum Hamiltonian is valid in any dimension". Our work can be viewed as a preliminary step along this direction, in which we obtain the high temperature expansions of the limiting free energy, for general covariance functions and in all dimensions.

Our approach is based on a formula that relates $\gamma_L(\beta)$ with the replica overlap of the polymer measure. The idea is to perform a semi-martingale decomposition of $\log Z_t$, see e.g. [15, Chapter 5]. After taking the expectation, the only contribution to $t^{-1}\log Z_t$ comes from the drift and can be expressed as a time average of the overlap fraction of two replicas. On the cylinder, the polymer endpoint distribution converges exponentially fast to the stationary distribution, see the proofs in [23, 37] and the related results for stochastic Burgers equation [40]. The overlap fraction of two replicas is simply related to the two-point correlation function of the stationary distribution. In this way, the limiting free energy can be written explicitly as an integral involving the two-point correlation function of the stationary distribution and the spatial covariance function of the random environment, see (2.2) below.

On the cylinder, the stationary distribution of the polymer endpoint is related to that of the KPZ equation (modulo a constant) and to the stochastic Burgers equation. It is well-known that for the $1+1$ spacetime white noise, the stationary distribution of the KPZ equation is the Brownian bridge [8, 19, 24, 25]. Using this connection and Yor's formula for the density of exponential functionals of Brownian bridge [41], the limiting free energy can be written down explicitly in this case, see Proposition 4.1 in Section 4. For the noise with a general covariance structure, which is the main interest of this paper, there are no explicit formulas of the invariant measure. We proceed in a different way, using a partial differential equation (PDE) hierarchy satisfied by the n−point correlation functions of the stationary distribution, see (2.6) below. The PDE hierarchy was derived in [22] on the whole space, and it admits a stationary solution on the cylinder. An asymptotic expansion in β^2 on the level of the hierarchy leads to the corresponding expansion of the limiting free energy. The approach is surprisingly simple, and we can actually obtain the expansion in β^2 up to any order, see the discussion in Section 4.2 below.

The same approach does not apply to the problem on the whole space. As t goes to infinity, the polymer endpoint spreads to infinity, hence there is no apparent equilibrium. Nevertheless, the replica overlap is invariant under the shift of the polymer endpoint. By embedding the endpoint distribution into an abstract space, which factors out the spatial shift, significant progress has been made recently on the localization properties of the endpoint distribution, see [6, 3, 10]. In this case, the limiting free energy can be expressed as the solution of a variational problem, generalizing (2.2) in a sense.

We mention two recent papers on a nonlinear version of (1.1) on torus [28, 27], where the dissipation rate was studied, i.e., how fast $u(t,x)$ decays to zero. Among other interesting results, a stronger version of (1.4) was estabilished in the linear setting, see [27, Theorem 1.3].

The rest of the paper is organized as follows. In Section 2, we prove some preliminary results on the endpoint distribution of the directed polymer and derive the PDE hierarchy satisfied by the $n-$point correlation functions. Section 3 is devoted to the asymptotic analysis of the PDE hierarchy and the proof of the main theorem. In Section 4, we discuss the case of the $1+1$ spacetime white noise and some further extensions.

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2. Endpoint distributions of directed polymers and a PDE hierarchy

With u solving (1.1) started from the initial data, that is given by a non-trivial Borel measure, define

$$\rho(t,x)=\frac{u(t,x)}{\int_{\mathbb{T}_L^d}u(t,x')dx'}, \tag{2.1}$$

which is the quenched density of the endpoint distribution of polymer of length t. We emphasize that $\rho(t,x)$ actually depends on β and we have kept the dependence implicit in our notation. Since we are interested in the high temperature regime, i.e. $\beta\ll 1$, throughout the rest of the paper we assume $\beta\in(0,1)$.

We first prove some results on $\rho(t,\cdot)$ and its relation to the free energy $\gamma_L(\beta)$, some of which were obtained in [23].

Let $\mathcal{M}_1(\mathbb{T}_L^d)$ be the space of Borel probability measures on $\mathbb{T}_L^d$ and $\mathbb{Z}_+$ be the set of non-negative integers. Denote by $D(\mathbb{T}_L^d)$ and $D_c(\mathbb{T}_L^d)$ the respective spaces of all Borel measurable and continuous densities on the torus $\mathbb{T}_L^d$.

Proposition 2.1. *$\{\rho(t,\cdot)\}_{t\geq 0}$ is an $\mathcal{M}_1(\mathbb{T}_L^d)$-valued Markov process. In fact, for any $t>0$ it takes values in $D_c(\mathbb{T}_L^d)$. The process has a unique invariant measure π that is supported on $D_c(\mathbb{T}_L^d)$.*

Let ϱ be a $D_c(\mathbb{T}_L^d)$-valued random variable with the distribution given by π, then the free energy can be expressed as

$$\gamma_L(\beta) = -\frac{1}{2}\beta^2\int_{(\mathbb{T}_L^d)^2} R(x-y)\mathbb{E}[\varrho(x)\varrho(y)]dxdy. \tag{2.2}$$

In addition, $\{\varrho(x): x\in\mathbb{T}_L^d\}$ is a continuous trajectory, stationary random field. For any $n\in\mathbb{Z}_+$ we have

$$C_*(n,R,L) := \mathbb{E}\Big[\sup_{x\in\mathbb{T}_L^d}\varrho(x)^n\Big] < +\infty. \tag{2.3}$$

Proof. Throughout the proof, we use C to denote a generic constant that may depend on $n, R(\cdot), L$, but not on $\beta\in(0,1)$, and may change from line to line.

The fact that the Markov process $\{\rho(t,\cdot)\}_{t\geq 0}$ has a unique invariant measure that is supported on the space of positive, continuous densities on $\mathbb{T}_L^d$ has been proved in [23, Theorem 2.3]. The expression of the free energy in (2.2) was given in [23, Equation (2.22)].

To show the stationarity of the field $\{\varrho(x): x\in\mathbb{T}_L^d\}$, we start from the flat initial data $u(0,x)\equiv 1$ so that $\rho(0,x)\equiv L^{-d}$. By [23, Theorem 2.3], we have $\rho(t,\cdot)\Rightarrow\varrho(\cdot)$ in distribution on $C(\mathbb{T}_L^d)$, as $t\to\infty$. Note that, for each fixed $t>0$, the field $\{\rho(t,x): x\in\mathbb{T}_L^d\}$ is stationary, and this in turn implies the stationarity of ϱ.

Estimate (2.10) is a consequence of [23, Lemma 4.9]. □

Throughout the rest of the paper, we assume that ϱ is sampled from π. For any $n\geq 1$ and $\mathbf{x}_{1:n} = (x_1,\ldots,x_n)\in\left(\mathbb{T}_L^d\right)^n$, define the n–point correlation function of the stationary random field $\varrho(x)$:

$$Q_n(\mathbf{x}_{1:n}) = Q_n(x_1,\ldots,x_n) = \mathbb{E}\Big[\prod_{j=1}^n \varrho(x_j)\Big]. \tag{2.4}$$

By Proposition 2.1, we have

$$\sup_{\mathbf{x}_{1:n}\in(\mathbb{T}_L^d)^n} Q_n(\mathbf{x}_{1:n}) \leq C(n,R,L), \tag{2.5}$$

and $Q_n(\cdot)$ is a continuous function jointly in all its variables on $(\mathbb{T}_L^d)^n$. Again to simplify the notations, we have kept the dependence of Q_n on β and

L implicit. Also note that since $\varrho(\cdot)$ is stationary and $\int_{\mathbb{T}_L^d} \varrho(x)dx = 1$, we actually have $Q_1(x) \equiv L^{-d}$.

Here is the main result of this section

Proposition 2.2. *For any $n \geq 1$, $Q_n : (\mathbb{T}_L^d)^n \to \mathbb{R}_+$ is a smooth function, and the sequence $\{Q_n\}_{n\geq 1}$ solves the following PDE hierarchy: for any $n \geq 1$,*

(2.6)
$$\begin{aligned}
&\frac{1}{2}\Delta Q_n + \beta^2 \sum_{1\leq i<j\leq n} R(x_i - x_j)Q_n \\
=&\beta^2 n \int_{\mathbb{T}_L^d} Q_{n+1}(\mathbf{x}_{1:n}, x_{n+1}) \sum_{i=1}^n R(x_i - x_{n+1})dx_{n+1} \\
&- \beta^2 \frac{n(n+1)}{2} \int_{\mathbb{T}_L^d \times \mathbb{T}_L^d} Q_{n+2}(\mathbf{x}_{1:n}, x_{n+1}, x_{n+2})R(x_{n+1} - x_{n+2})dx_{n+1}dx_{n+2}.
\end{aligned}$$

Proof. To show (2.6), we make use of a dynamic version proved in [22]. Define

$$\mathcal{Q}_n(t, \mathbf{x}_{1:n}) = \mathbb{E}\Big[\prod_{j=1}^n \rho(t, x_j)\Big],$$

with $\rho(t, x)$ given by (2.1). Then [22, Theorem 1.1] shows that $\{\mathcal{Q}_n\}_{n\geq 1}$ satisfies the following hierarchy: for any $n \geq 1, T > 0$, and $f \in C^\infty\big((\mathbb{T}_L^d)^n\big)$,

(2.7)
$$\langle f, \mathcal{Q}_n(T)\rangle = \langle f, \mathcal{Q}_n(0)\rangle + \int_0^T \langle \frac{1}{2}\Delta f, \mathcal{Q}_n(t)\rangle dt + \beta^2 \sum_{k=0}^2 \int_0^T \langle f_k, \mathcal{Q}_{n+k}(t)\rangle dt,$$

where

$$\begin{aligned}
f_0(\mathbf{x}_{1:n}) &= f(\mathbf{x}_{1:n}) \sum_{1\leq i<j\leq n} R(x_i - x_j), \\
f_1(\mathbf{x}_{1:n}, x_{n+1}) &= -nf(\mathbf{x}_{1:n}) \sum_{i=1}^n R(x_i - x_{n+1}), \\
f_2(\mathbf{x}_{1:n}, x_{n+2}) &= \frac{1}{2}n(n+1)f(\mathbf{x}_{1:n})R(x_{n+1} - x_{n+2}),
\end{aligned}$$

and the brackets $\langle \cdot, \cdot \rangle$ in (2.7) are the corresponding L^2 inner product. Assume $\rho(0, \cdot)$ is sampled from the invariant measure π, then we have

$$\mathcal{Q}_n(t, \mathbf{x}_{1:n}) = Q_n(\mathbf{x}_{1:n}), \qquad \text{for all } t \geq 0, n \geq 1,$$

so (2.7) actually becomes

$$\langle \frac{1}{2}\Delta f, Q_n\rangle + \beta^2 \sum_{k=0}^2 \langle f_k, Q_{n+k}\rangle = 0, \tag{2.8}$$

which is the weak formulation of (2.6). Let

$$\begin{aligned} F(\mathbf{x}_{1:n}) := \beta^2\Big\{ & n\int_{\mathbb{T}_L^d} Q_{n+1}(\mathbf{x}_{1:n},x_{n+1})\sum_{i=1}^n R(x_i-x_{n+1})dx_{n+1} \\ & -\frac{n(n+1)}{2}\int_{\mathbb{T}_L^d\times\mathbb{T}_L^d} Q_{n+2}(\mathbf{x}_{1:n},x_{n+1},x_{n+2})R(x_{n+1}-x_{n+2})dx_{n+1}dx_{n+2} \\ & -\sum_{1\le i<j\le n} R(x_i-x_j)Q_n(\mathbf{x}_{1:n})\Big\}. \end{aligned} \tag{2.9}$$

The function F is continuous and, substituting $f\equiv 1$ into (2.8), we conclude that

$$\int_{(\mathbb{T}_L^d)^n} F(\mathbf{x}_{1:n})d\mathbf{x}_{1:n}=0.$$

Therefore the Poisson equation $\frac{1}{2}\Delta\tilde{Q}_n=F$ has a unique, up to a constant, solution $\tilde{Q}_n$ that belongs to any Sobolev space $W^{2,p}\big((\mathbb{T}_L^d)^n\big)$, $p\in[1,+\infty)$ - consisting of functions with two generalized derivatives that are L^p integrable. The function $Q_n-\tilde{Q}_n$ is harmonic on $(\mathbb{T}_L^d)^n$, in the weak sense, therefore, by the Weyl lemma, see e.g. [32, Theorem 2.3.1, p. 42], it is harmonic in the strong sense. As a result $Q_n\in W^{2,p}\big((\mathbb{T}_L^d)^n\big)$ for any $p\in[1,+\infty)$ and $n\ge 1$. Hence also $F\in W^{2,p}\big((\mathbb{T}_L^d)^n\big)$. This, by an application of the apriori estimates, allows us to conclude that in fact $Q_n\in W^{4,p}\big((\mathbb{T}_L^d)^n\big)$ for any $p\in[1,+\infty)$ and $n\ge 1$. Using a bootstrap argument, we can conclude that $Q_n\in C^\infty\big((\mathbb{T}_L^d)^n\big)$ for any $n\ge 1$. Since Q_n are smooth functions, we know that they are classical solutions to (2.6), which completes the proof. □

The next result is on the sample path properties of ϱ. We first introduce some notation: let

$$\begin{aligned} D_\infty(\mathbb{T}_L^d) &:= D(\mathbb{T}_L^d)\cap C^\infty(\mathbb{T}_L^d) \\ &= \{f:\mathbb{T}_L^d\to\mathbb{R}: 0\le f\in C^\infty(\mathbb{T}_L^d), \int_{\mathbb{T}_L^d} f(x)dx=1\}. \end{aligned}$$

Corollary 2.3. *The field $\{\varrho(x): x\in\mathbb{T}_L^d\}$ has smooth realizations, i.e. the invariant measure π is supported on $D_\infty(\mathbb{T}_L^d)$. In addition, for any $n\in\mathbb{Z}_+$ and multiindex $\alpha=(\alpha_1,\dots,\alpha_d)$ with $\alpha_j\in\mathbb{Z}_+$, $j=1,\dots,d$, there exists a constant $C=C(n,\alpha,R,d,L)$ such that*

$$\mathbb{E}\Big[\sup_{x\in\mathbb{T}_L^d}|\partial^\alpha\varrho(x)|^n\Big]\le C(n,\alpha,R,L). \tag{2.10}$$

Proof. Since the covariance functions $Q_n(\cdot)$ are smooth, [1, Theorem 2.2.2, p. 27] implies that $\partial^\alpha\varrho(x)$ exists for each x in the $L^2(\pi)$ sense. An application of [29, Theorem 1.4.1, p. 31] allows us to conclude that in fact the derivative field has a.s. continuous modification for each multiindex α. This proves the existence of smooth realizations of the field $\{\varrho(x): x\in\mathbb{T}_L^d\}$.

It remains to show (2.10). Note that it suffices to prove a weaker statement: that for any $n \in \mathbb{Z}_+$ and multiindex $\alpha = (\alpha_1, \dots, \alpha_d)$ with $\alpha_j \in \mathbb{Z}_+, j = 1, \dots, d$ we have

$$C_{*,w}(n, \alpha, R, L) := \mathbb{E}\big[|\partial^\alpha \varrho(x)|^n\big] < +\infty. \tag{2.11}$$

The latter is a simple consequence of the existence of an appropriate derivative of the function Q_n. Note that by stationarity the right hand side does not depend on $x \in \mathbb{T}_L^d$.

To prove that (2.11) implies (2.10), observe that by the Sobolev embedding there exists a deterministic constant C such that

$$\sup_{x \in \mathbb{T}_L^d} |\partial^\alpha \varrho(x)|^n \le C \|\varrho\|^n_{W^{k,p}(\mathbb{T}_L^d)} \tag{2.12}$$

for all realizations of $\varrho(\cdot)$, provided $k > d/p + |\alpha|$. Here $|\alpha| = \sum_{j=1}^d \alpha_j$ and $\|\varrho\|_{W^{k,p}(\mathbb{T}_L^d)} = \sum_{|\alpha| \le k} \|\partial^\alpha \varrho\|_{L^p(\mathbb{T}_L^d)}$ is the Sobolev norm. The estimate (2.10) is then a consequence of (2.12) and (2.11). □

By (2.2), the free energy can be expressed in terms of the two-point correlation Q_2:

$$\gamma_L(\beta) = -\frac{1}{2}\beta^2 \int_{(\mathbb{T}_L^d)^2} R(x_1 - x_2) Q_2(x_1, x_2) dx_1 dx_2. \tag{2.13}$$

Thus, to obtain the asymptotics of $\gamma_L(\beta)$ in the high temperature regime of $\beta \to 0$, it reduces to studying the asymptotic behaviors of Q_2 as $\beta \to 0$. For $\beta = 0$, the polymer measure degenerates to the Wiener measure, and the Brownian motion on $\mathbb{T}_L^d$ has the unique stationary distribution given by the uniform measure, in which case the n−point correlation function Q_n equals to L^{-nd}. The following lemma provides preliminary estimates on the difference between Q_n and L^{-nd} for $\beta \ll 1$.

Denote $L_0^2\big((\mathbb{T}_L^d)^n\big)$ the space of square integrable functions on $(\mathbb{T}_L^d)^n$ with zero mean, i.e.

$$L_0^2\big((\mathbb{T}_L^d)^n\big) = \left\{ f \in L^2\big((\mathbb{T}_L^d)^n\big) : \int_{(\mathbb{T}_L^d)^n} f(\mathbf{x}_{1:n}) d\mathbf{x}_{1:n} = 0 \right\}.$$

Define

$$\bar{Q}_n = Q_n - L^{-nd}. \tag{2.14}$$

Lemma 2.4. *For any $n \ge 2$, we have*

$$\|\bar{Q}_n\|_{L^2((\mathbb{T}_L^d)^n)} = O(\beta^2), \qquad \text{as } \beta \to 0. \tag{2.15}$$

Proof. For any n, we rewrite (2.6) as $\frac{1}{2}\Delta Q_n = \beta^2 F$, where F is a smooth function in $L_0^2\big((\mathbb{T}_L^d)^n\big)$, given by (2.9). Since $\int_{(\mathbb{T}_L^d)^n} Q_n d\mathbf{x}_{1:n} = 1$, we have $\bar{Q}_n \in L_0^2\big((\mathbb{T}_L^d)^n\big)$, which implies

$$\bar{Q}_n = 2\beta^2 \Delta^{-1} F,$$

where Δ^{-1} is the inverse of Δ, which is a bounded operator from $L^2_0\big((\mathbb{T}^d_L)^n\big)$ to $L^2_0\big((\mathbb{T}^d_L)^n\big)$. It remains to use the fact that $\|F\|_{L^2((\mathbb{T}^d_L)^n)}$ is bounded uniformly in $\beta \in (0,1)$ to complete the proof. □

Recall that $\gamma_L(\beta)$ is related to Q_2 through (2.13) and we have assumed $\int_{\mathbb{T}^d_L} R(x)dx = 1$, the above lemma gives the leading order of $\gamma_L(\beta)$

$$\gamma_L(\beta) + \frac{\beta^2}{2L^d} = O(\beta^4), \qquad \text{as } \beta \to 0.$$

3. Proof of Theorem 1.1

We start from the equation satisfied by Q_2

$$\begin{aligned}
&\frac{1}{2}\Delta Q_2 + \beta^2 R(x_1 - x_2)Q_2 \\
&=2\beta^2 \int_{\mathbb{T}^d_L} Q_3(x_1,x_2,x_3)[R(x_1 - x_3) + R(x_2 - x_3)]dx_3 \\
&\quad - 3\beta^2 \int_{\mathbb{T}^d_L\times\mathbb{T}^d_L} Q_4(x_1,x_2,x_3,x_4)R(x_3 - x_4)dx_3dx_4.
\end{aligned} \tag{3.1}$$

In light of Lemma 2.4, we rewrite the equation for Q_2 in terms of $\bar{Q}_2$ (see (2.14)), stated in the following lemma

Lemma 3.1. *We have*

$$\frac{1}{2}\Delta\bar{Q}_2 = \frac{\beta^2}{L^{2d}}\Big(\frac{1}{L^d} - R(x_1 - x_2)\Big) + \beta^2\mathcal{E}_\beta(x_1,x_2), \tag{3.2}$$

where $\mathcal{E}_\beta(x_1,x_2)$ is a smooth function in $L^2_0(\mathbb{T}^d_L\times\mathbb{T}^d_L)$ such that

$$\|\mathcal{E}_\beta\|_{L^2(\mathbb{T}^d_L\times\mathbb{T}^d_L)} = O(\beta^2). \tag{3.3}$$

Proof. It is straightforward to check that

$$\begin{aligned}
\mathcal{E}_\beta(x_1,x_2) =&2\int_{\mathbb{T}^d_L} \bar{Q}_3(x_1,x_2,x_3)[R(x_1 - x_3) + R(x_2 - x_3)]dx_3 \\
&- 3\int_{\mathbb{T}^d_L\times\mathbb{T}^d_L} \bar{Q}_4(x_1,x_2,x_3,x_4)R(x_3 - x_4)dx_3dx_4 - R(x_1 - x_2)\bar{Q}_2(x_1,x_2).
\end{aligned}$$

Since $\int_{\mathbb{T}^d_L} R(x)dx = 1$, we have $\int_{\mathbb{T}^d_L\times\mathbb{T}^d_L} \mathcal{E}_\beta dx_1dx_2 = 0$ from the equation (3.2). Then we only need to invoke Lemma 2.4 to complete the proof. □

Define $g_L : \mathbb{T}^d_L\times\mathbb{T}^d_L \to \mathbb{R}$ as the unique solution in $L^2_0(\mathbb{T}^d_L\times\mathbb{T}^d_L)$ to

$$\frac{1}{2}\Delta g_L = \frac{1}{L^{2d}}\Big(\frac{1}{L^d} - R(x_1 - x_2)\Big). \tag{3.4}$$

We have $g_L(x_1,x_2) = G_L(x_1 - x_2)$ with $G_L : \mathbb{T}_L \to \mathbb{R}$ solving

$$\Delta G_L(x) = \frac{1}{L^{2d}}\Big(\frac{1}{L^d} - R(x)\Big), \qquad x \in \mathbb{T}^d_L. \tag{3.5}$$

Using g_L, we can refine Lemma 2.4 when $n = 2$.

Lemma 3.2. *We have*

$$\|\bar{Q}_2 - \beta^2 g_L\|_{L^2(\mathbb{T}_L^d\times\mathbb{T}_L^d)} = O(\beta^4), \qquad \text{as } \beta \to 0.$$

Proof. By (3.2) and (3.4), we know that $f = \bar{Q}_2 - \beta^2 g_L$ is the solution to

$$\frac{1}{2}\Delta f = \beta^2 \mathcal{E}_\beta,$$

with $\int_{\mathbb{T}_L^d\times\mathbb{T}_L^d} f dx_1 dx_2 = 0$. Using (3.3), we can apply the same argument as for Lemma 2.4 to complete the proof. □

Now we can complete the proof of the main theorem.

Proof of Theorem 1.1. Recall that

$$\gamma_L(\beta) = -\frac{1}{2}\beta^2 \int_{\mathbb{T}_L^d\times\mathbb{T}_L^d} R(x_1-x_2)Q_2(x_1,x_2)dx_1dx_2.$$

We can rewrite the above in the form

$$\begin{aligned}\gamma_L(\beta) = &-\frac{1}{2}\beta^2 \int_{\mathbb{T}_L^d\times\mathbb{T}_L^d} R(x_1-x_2)L^{-2d}dx_1dx_2 - \frac{1}{2}\beta^4\int_{\mathbb{T}_L^d\times\mathbb{T}_L^d} R(x_1-x_2)g_L(x_1,x_2)dx_1dx_2\\ &-\frac{1}{2}\beta^4\int_{\mathbb{T}_L^d\times\mathbb{T}_L^d} R(x_1-x_2)(\beta^{-2}\bar{Q}_2(x_1,x_2)-g_L(x_1,x_2))dx_1dx_2.\end{aligned}$$

Applying Lemma 3.2 and using the fact that

$$\int_{\mathbb{T}_L^d\times\mathbb{T}_L^d} R(x_1-x_2)g_L(x_1,x_2)dx_1dx_2 = L^d\int_{\mathbb{T}_L^d} R(x)G_L(x)dx,$$

we immediately derive that

$$\gamma_L(\beta) = -\frac{\beta^2}{2L^d} - \frac{1}{2}\beta^4 L^d \int_{\mathbb{T}_L^d} R(x)G_L(x)dx + O(\beta^6).$$

By (3.5), we have that the Fourier coefficients of G equal

$$\hat{G}_L(n) = \int_{\mathbb{T}_L^d} G_L(x)e^{-i2\pi n\cdot x/L}dx = \frac{\hat{R}(n/L)}{4\pi^2|n|^2L^{2d-2}}, \quad n\neq 0,$$

with $\hat{R}(\cdot)$ defined in (1.5), and $\hat{G}_L(0) = 0$. By the Parseval identity, we obtain

$$\int_{\mathbb{T}_L^d} R(x)G_L(x)dx = \frac{1}{L^d}\sum_{n\neq0}\hat{G}_L(n)\hat{R}(n/L) = \sum_{n\neq0}\frac{\hat{R}(n/L)^2}{4\pi^2|n|^2L^{3d-2}}$$

which completes the proof of (1.6).

In $d=1$, $\hat{R}(\frac{n}{L}) \to \hat{R}(0) = 1$, as $L\to\infty$, which implies

$$\gamma_L^{(4)} = -\frac{1}{8\pi^2}\sum_{n\neq0}|n|^{-2}\hat{R}(\frac{n}{L})^2 \to -\frac{1}{8\pi^2}\sum_{n\neq0}|n|^{-2} = -\frac{1}{24}.$$

In $d=2$, we divide the summation into two parts:

$$\gamma_L^{(4)} = -\frac{1}{8\pi^2 L^2}\Bigg(\sum_{0\neq|n|<\delta L} + \sum_{|n|\geq \delta L}\Bigg)|n|^{-2}\hat{R}\Big(\frac{n}{L}\Big)^2 =: A_1 + A_2,$$

where $\delta>0$ is a constant to be sent to zero after sending $L\to\infty$. For the second part, we have

$$L^2 A_2 = -\frac{1}{8\pi^2 L^2}\sum_{|n/L|>\delta}\Big|\frac{n}{L}\Big|^{-2}\hat{R}\Big(\frac{n}{L}\Big)^2 \to -\frac{1}{8\pi^2}\int_{|\xi|>\delta}|\xi|^{-2}\hat{R}(\xi)d\xi, \quad \text{as } L\to\infty.$$

For the first part, we write it as

$$L^2 A_1 = -\frac{1}{8\pi^2}\sum_{0\neq|n|<\delta L}|n|^{-2} + \frac{1}{8\pi^2}\sum_{|n|<\delta L}|n|^{-2}\left(1-\hat{R}\Big(\frac{n}{L}\Big)^2\right) =: C_1 + C_2.$$

Since R is smooth and $\hat{R}(0)=1$, we have

$$\lim_{\delta\to 0}\limsup_{L\to\infty}\frac{C_2}{\log L} = 0.$$

For C_1, by an elementary calculation we have

$$\lim_{\delta\to 0}\lim_{L\to\infty}\frac{C_1}{\log L} = -\frac{1}{4\pi},$$

and this completes the proof of the case in $d=2$.

In $d\geq 3$, we have

$$L^d\gamma_L^{(4)} = -\frac{1}{8\pi^2 L^d}\sum_{n\neq 0}\Big|\frac{n}{L}\Big|^{-2}\hat{R}\Big(\frac{n}{L}\Big)^2 \to -\frac{1}{8\pi^2}\int_{\mathbb{R}^d}|\xi|^{-2}\hat{R}(\xi)^2 d\xi, \quad \text{as } L\to\infty.$$

The proof is complete. □

4. Discussions

4.1. **Spacetime white noise.** In this section, we consider the case when $d=1$ and the random potential is a $1+1$ spacetime white noise, in which case $R(\cdot)=\delta(\cdot)$. Define $h(t,x)=\beta^{-1}\log u(t,x)$, which is the formal solution to the KPZ equation, see (4.10) below. By the results in [8, 19, 24, 25], we know that the invariant measure for the process $\{h(t,x)-h(t,0): x\in\mathbb{T}_L\}_{t\geq 0}$ is given by the law of the Brownian bridge $\mathcal{B}_{0,L}(\cdot)$ with $\mathcal{B}_{0,L}(0)=\mathcal{B}_{0,L}(L)=0$. Here for the notational convenience, we extend the Brownian bridge periodically and also view it as a process on $\mathbb{T}_L$. For the polymer endpoint density, if we write it as

$$\rho(t,x) = \frac{u(t,x)}{\int_{\mathbb{T}_L} u(t,x')dx'} = \frac{e^{\beta(h(t,x)-h(t,0))}}{\int_{\mathbb{T}_L} e^{\beta(h(t,x')-h(t,0))}dx'},$$

it is immediate to conclude that the invariant measure is given by the law of random densities on $\mathbb{T}_L$

$$\varrho(x) = \frac{e^{\beta\mathcal{B}_{0,L}(x)}}{\int_0^L e^{\beta\mathcal{B}_{0,L}(x')}dx'}.$$

The random field $\{\varrho(x) : x \in \mathbb{T}_L\}$ is stationary, therefore, the limiting free energy in (2.2) (with $R(\cdot) = \delta(\cdot)$) reduces to

$$\begin{aligned}\gamma_L(\beta) &= -\frac{1}{2}\beta^2 \int_{\mathbb{T}_L^2} R(x-y)\mathbb{E}[\varrho(x)\varrho(y)]dxdy \\ &= -\frac{1}{2}\beta^2 \int_{\mathbb{T}_L} \mathbb{E}[\varrho(x)^2]dx = -\frac{1}{2}\beta^2 L\mathbb{E}[\varrho(0)^2] \\ &= -\frac{1}{2}\beta^2 L\mathbb{E}(\int_0^L e^{\beta\mathcal{B}_{0,L}(x)}dx)^{-2}.\end{aligned} \tag{4.1}$$

The random variable $\int_0^L e^{\beta\mathcal{B}_{0,L}(x)}dx$ appears frequently in physics and mathematical finance, and we refer to [31] for an extensive discussion. Its density function can be written explicitly, see [41, Proposition 6.2, p. 527], using which we obtain the following proposition:

Proposition 4.1. *In the case of a $1+1$ spacetime white noise, we have*

$$\gamma_L(\beta) = -\frac{\beta^6 L}{4\pi}\exp\left\{\frac{2\pi^2}{\beta^2 L}\right\}\int_0^\infty \frac{(e^y - e^{-y})}{(e^{y/2}+e^{-y/2})^6}\exp\left\{-\frac{2y^2}{\beta^2 L}\right\}\sin\left(\frac{4\pi y}{\beta^2 L}\right)dy. \tag{4.2}$$

For fixed $\beta > 0$, we have

$$\gamma_L(\beta) \to -\frac{\beta^4}{24}, \qquad \text{as } L \to \infty. \tag{4.3}$$

For fixed $L > 0$, we have

$$\gamma_L(\beta) = -\frac{\beta^2}{2L} - \frac{\beta^4}{24} - \frac{1781\beta^6 L}{840} + O(\beta^8), \qquad \text{as } \beta \to 0. \tag{4.4}$$

Proof. First, by the scaling property of the Brownian bridge

$$\int_0^L e^{\beta\mathcal{B}_{0,L}(x)}dx \overset{\text{law}}{=} L\int_0^1 e^{\beta\sqrt{L}\mathcal{B}_{0,1}(x)}dx.$$

To simplify the notation, define $Y_\lambda = \int_0^1 e^{\lambda\mathcal{B}(x)}dx$, so it remains to compute $\mathbb{E}Y_\lambda^{-2}$. Denote the density of Y_λ by $f_\lambda(z)$, by [41, Proposition 6.2, p. 527] we have

$$f_\lambda(z) = \frac{4}{\pi\lambda^2 z^2}\exp\left\{-\frac{4}{\lambda^2 z}+\frac{2\pi^2}{\lambda^2}\right\}\int_0^\infty \exp\left\{-\frac{2y^2}{\lambda^2}-\frac{4\cosh y}{\lambda^2 z}\right\}\left(\sinh y\right)\sin\left(\frac{4\pi y}{\lambda^2}\right)dy. \tag{4.5}$$

Using the above density formula, we have

$$\mathbb{E}Y_\lambda^{-2} = \int_0^\infty z^{-2} f_\lambda(z)dz = \frac{4}{\pi\lambda^2}\exp\left\{\frac{2\pi^2}{\lambda^2}\right\}\int_0^\infty \exp\left\{-\frac{4}{\lambda^2 z}\right\}\frac{1}{z^4} \times\left(\int_0^\infty \exp\left\{-\frac{2y^2}{\lambda^2}-\frac{4\cosh y}{\lambda^2 z}\right\}\big(\sinh y\big)\sin\left(\frac{4\pi y}{\lambda^2}\right)dy\right)dz. \tag{4.6}$$

Changing variables $z' = (z\lambda^2)^{-1}$ we get

$$\mathbb{E}Y_\lambda^{-2} = \frac{4\lambda^4}{\pi}\exp\left\{\frac{2\pi^2}{\lambda^2}\right\}\int_0^\infty z^2\exp\{-4z\} \times\left(\int_0^\infty \exp\left\{-\frac{2y^2}{\lambda^2}-4z\cosh y\right\}\big(\sinh y\big)\sin\left(\frac{4\pi y}{\lambda^2}\right)dy\right)dz. \tag{4.7}$$

Note that

$$\int_0^\infty z^2\exp\{-4z(1+\cosh y)\}\,dz = \frac{1}{32(1+\cosh y)^3}.$$

Hence, we get

$$\begin{aligned}\mathbb{E}Y_\lambda^{-2} &= \frac{\lambda^4}{8\pi}\exp\left\{\frac{2\pi^2}{\lambda^2}\right\}\int_0^\infty \frac{\sinh y}{(1+\cosh y)^3}\exp\left\{-\frac{2y^2}{\lambda^2}\right\}\sin\left(\frac{4\pi y}{\lambda^2}\right)dy\\ &= \frac{\lambda^4}{2\pi}\exp\left\{\frac{2\pi^2}{\lambda^2}\right\}\int_0^\infty \frac{(e^y-e^{-y})}{(e^{y/2}+e^{-y/2})^6}\exp\left\{-\frac{2y^2}{\lambda^2}\right\}\sin\left(\frac{4\pi y}{\lambda^2}\right)dy. \qquad (4.8)\end{aligned}$$

Recall that

$$\gamma_L(\beta) = -\frac{\beta^2}{2L}\mathbb{E}Y_{\beta\sqrt{L}}^{-2}$$

Combining with (4.8), we complete the proof of (4.2).

To show (4.3), by the fact that $\frac{\sin x}{x} \to 1$ as $x \to 0$, we conclude that

$$\mathbb{E}Y_\lambda^{-2} = 2\lambda^2\Big(c+o(1)\Big), \quad \text{where} \tag{4.9}$$

$$c := \int_0^\infty \frac{y(e^y-e^{-y})}{(e^{y/2}+e^{-y/2})^6}dy,$$

with the $o(1)$ term going to zero as $\lambda\to\infty$. It is an elementary calculation to compute the value $c = \frac{1}{24}$, which completes the proof.

To show (4.4), we can directly start from the expression

$$\gamma_L(\beta) = -\frac{\beta^2}{2L}\mathbb{E}\left(\int_0^1 e^{\beta\sqrt{L}\mathcal{B}_{0,1}(x)}dx\right)^{-2}.$$

For fixed L, it is clear that $\gamma_L(\cdot)$ is a smooth function in β, so one can compute explicitly the derivatives of γ_L at $\beta=0$ to obtain (4.4). Since it is a straightforward calculation, we do not provide the detail here. □

4.2. **Asymptotic expansion of the invariant measure.** From the proof of Theorem 1.1, it is clear that the expansion in (1.6) can be extended to an arbitrary high order. We only kept the first two terms since their expressions are more explicit. Our expansion is based on the the two-point correlation function Q_2, because that is what the limiting free energy depends on. It actually corresponds to an expansion of the invariant measure ϱ in the parameter β. Below we sketch the heuristic connections.

Suppose that $d \geq 1$ and R is a smooth and compactly supported function. Define $h(t,x) = \frac{1}{\beta}\log u(t,x)$, which is the solution to the KPZ equation

$$\partial_t h = \frac{1}{2}\Delta h + \frac{1}{2}\beta|\nabla h|^2 + V - \frac{1}{2}R(0)\beta. \tag{4.10}$$

We can write the polymer endpoint distribution in terms of h as

$$\rho(t,x) = \frac{u(t,x)}{\int_{\mathbb{T}_L^d} u(t,x')dx'} = \frac{e^{\beta h(t,x)}}{\int_{\mathbb{T}_L^d} e^{\beta h(t,x')}dx'} = \frac{e^{\beta(h(t,x)-\bar{h}(t))}}{\int_{\mathbb{T}_L^d} e^{\beta(h(t,x')-\bar{h}(t))}dx'}. \tag{4.11}$$

Here $\bar{h}(t) = L^{-d}\int_{\mathbb{T}_L^d} h(t,x)dx$ is the average of $h(t,x)$. Thus, an expansion of the stationary distribution of $h(t,\cdot) - \bar{h}(t)$ in β would lead to a corresponding expansion of ϱ. For $\beta \ll 1$, we approximate (4.10) by the Edwards-Wilkinson equation

$$\partial_t h = \frac{1}{2}\Delta h + V.$$

There are no stationary invariant probability measures for the above equation on the torus, as a result of the growth of the zero mode. If we remove the zero mode and consider the following equation

$$\partial_t \tilde{h} = \frac{1}{2}\Delta \tilde{h} + V - \bar{V}, \tag{4.12}$$

where $\bar{V}(t) = L^{-d}\int_{\mathbb{T}_L^d} V(t,x)dx$, then as a Markov process it admits a stationary distribution with an explicit density. Replacing $h(t,x) - \bar{h}(t)$ in (4.11) by the stationary solution $\tilde{h}$, i.e. the one where the initial data is sampled from the invariant distribution, we obtain the first order approximation of the stationary measure ϱ. It is straightforward to check that the two-point correlation function of the stationary solution to (4.12) is directly related to the solution to (3.4).

To make the above argument rigorous, one needs to control the error in the approximation of the KPZ equation by the Edwards-Wilkinson equation. For us, it seems more convenient to do it on the level of Q_2 through the PDE hierarchy (2.6), where we may borrow analytic tools.

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