

GAUSSIAN FLUCTUATIONS FROM RANDOM SCHRÖDINGER EQUATION

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ABSTRACT. We study the Schrödinger equation driven by a weak Brownian forcing, and derive Gaussian fluctuations in the form of a time-inhomogeneous Ornstein-Uhlenbeck process. As a result, when evaluated at a fixed frequency, the intensity of the incoherent wave is of exponential distribution.

1. INTRODUCTION

Wave propagation in random media is a complex phenomenon due to the existence of multiple scales, including the propagation distance, the correlation length and the strength of the media, the wave length, etc. Ideally, one would like to know about the statistical properties of the wave field, in particular its moments information that is needed in practice. This depends in a highly nonlinear way on the statistical properties of the media, and it is usually impossible to resolve all the scales and study the wave equation directly. Thus, in most approaches, effective and simplified models are derived which only involve a few parameters related to the media. There is a large body of literature on the subject and various approximations have been proposed in different asymptotic regimes, see [2, 6, 7] and the references therein.

In a high frequency regime, the backscattering of the wave is neglected and the forward approximation in a privileged direction leads to a Schrödinger equation with a random potential. This approach is used e.g. to describe the propagation of a wave beam in a turbulent medium in the forward scattering approximation of the full wave equation, see [18]. The refraction index then plays the role of a potential.

A direct study of the random Schrödinger equation is still a challenging task, with part of the reason being that the moments of the solution do not solve closed-form equations, and this makes it hard to extract statistical information on the wave field, see e.g. [2, Chapter 5,6] and the references therein.

One can further perform a Markovian approximation of the randomness and assume it is δ -correlated in the privileged direction. It leads to the so-called Itô-Schrödinger model. This model appears e.g. as a diffusive approximation for linear acoustic waves propagating in $1 + d$ spatial dimensions in a random medium, when the correlation length of the medium and the typical wavelength is much smaller than the propagation distance, see [8]. Using Itô calculus one can show in this particular case that the moments of the wave function solve closed-form equations.

For the Itô-Schrödinger model, the first and second moment equations are straightforward to solve, with the explicit solutions available and corresponding to the ballistic and the scattering component of the wave field respectively. Another important quantity is the fourth moment, as it is related to fluctuations of the intensity of the wave field. The corresponding moment equation is more complicated and can not be solved explicitly. Various approximations were derived from both theoretical and numerical points of view.

In applications such as light passing through a turbulent atmosphere or sound waves propagating in the ocean, it is a well-accepted fact that the distribution of the complex wave field becomes approximately a complex Gaussian, that is, the real and imaginary parts are independent Gaussians with the same variance, and as a consequence, the intensity of the wave field (given by the square of its absolute value) is of exponential distribution [19, 20]. This has been proved in $d = 1$ in a randomly layered medium [6, Chapter 9]. Progress has also been made in high dimensions, focusing on estimating of the fourth moment to verify the Gaussian summation rule, see [9, 10].

In the present paper, we focus on the Itô-Schrödinger model, and our main contribution is to prove Gaussianity of the wave field in an asymptotic regime where the medium has a weak strength and the propagation distance is large. More precisely, we consider the asymptotics of the compensated wave field, see (2.6) below. This object has been introduced in [1]. It is a field, in both time and momentum variables, that is obtained from the Fourier transform of the solution of a random Schrödinger equation by removing the fast oscillations of its phase. This is done by “recentering” the phase through solving the free Schrödinger equation backward (with no potential). We prove, see Theorem 2.2, that asymptotically the compensated wave field converges in law to a complex Gaussian field that is the solution of an time-inhomogenous Ornstein-Uhlenbeck equation, see (2.7) and Section 3.2 below.

Concerning the method of our proof, we use a martingale representation for the compensated wave field, see (5.4) and (5.5) below. The limit is then verified by proving the convergence of the respective martingale field appearing in (5.5). This is achieved by proving the convergence of its quadratic variation, which constitutes the main thrust of our argument. On the other hand, as the moments of the wave field solve deterministic equations, one can in principle analyze those equations and try to establish the Gaussian limit in a more analytic way. This perspective has actually been adopted in [9, 10] and many previous works. A message we want to convey here is that, the convergence of the martingale quadratic variation only involves a fourth moment calculation, which simplifies the analysis a bit.

The paper is organized as follows: in Section 2 we formulate our main result. Some of its aspects are discussed in Section 3. The proof of the main result is carried out in Sections 4, 6 and 5.

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2. MAIN RESULT

The equation we study takes the form

$$(2.1) \quad \begin{aligned} i\partial_t\phi + \frac{1}{2}\Delta\phi - \varepsilon V(t, x) \circ \phi &= 0, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d, \\ \phi(0, x) &= \phi_0(x). \end{aligned}$$

Here $\mathbb{R}_+ = (0, +\infty)$, the random potential $V(t, x)$ is a real distribution-valued, Gaussian process over some probability space $(\Omega, \mathcal{V}, \mathbb{P})$ that is white in time and smooth in space, with the covariance function

$$\mathbb{E}[V(t, x)V(s, y)] = \delta(t - s)R(x - y), \quad (t, x), (s, y) \in \mathbb{R} \times \mathbb{R}^d.$$

The parameter $\varepsilon > 0$ regulates the strength of the field and \mathbb{E} is the expectation with respect to \mathbb{P} . We assume that $R(\cdot)$ belongs to the Schwarz class $\mathcal{S}(\mathbb{R}^d)$. The product \circ between the solution and the noise is in the Stratonovitch sense. Equation (2.1) is understood via the corresponding Itô stochastic partial differential equation

$$(2.2) \quad d\phi = \left(\frac{i}{2} \Delta \phi - \frac{\varepsilon^2}{2} R(0) \phi \right) dt - i\varepsilon B(dt) \phi,$$

where $(B(t))_{t \geq 0}$ is a smooth in space Wiener process such that $\dot{B}(t, x) = V(t, x)$, i.e. it is a Gaussian random field with the covariance function

$$(2.3) \quad \mathbb{E}[B(t, x)B(s, y)] = (t \wedge s)R(x - y), \quad (t, x), (s, y) \in \bar{\mathbb{R}}_+ \times \mathbb{R}^d.$$

The equation was analyzed in the early work [5], and it was shown that the L^2 norm of the wave function is conserved [5, Equation (2.19)], i.e.

$$\|\phi(t, \cdot)\|_{L^2(\mathbb{R}^d)} = \|\phi_0\|_{L^2(\mathbb{R}^d)}, \quad t \geq 0, \quad \mathbb{P} \text{ a.s.}$$

We shall denote by $\|\cdot\|_{L^p(\mathbb{R}^d)}$ the L^p norm with respect to the Lebesgue measure over \mathbb{R}^d . The Fourier transform of a given function $f \in L^2(\mathbb{R}^d)$ shall be denoted by $\hat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{-i\xi \cdot x} dx$, $\xi \in \mathbb{R}^d$.

Before stating our main result, let us introduce some definitions. Let $w(t, \cdot, \xi)$ be the finite measure-valued solution of the linear kinetic equation

$$(2.4) \quad \begin{aligned} \partial_t w(t, \cdot, \xi) + \xi \cdot \nabla_x w(t, \cdot, \xi) &= \int_{\mathbb{R}^d} \frac{\hat{R}(p)}{(2\pi)^d} [w(t, \cdot, \xi - p) - w(t, \cdot, \xi)] dp, \\ w(0, dx, \xi) &= |\hat{\phi}_0(\xi)|^2 \delta_0(dx). \end{aligned}$$

Here δ_0 is the Dirac measure at 0. Define the measure

$$(2.5) \quad u(t, dx, \xi) := \int_{\mathbb{R}^d} w(t, dx, \xi - p) \frac{\hat{R}(p) dp}{(2\pi)^d}.$$

Assume throughout the paper that $\hat{\phi}_0 \in C_b(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \cap \text{Lip}(\mathbb{R}^d)$, where $C_b(\mathbb{R}^d)$ and $\text{Lip}(\mathbb{R}^d)$ denote the spaces of bounded and continuous, and Lipschitz continuous functions, respectively. The following simple fact holds.

Proposition 2.1. *For each $(t, \xi) \in \mathbb{R}_+ \times \mathbb{R}^d$ the measure $u(t, \cdot, \xi)$ is absolutely continuous with respect to the Lebesgue measure. Its density $U(t, \cdot, \xi)$ is strictly positive and smooth.*

The proof of the proposition is presented in Section 4.

Next we define the *compensated wave field*

$$(2.6) \quad X_\xi^\varepsilon(t, \eta) := \hat{\phi} \left(\frac{t}{\varepsilon^2}, \xi + \varepsilon^2 \eta \right) \exp \left\{ \frac{it}{2\varepsilon^2} |\xi + \varepsilon^2 \eta|^2 \right\}, \quad (t, \eta) \in \bar{\mathbb{R}}_+ \times \mathbb{R}^d,$$

see Remark 3.5 below for a discussion on the interpretation of the field.

Our main results concerns the long time, large scale behavior of the Fourier transform $\hat{\phi}(t, \xi)$ of the wave function and can be stated as follows.

Theorem 2.2. *Fix $\xi \in \mathbb{R}^d$. The following convergence holds:*

$$\left\{ X_\xi^\varepsilon(t, \eta) \right\}_{(t, \eta) \in \bar{\mathbb{R}}_+ \times \mathbb{R}^d} \Rightarrow \left\{ X_\xi(t, \eta) \right\}_{(t, \eta) \in \bar{\mathbb{R}}_+ \times \mathbb{R}^d}, \quad \text{as } \varepsilon \rightarrow 0,$$

in law over $C(\bar{\mathbb{R}}_+ \times \mathbb{R}^d)$. The limit X_ξ is a complex valued Gaussian process admitting the representation

$$(2.7) \quad X_\xi(t, \eta) = \hat{\phi}_0(\xi) e^{-\frac{1}{2}R(0)t} + \int_0^t e^{-\frac{1}{2}R(0)(t-s)} B_\xi(ds, \eta),$$

where B_ξ is a zero mean complex Gaussian process with the covariance function

$$(2.8) \quad \begin{aligned} \mathbb{E}[B_\xi(t_1, \eta_1)B_\xi^*(t_2, \eta_2)] &= \int_0^{t_1 \wedge t_2} \hat{U}(s, \eta_1 - \eta_2, \xi) ds, \\ \mathbb{E}[B_\xi(t_1, \eta_1)B_\xi(t_2, \eta_2)] &= 0, \quad (t_j, \eta_j) \in \bar{\mathbb{R}}_+ \times \mathbb{R}^d, \quad j = 1, 2. \end{aligned}$$

The function $(t, \eta) \mapsto \hat{U}(t, \eta, \xi)$, $(t, \eta) \in \bar{\mathbb{R}}_+ \times \mathbb{R}^d$ is given by

$$(2.9) \quad \hat{U}(t, \eta, \xi) := \int_{\mathbb{R}^d} e^{-i\eta \cdot y} U(t, y + \xi t, \xi) dy.$$

The proof of the theorem is presented in Section 5.

3. DISCUSSION

3.1. On the interpretation of the result.

3.1.1. *On the initial data.* Suppose that the initial data varies on the microscopic scale and is described by the family of wave functions

$$(3.1) \quad \tilde{\chi}_\varepsilon(y) := \frac{1}{\varepsilon^d} \phi_0(y), \quad y \in \mathbb{R}^d,$$

where y is the spatial coordinate in the microscopic units. We assume that the macroscopic coordinate is given by $x = \varepsilon^2 y$, so the prefactor ε^{-d} in the left hand side of (3.1) assures that the macroscopic energy density of the wave is of order $O(1)$, provided that $\phi_0 \in L^2(\mathbb{R}^d)$. The initial data is fast oscillating on the macroscopic scale and is described by the initial profile

$$(3.2) \quad \tilde{\phi}_\varepsilon(x) := \frac{1}{\varepsilon^d} \phi_0\left(\frac{x}{\varepsilon^2}\right), \quad x \in \mathbb{R}^d,$$

with $\phi_0 \in L^2(\mathbb{R}^d)$. The family $(\tilde{\phi}_\varepsilon)_{\varepsilon \in (0,1]}$ forms a bounded set in $L^2(\mathbb{R}^d)$.

3.1.2. *Compensated wave-function, Wigner and smoothed Wigner functions.* Consider now $\tilde{\phi}_\varepsilon(t, x) = \varepsilon^{-d} \phi\left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon^2}\right)$, where ϕ is the solution of (2.1). Since the laws of the noise $\frac{1}{\varepsilon} V\left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon^2}\right)$ and that of $V\left(t, \frac{x}{\varepsilon^2}\right)$ are identical, the law of $\tilde{\phi}_\varepsilon(t, x)$ coincides with that of the solution of the equation

$$(3.3) \quad \begin{aligned} i\partial_t \tilde{\phi}_\varepsilon(t, x) + \frac{\varepsilon^2}{2} \Delta \tilde{\phi}_\varepsilon(t, x) - V\left(t, \frac{x}{\varepsilon^2}\right) \circ \tilde{\phi}_\varepsilon(t, x) &= 0, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d, \\ \tilde{\phi}_\varepsilon(0, x) &= \tilde{\phi}_\varepsilon(x). \end{aligned}$$

Furthermore, the Fourier transform is related to the unscaled wave function through

$$(3.4) \quad \hat{\tilde{\phi}}_\varepsilon\left(t, \frac{\xi}{\varepsilon^2}\right) = \varepsilon^d \hat{\phi}\left(\frac{t}{\varepsilon^2}, \xi\right).$$

The compensated wave-function, defined as

$$(3.5) \quad X_\xi^\varepsilon(t, \eta) := \frac{1}{\varepsilon^d} \hat{\tilde{\phi}}_\varepsilon\left(t, \frac{\xi + \varepsilon^2 \eta}{\varepsilon^2}\right) \exp\left\{\frac{it}{2\varepsilon^2} |\xi + \varepsilon^2 \eta|^2\right\}, \quad (t, \eta) \in \bar{\mathbb{R}}_+ \times \mathbb{R}^d.$$

is given by the expression (2.6). Its inverse Fourier transform is a spectral measure defined by the equality

$$\int_{\mathbb{R}^d} \mathcal{X}_\varepsilon(t, dx, \xi) J^*(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} X_\xi^\varepsilon(t, \eta) \hat{J}^*(\eta) d\eta$$

for any J that belongs to the Schwartz class $\mathcal{S}(\mathbb{R}^d)$. For fixed $\varepsilon > 0, \xi \in \mathbb{R}^d, t > 0$, $X_\xi^\varepsilon(t, \cdot)$ belongs to $L^2(\mathbb{R}^d)$ almost surely. Therefore, we know that $\mathcal{X}_\varepsilon(t, dx, \xi)$ actually has a density in x , which we shall also denote, with some abuse of the

notation, by $\mathcal{X}_\varepsilon(t, x, \xi)$. Denote by $\mathcal{X}(t, dx, \xi)$ the respective spectral measure associated with the stationary field $\eta \mapsto X_\xi(t, \eta)$.

Note that $X_\xi^\varepsilon(0, \eta) = \hat{\phi}_0(\xi + \varepsilon^2 \eta)$, so obviously we have

$$(3.6) \quad \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^{2d}} \mathcal{X}_\varepsilon(0, x, \xi) J^*(x, \xi) dx d\xi = \int_{\mathbb{R}^d} \hat{\phi}_0(\xi) \hat{J}^*(0, \xi) d\xi$$

for any test function $J \in \mathcal{S}(\mathbb{R}^{2d})$. Hence

$$(3.7) \quad \lim_{\varepsilon \rightarrow 0} \mathcal{X}_\varepsilon(0, x, \xi) = \hat{\phi}_0(\xi) \delta(x),$$

*-weakly in $\mathcal{S}'(\mathbb{R}^{2d})$. The following result holds.

Proposition 3.1. *Fix any test function $J \in \mathcal{S}(\mathbb{R}^{2d})$ and $t \geq 0$. Then, for any $\xi \in \mathbb{R}^d$ we have*

$$(3.8) \quad \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^d} \mathcal{X}_\varepsilon(t, x, \xi) J^*(x, \xi) dx = \int_{\mathbb{R}^d} \mathcal{X}(t, dx, \xi) J^*(x, \xi)$$

in law. In addition

$$(3.9) \quad \begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^{2d}} \mathcal{X}_\varepsilon(t, x, \xi) J^*(x, \xi) dx d\xi &= \frac{e^{-\frac{1}{2}tR(0)}}{(2\pi)^d} \int_{\mathbb{R}^{2d}} \hat{J}^*(\eta, \xi) \hat{\phi}_0(\xi) d\eta d\xi \\ &= e^{-\frac{1}{2}tR(0)} \int_{\mathbb{R}^d} J^*(0, \xi) \hat{\phi}_0(\xi) d\xi \end{aligned}$$

in $L^2(\Omega)$, as $\varepsilon \rightarrow 0$.

The proof of the proposition is contained in Section 6.

It is worthwhile to compare the behavior of the compensated wave function with that of the Wigner functions corresponding to the family $(\tilde{\phi}_\varepsilon)_{\varepsilon \in (0,1]}$, cf e.g. [14, Item 1), p. 557],

$$(3.10) \quad w_\varepsilon(t, x, \xi) := \int_{\mathbb{R}^d} \tilde{\phi}_\varepsilon \left(t, x + \frac{\varepsilon^2 y}{2} \right) \tilde{\phi}_\varepsilon^* \left(t, x - \frac{\varepsilon^2 y}{2} \right) e^{-i\xi \cdot y} dy.$$

By taking the Fourier transform, we obtain

$$(3.11) \quad \begin{aligned} w_\varepsilon(t, x, \xi) &= \frac{1}{(2\pi\varepsilon^2)^d} \int_{\mathbb{R}^d} \hat{\phi}_\varepsilon \left(t, \frac{\xi}{\varepsilon^2} + \frac{\eta}{2} \right) \hat{\phi}_\varepsilon^* \left(t, \frac{\xi}{\varepsilon^2} - \frac{\eta}{2} \right) e^{i\eta \cdot x} d\eta \\ &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{\phi} \left(\frac{t}{\varepsilon^2}, \xi + \frac{\varepsilon^2 \eta}{2} \right) \hat{\phi}^* \left(\frac{t}{\varepsilon^2}, \xi - \frac{\varepsilon^2 \eta}{2} \right) e^{i\eta \cdot x} d\eta. \end{aligned}$$

Using (2.6) we get

$$(3.12) \quad w_\varepsilon(t, x, \xi) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} X_\xi^\varepsilon \left(t, \frac{\eta}{2} \right) (X_\xi^\varepsilon)^* \left(t, -\frac{\eta}{2} \right) e^{i\eta \cdot (x - \xi t)} d\eta.$$

Theorem 2.2 implies the following, see Section 6 below for the proof.

Proposition 3.2. *Fix any test function $J \in \mathcal{S}(\mathbb{R}^{2d})$ and $t > 0$. Then, for any $\xi \in \mathbb{R}^d$ we have*

$$(3.13) \quad \int_{\mathbb{R}^d} w_\varepsilon(t, x, \xi) J^*(x, \xi) dx \Rightarrow \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} X_\xi \left(t, -\frac{\eta}{2} \right) X_\xi^* \left(t, \frac{\eta}{2} \right) e^{-i\eta \cdot \xi t} \hat{J}^*(\eta, \xi) d\eta,$$

in distribution, as $\varepsilon \rightarrow 0$. In addition, (cf (3.29) below)

$$(3.14) \quad \begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^{2d}} w_\varepsilon(t, x, \xi) J^*(x, \xi) dx d\xi &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathbb{E} \left[X_\xi \left(t, -\frac{\eta}{2} \right) X_\xi^* \left(t, \frac{\eta}{2} \right) \right] e^{-i\eta \cdot \xi t} \hat{J}^*(\eta, \xi) d\eta d\xi \\ &= \int_{\mathbb{R}^{2d}} w(t, x, \xi) J^*(x, \xi) dx d\xi, \end{aligned}$$

in $L^2(\Omega)$, as $\varepsilon \rightarrow 0$.

Formula (3.7) shows that our highly oscillatory initial data localizes at $x = 0$. To “smear” the observed position around 0 we introduce also the *smoothed Wigner function*, cf [9, formula (89), p. 65], defined as follows

$$(3.15) \quad w_\varepsilon^s(t, x, \xi) := \frac{1}{2^{d/2}\pi^{3d/2}\varepsilon^{2d}} \left| \int_{\mathbb{R}^d} e^{i\eta \cdot x} e^{-|\eta|^2} \hat{\phi}_\varepsilon^* \left(t, \frac{\xi}{\varepsilon^2} - \eta \right) d\eta \right|^2.$$

A simple calculation shows that

$$(3.16) \quad w_\varepsilon^s(t, x, \xi) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} w_\varepsilon \left(t, x - y, \xi - \frac{\varepsilon^2 \eta}{2} \right) \exp \left\{ -\frac{|y|^2}{2} - \frac{|\eta|^2}{2} \right\} dy d\eta.$$

In other words, w_ε^s is an average of w_ε on the $O(1)$ -scale of the spatial variable and on the $O(\varepsilon^2)$ -scale of the frequency variable. Using (3.12) and (3.15), for the family $(\tilde{\phi}_\varepsilon)_{\varepsilon \in (0,1]}$ given by (3.4), we get therefore

$$(3.17) \quad \begin{aligned} w_\varepsilon^s(t, x, \xi) &= \frac{1}{2^{d/2}\pi^{3d/2}} \int_{\mathbb{R}^{2d}} (X_\xi^\varepsilon)^*(t, -\eta) X_\xi^\varepsilon(t, -\eta') \\ &\times e^{i(\eta - \eta') \cdot (x - \xi t)} \exp \left\{ -|\eta|^2 \left(1 - \frac{i\varepsilon^2 t}{2} \right) - |\eta'|^2 \left(1 + \frac{i\varepsilon^2 t}{2} \right) \right\} d\eta d\eta'. \end{aligned}$$

From Theorem 2.2, we conclude immediately the following.

Proposition 3.3. *Fix any $(t, x, \xi) \in \mathbb{R}_+ \times \mathbb{R}^{2d}$. Then,*

$$(3.18) \quad w_\varepsilon^s(t, x, \xi) \Rightarrow \frac{1}{2^{d/2}\pi^{3d/2}} \left| \int_{\mathbb{R}^d} e^{i\eta \cdot (x - \xi t)} e^{-|\eta|^2} X_\xi^*(t, -\eta) d\eta \right|^2,$$

in distribution as $\varepsilon \rightarrow 0$.

3.2. $X_\xi(t, \eta)$ as an inhomogeneous Ornstein-Uhlenbeck process. Let us make a few comments about the limiting equation (2.7). Note that in light of Proposition 2.1, we can write

$$(3.19) \quad B_\xi(t, \eta) = \int_0^t \int_{\mathbb{R}^d} e^{-i\eta \cdot y} \mathbb{U}^{1/2}(s, y + \xi s, \xi) B_w(ds, dy),$$

where $B_w(ds, dy)$ is complex valued space-time white noise, i.e.

$$(3.20) \quad \begin{aligned} \mathbb{E}[B_w(dt, dx) B_w^*(ds, dy)] &= \delta(t - s) \delta(x - y) dt ds dx dy, \\ \mathbb{E}[B_w(dt, dx) B_w(ds, dy)] &= 0, \quad (t, x), (s, y) \in \mathbb{R} \times \mathbb{R}^d. \end{aligned}$$

We can write therefore

$$(3.21) \quad \begin{cases} dX_\xi(t, \eta) = -\frac{1}{2} R(0) X_\xi(t, \eta) dt + \int_{\mathbb{R}^d} e^{-i\eta \cdot y} \mathbb{U}^{1/2}(t, y + \xi t, \xi) B_w(dt, dy), \\ X_\xi(0, \eta) = \hat{\phi}_0(\xi), \end{cases}$$

so for fixed $\xi, \eta \in \mathbb{R}^d$, $X_\xi(t, \eta)$ is actually a time-inhomogeneous Ornstein-Uhlenbeck process.

3.3. The covariance structure of $X_\xi(t, \eta)$. For fixed $(t, \xi) \in \mathbb{R}_+ \times \mathbb{R}^d$, the field $\eta \mapsto X_\xi(t, \eta)$, $\eta \in \mathbb{R}^d$ is a stationary complex-valued Gaussian. A direct calculation, using (3.21) and (2.4), shows that its second absolute moment $\tilde{w}(t, \xi) := \mathbb{E}[|X_\xi(t, \eta)|^2]$ satisfies the homogeneous linear Boltzmann equation

$$(3.22) \quad \partial_t \tilde{w}(t, \xi) = \int_{\mathbb{R}^d} \frac{\hat{R}(p)}{(2\pi)^d} [\tilde{w}(t, \xi - p) - \tilde{w}(t, \xi)] dp, \quad \tilde{w}(0, \xi) = |\hat{\phi}_0(\xi)|^2.$$

We can also compute its mean and the covariance function:

$$(3.23) \quad \begin{aligned} \mathbb{E}X_\xi(t, \eta) &= \hat{\phi}_0(\xi) e^{-\frac{1}{2} R(0)t}, \\ \text{Cov}(X_\xi(t, \eta_1), X_\xi(t, \eta_2)) &= \int_0^t \int_{\mathbb{R}^d} e^{-R(0)(t-s)} \hat{\mathbb{U}}(s, \eta_1 - \eta_2, \xi) ds. \end{aligned}$$

From the equation satisfied by w , see (2.4), we conclude that \hat{w} - its Fourier transform in the x variable - satisfies the equation:

$$(3.24) \quad \begin{aligned} \partial_t \hat{w}(t, \eta, \xi) + i\xi \cdot \eta \hat{w}(t, \eta, \xi) &= \int_{\mathbb{R}^d} \frac{\hat{R}(p)}{(2\pi)^d} \hat{w}(t, \eta, \xi - p) dp - R(0) \hat{w}(t, \eta, \xi), \\ \hat{w}(t, \eta, \xi) &= |\hat{\phi}_0(\xi)|^2. \end{aligned}$$

Define

$$(3.25) \quad \hat{f}(t, \eta, \xi) := \hat{w}(t, \eta, \xi) e^{i\xi \cdot \eta t}.$$

From (3.25) we conclude that it satisfies the integral equation

$$(3.26) \quad \hat{f}(t, \eta, \xi) = |\hat{\phi}_0(\xi)|^2 e^{-R(0)t} + \int_0^t e^{-R(0)(t-s)} \int_{\mathbb{R}^d} \frac{\hat{R}(p)}{(2\pi)^d} e^{ip \cdot \eta s} \hat{f}(s, \eta, \xi - p) dp ds.$$

The solution can be written as an infinite series expansion

$$(3.27) \quad \begin{aligned} \hat{f}(t, \eta, \xi) &= e^{-R(0)t} \left\{ |\hat{\phi}_0(\xi)|^2 \right. \\ &\quad \left. + \sum_{n \geq 1} \int_{[0, t]_<^n} \int_{\mathbb{R}^{nd}} \prod_{j=1}^n \frac{\hat{R}(p_j) e^{i\eta \cdot p_j s_j}}{(2\pi)^d} |\hat{\phi}_0(\xi - p_1 - \dots - p_n)|^2 d\mathbf{p}_{1,n} d\mathbf{s}_{1,n} \right\}. \end{aligned}$$

Here

$$[0, t]_<^n = \{0 < s_n < \dots < s_1 < t\}$$

is the n dimensional simplex, $d\mathbf{p}_{1,n} := dp_1 \dots dp_n$, $d\mathbf{s}_{1,n} = ds_1 \dots ds_n$.

Comparing (3.25) with (2.9), we obtain

$$(3.28) \quad \int_{\mathbb{R}^d} \frac{\hat{R}(p)}{(2\pi)^d} e^{ip \cdot \eta s} \hat{f}(s, \eta, \xi - p) dp = \hat{U}(s, \eta, \xi),$$

therefore, from (3.23) and (3.26), we conclude that

$$(3.29) \quad \mathbb{E}[X_\xi(t, \eta_1) X_\xi^*(t, \eta_2)] = \hat{f}(t, \eta_1 - \eta_2, \xi) = \hat{w}(t, \eta_1 - \eta_2, \xi) e^{i\xi \cdot (\eta_1 - \eta_2)t}.$$

3.4. Some additional remarks.

Remark 3.4. The limit of $X_\xi^\varepsilon(t, \eta)$ can be written as

$$X_\xi(t, \eta) = \mathbb{E}X_\xi(t, \eta) + \tilde{X}_\xi(t, \eta).$$

The mean represents the ballistic component of the (compensated) wave field and the fluctuating part corresponds to the scattering (random) component and is given by a stochastic convolution, see formula (2.7). Since the latter term is a complex Gaussian, i.e., its real and imaginary part are independent zero mean Gaussians with the same variance, given by

$$\sigma^2(t, \xi) = \frac{1}{2} \mathbb{E}|\tilde{X}_\xi^\varepsilon(t, \eta)|^2 = \frac{1}{2} \left(\tilde{w}(t, \xi) - |\hat{\phi}_0(\xi)|^2 e^{-R(0)t} \right),$$

where \tilde{w} solves (3.22). It is an elementary fact that the intensity of the scattering component, defined as $|\tilde{X}_\xi^\varepsilon(t, \eta)|^2$, is of exponential distribution $\text{Exp}(\frac{1}{2\sigma^2(t, \xi)})$.

Remark 3.5. Theorem 2.2 concerns the asymptotics of the compensated wave function $X_\xi^\varepsilon(t, \eta)$, defined in (2.6). As can be seen from its definition, the field is obtained from the Fourier transform of the solution of (2.1) by ‘‘compensating’’ with the fast oscillating phase corresponding to the free Schrödinger equation (with no potential present). This object has been introduced in [1]. To prove Gaussianity of the scaled limit, we use the white-in-time structure of the potential and study the

relevant martingales. By invoking a martingale central limit theorem, the argument reduces to proving the convergence of quadratic variations. This in turn involves a fourth moment calculation and we deal with it via a diagram expansion method. The δ -correlation in time simplifies the diagrams significantly, compared to the smoothly correlated case. On a heuristic level, the convergence of the quadratic variation to a deterministic limit comes from a self-averaging effect, which roughly says that for any two distinct frequencies $\xi_1 \neq \xi_2$ the wave function evaluated at ξ_1 and ξ_2 are asymptotically independent hence the randomness disappears after the averaging. We will discuss this phenomenon in more details in Remark 5.2.

Remark 3.6. Although we do not provide the details, analogous results for other dispersion relations can be derived almost verbatim. More precisely we can replace the laplacian Δ by its fractional counterpart $-|\Delta|^\alpha$ for any $\alpha > 0$, or operators defined by other Fourier multipliers, and prove a similar result, with an appropriately adjusted scaling. An interesting feature is that the dispersion relation neither affects the limiting Gaussian nature nor the marginal distribution. As it will become clear later on, the Gaussianity comes from the magnitude of the phase, rather than its specific structure, and the second moment calculation in (3.22) does not involve the phase information due to the unitary evolution of the Schrödinger equation, so the marginal distribution is always the same. The dispersion relation only shows up in the limiting covariance structure.

Remark 3.7. The rescaled wave function $\phi_\varepsilon(t, x) = \phi(\frac{t}{\varepsilon^2}, x)$ satisfies

$$(3.30) \quad i\partial_t \phi_\varepsilon + \frac{1}{2\varepsilon^2} \Delta \phi_\varepsilon - \frac{1}{\varepsilon} V(\frac{t}{\varepsilon^2}, x) \circ \phi_\varepsilon = 0.$$

Since the potential is δ -correlated in time, the laws of the fields $(\frac{1}{\varepsilon} V(\frac{t}{\varepsilon^2}, x))_{(t,x) \in \mathbb{R}^{1+d}}$ and that of $(V(t, x))_{(t,x) \in \mathbb{R}^{1+d}}$ coincide. Therefore, the law of $(\phi_\varepsilon(t, x))_{(t,x) \in \mathbb{R}^{1+d}}$ is the same as that of the solution of

$$(3.31) \quad i\partial_t \phi_\varepsilon + \frac{1}{2\varepsilon^2} \Delta \phi_\varepsilon - V(t, x) \circ \phi_\varepsilon = 0.$$

It is clear that we can replace the factor $\frac{1}{\varepsilon^2}$ in the above equation by $\frac{1}{\gamma}$, send $\gamma \rightarrow 0$, and adjust accordingly the formulation of our result.

Remark 3.8. From an application point of view, we start with the Schrödinger equation of the form

$$i\partial_t \Phi + \frac{c_0}{2\omega} \Delta \Phi - \frac{\omega\sigma}{2c_0} V(t, x) \circ \Phi = 0,$$

which is the standard form that comes from the paraxial approximation, see e.g. [17]. Here $c_0 \sim O(1)$ is a constant of describing the average wave speed, $\omega \gg 1$ is the wave frequency, and σ is the strength of random media. Now suppose we consider a propagation distance of order $L \gg 1$, so $\Phi_L(t, x) = \Phi(tL, x)$ satisfies

$$i\partial_t \Phi_L + \frac{c_0 L}{2\omega} \Delta \Phi - \frac{L\omega\sigma}{2c_0} V(tL, x) \circ \Phi_L = 0.$$

By the scaling property of V , Φ_L has the same law as the solution to

$$i\partial_t \Phi_L + \frac{c_0 L}{2\omega} \Delta \Phi - \frac{\sqrt{L}\omega\sigma}{2c_0} V(t, x) \circ \Phi_L = 0.$$

Thus, if the parameters (ω, σ, L) satisfy

$$\sqrt{L}\omega\sigma \sim O(1), \quad \text{and} \quad \frac{L}{\omega} \gg 1,$$

we are in the regime given by (3.31) and the result in the paper shows that the compensated wave function has then an approximate Gaussian distribution.

Remark 3.9. Another interesting scaling regime one can consider is the following. Suppose we start with a Schrodinger equation with a random driving force of order $O(1)$:

$$i\partial_t\phi + \frac{1}{2}\Delta\phi - V(t, x) \circ \phi = 0.$$

The goal is to study the long time behavior of $\phi(t, x)$. It can be checked that the second moment $\mathbb{E}[|\hat{\phi}(t, \xi)|^2]$ equals to $\tilde{w}(t, \xi)$, which evolves according to (3.22). From the probabilistic point of view, the equation is associated with a Markov jump process corresponding to the momentum variable. It performs a jump process with the kernel $(2\pi)^{-d}\hat{R}(p)$. A standard diffusion approximation yields that $\tilde{w}_\varepsilon(t, \xi) := \tilde{w}(\frac{t}{\varepsilon^2}, \frac{\xi}{\varepsilon})$ satisfies

$$\partial_t \tilde{w}_\varepsilon \approx \frac{1}{2} \nabla \cdot (D \nabla \tilde{w}_\varepsilon), \quad \text{with } D := \int_{\mathbb{R}^d} \frac{\hat{R}(p)p \otimes p}{(2\pi)^d} dp.$$

In other words, the second moment actually converges to the solution of a heat equation in the high frequency regime, which is very natural as the Schrödinger dynamics mixes low and high frequencies and we are looking at an “infinite” long time scale. In this case, to study the behavior of $\hat{\phi}(\frac{t}{\varepsilon^2}, \frac{\xi}{\varepsilon})$ is a challenging problem. To go from the jump process to a diffusion process, the number of jumps needs to go to infinity and the effect of each individual jump goes to zero, in other words, the effective contribution to the second moment in the diffusive regime comes from infinitely many negligible jumps. From a mathematical point of view, if we write $\hat{\phi}(\frac{t}{\varepsilon^2}, \frac{\xi}{\varepsilon})$ by a Wiener chaos expansion, then the main contribution to its second moment comes from those chaos of very high order, each of which goes to zero while the sum converges as a Riemann sum. As a result, the dependence of the wave function on the random media becomes increasingly nonlinear as $\varepsilon \rightarrow 0$. To study the weak convergence of such random variables is difficult, and this also appears in the study of long time behaviors of random heat equations which requires new ideas and tools. In the weak forcing regime we consider here, the randomness does not escape to the tail of the chaos expansion, so it suffices to pass to the limit on the term by term basis.

Remark 3.10. While Theorem 2.2 is on the compensated wave function restricted to a small neighborhood of a fixed frequency ξ , the same proof in the paper applies to finitely many different frequencies ξ_1, \dots, ξ_n . In particular, the following result holds: if $\xi_i \neq \xi_j$ for $i \neq j \in \{1, \dots, n\}$, then

$$(X_{\xi_1}^\varepsilon(\cdot, \cdot), \dots, X_{\xi_n}^\varepsilon(\cdot, \cdot)) \Rightarrow (X_{\xi_1}^{(1)}(\cdot, \cdot), \dots, X_{\xi_n}^{(n)}(\cdot, \cdot))$$

in distribution in $\underbrace{C(\bar{\mathbb{R}}_+ \times \mathbb{R}^d) \times \dots \times C(\bar{\mathbb{R}}_+ \times \mathbb{R}^d)}_{n\text{-copies}}$, where $\{X_{\xi_j}^{(j)}(\cdot, \cdot)\}_{j=1}^n$ are independent

and each component $X_{\xi_j}^{(j)}(\cdot, \cdot)$ has the same law as that of $X_{\xi_j}(\cdot, \cdot)$.

Remark 3.11. Let us discuss two related works here. In [1], the authors studied the same problem except that the random potential has a smooth temporal covariance function, which creates extra technical difficulties in diagram expansions that we do not encounter here. The result in [1] is on the convergence of marginal distributions and the proof is based on the moment convergence. Thus, the main contribution of this paper is to show the convergence on the process level and to present a simpler proof using the martingale structure. In [9], the authors considered the same weak forcing regime of the Itô-Schrödinger model as ours (referred as the scintillation regime in the paper), i.e.

$$\text{propagation distance} = \frac{1}{\text{size of forcing}^2},$$

but with a low frequency initial condition, see [9, Equations (45)-(47)]. One of the results in [9] is that the fourth moments of the wave field approximately satisfy the Gaussian summation rule. Our conclusion is that the approximation holds also for the respective laws.

4. PROOF OF PROPOSITION 2.1

The Duhamel solution of (2.4) is given by the following series expansion

$$(4.1) \quad \mathbf{w}(t, dx, \xi) = e^{-R(0)t} \left\{ |\hat{\phi}_0(\xi)|^2 \delta_0(dx - \xi t) + \sum_{n=1}^{+\infty} \int_{\Delta_n(t)} d\tau_{1,n} \int_{\mathbb{R}^{nd}} |\hat{\phi}_0(p_n)|^2 \right. \\ \left. \times \delta_0(dx - \xi\tau_1 - p_1\tau_2 - \dots - p_{n-1}\tau_n - p_n(t - \tau_n)) \prod_{j=1}^n \frac{\hat{R}(p_{j-1} - p_j)}{(2\pi)^d} d\mathbf{p}_{1,n} \right\}.$$

Here $p_0 = \xi$ and $d\mathbf{p}_{1,n} := dp_1 \dots dp_n$, $d\tau_{1,n} = d\tau_1 \dots d\tau_n$ and

$$\Delta_n(t) := \left\{ (\tau_1, \dots, \tau_n) : \tau_j \geq 0, j = 1, \dots, n, \sum_{j=1}^n \tau_j \leq t \right\}.$$

Using the formula

$$\int_{\mathbb{R}^d} \delta_0(dx - (\xi - p)\tau - z) \frac{\hat{R}(p)dp}{(2\pi)^d} = \left(\frac{1}{2\pi\tau} \right)^d \hat{R}\left(\xi - \frac{x-z}{\tau}\right) dx, \quad (\tau, z, \xi) \in \mathbb{R}_+ \times \mathbb{R}^{2d},$$

we conclude that, cf (2.5),

$$(4.2) \quad \mathbf{u}(t, dx, \xi) = e^{-R(0)t} \left\{ \left(\frac{1}{2\pi t} \right)^d |\hat{\phi}_0(\xi)|^2 \hat{R}\left(\xi - \frac{x}{t}\right) + \sum_{n=1}^{+\infty} \int_{\Delta_n(t)} d\tau_{1,n} \int_{\mathbb{R}^{nd}} \left(\frac{1}{2\pi\tau_1} \right)^d |\hat{\phi}_0(p_n)|^2 \right. \\ \left. \times \hat{R}\left(\xi - \frac{1}{\tau_1}(x - p_1\tau_2 - \dots - p_{n-1}\tau_n - p_n(t - \tau_n))\right) \prod_{j=1}^n \frac{\hat{R}(p_{j-1} - p_j)}{(2\pi)^d} d\mathbf{p}_{1,n} \right\} dx,$$

and the conclusion of the proposition follows.

5. PROOF OF THEOREM 2.2

5.1. Preliminaries. The rescaled wave function $\phi_\varepsilon(t, x) = \phi\left(\frac{t}{\varepsilon^2}, x\right)$ satisfies (3.30). Due to the scale invariance of the white noise, see Remark 3.7 above, the solution coincides, up to the law, with the solution of the Itô equation

$$id\phi_\varepsilon(t, x) + \left(\frac{1}{2\varepsilon^2} \Delta\phi_\varepsilon(t, x) + \frac{i}{2} R(0)\phi_\varepsilon \right) dt - B(dt, x)\phi_\varepsilon(t, x) = 0,$$

where $B(t, x)$ is the Wiener process with the covariance given by (2.3). We rewrite the above equation in the Fourier domain as

$$(5.1) \quad id\hat{\phi}_\varepsilon + \left(-\frac{|\xi|^2}{2\varepsilon^2} \hat{\phi} + \frac{i}{2} R(0)\hat{\phi}_\varepsilon \right) dt - \int_{\mathbb{R}^d} \frac{\hat{B}(dt, dp)}{(2\pi)^d} \hat{\phi}_\varepsilon(t, \xi - p) = 0,$$

where $\hat{B}(dt, dp)$ is the Gaussian noise with the correlation

$$(5.2) \quad \mathbb{E}[\hat{B}(dt, dp)\hat{B}^*(ds, dq)] = (2\pi)^d \hat{R}(p)\delta(t-s)\delta(p-q) dt ds dp dq, \\ \hat{B}^*(dt, dp) = \hat{B}(dt, -dp).$$

Define

$$(5.3) \quad \hat{\psi}_\varepsilon(t, \xi) := \hat{\phi}_\varepsilon(t, \xi) \exp \left\{ \left(\frac{i|\xi|^2}{\varepsilon^2} + R(0) \right) \frac{t}{2} \right\}.$$

Note that by (2.6)

$$(5.4) \quad X_\xi^\varepsilon(t, \eta) = \hat{\psi}_\varepsilon(t, \xi + \varepsilon^2 \eta) e^{-\frac{1}{2} R(0)t}.$$

The random field $\hat{\psi}_\varepsilon(\cdot)$ is a solution of the integral equation

$$(5.5) \quad \hat{\psi}_\varepsilon(t, \xi) = \hat{\phi}_0(\xi) + M_\varepsilon(t, \xi).$$

Here

$$(5.6) \quad M_\varepsilon(t, \xi) := \frac{1}{i(2\pi)^d} \int_0^t \int_{\mathbb{R}^d} \exp\left\{i(2\xi \cdot p - |p|^2) \frac{s}{2\varepsilon^2}\right\} \hat{\psi}_\varepsilon(s, \xi - p) \hat{B}(ds, dp).$$

Note that $\{M_\varepsilon(t, \xi)\}_{t \geq 0}$ is a continuous trajectory square integrable martingale, with respect to the filtration

$$\mathcal{F}_t = \sigma\left(\{B(s, x), s \in [0, t], x \in \mathbb{R}^d\}\right), \quad t \geq 0.$$

Its quadratic variations are denoted by

$$(5.7) \quad \begin{aligned} Q_\xi^\varepsilon(t) &:= \langle M_\varepsilon(\cdot, \xi), M_\varepsilon^*(\cdot, \xi) \rangle_t = \frac{1}{(2\pi)^d} \int_0^t \int_{\mathbb{R}^d} |\hat{\psi}_\varepsilon(s, \xi - p)|^2 \hat{R}(p) dp ds, \\ \mathcal{Q}_\xi^\varepsilon(t) &:= \langle M_\varepsilon(\cdot, \xi), M_\varepsilon(\cdot, \xi) \rangle_t \\ &= -\frac{1}{(2\pi)^d} \int_0^t \int_{\mathbb{R}^d} \exp\left\{-\frac{is|p|^2}{\varepsilon^2}\right\} \hat{\psi}_\varepsilon(s, \xi - p) \hat{\psi}_\varepsilon(s, \xi + p) \hat{R}(p) dp ds. \end{aligned}$$

The following simple lemma holds.

Lemma 5.1. *We have*

$$(5.8) \quad \mathbb{E}[|\hat{\psi}_\varepsilon(t, \xi)|^2] = \tilde{w}(t, \xi) e^{R(0)t}, \quad (t, \xi) \in \bar{\mathbb{R}}_+ \times \mathbb{R}^d.$$

Proof. By the Itô isometry, we have

$$\mathbb{E}[|\hat{\psi}_\varepsilon(t, \xi)|^2] = |\hat{\phi}_0(\xi)|^2 + \int_0^t \int_{\mathbb{R}^d} \frac{\hat{R}(p)}{(2\pi)^d} \mathbb{E}[|\hat{\psi}_\varepsilon(s, \xi - p)|^2] dp ds.$$

It is straightforward to check that the deterministic function $\mathbb{E}[|\hat{\psi}_\varepsilon(t, \xi)|^2] e^{-R(0)t}$ solves (3.22) - the equation satisfied by $\tilde{w}(t, \xi)$. By the uniqueness of the solution, we conclude (5.8). \square

Remark 5.2. Our goal is to show that $X_\xi^\varepsilon(t, \eta)$ converges in distribution to $X_\xi(t, \eta)$, which solves the integral equation (2.7). Given (5.4) and (5.5), it reduces to the convergence of

$$M_\varepsilon(t, \xi + \varepsilon^2 \eta) \Rightarrow \int_0^t e^{\frac{1}{2} R(0)s} B_\xi(ds, \eta).$$

The limiting Gaussianity of the martingale comes from the self averaging of the quadratic variation. Take $\eta = 0$ for example. Our proof shows that $\hat{\psi}_\varepsilon(s, \xi_1)$ and $\hat{\psi}_\varepsilon(s, \xi_2)$ becomes asymptotically independent due to the high oscillations of the wave field, for any $\xi_1 \neq \xi_2$. Thus, the term

$$\int_{\mathbb{R}^d} |\hat{\psi}_\varepsilon(s, \xi - p)|^2 \hat{R}(p) dp,$$

appearing in the integral expression of $Q_\xi^\varepsilon(t)$, behaves like a sum of independent random variables. As a result, its limit, as $\varepsilon \rightarrow 0$, is deterministic and given by

$$\int_{\mathbb{R}^d} \mathbb{E}[|\hat{\psi}_\varepsilon(s, \xi - p)|^2] \hat{R}(p) dp = e^{R(0)s} \int_{\mathbb{R}^d} \tilde{w}(s, \xi - p) \hat{R}(p) dp,$$

due to Lemma 5.1. We refer to this phenomenon as *self-averaging*. This also explains that in order to see some nontrivial correlation structure of $\hat{\psi}_\varepsilon$, one needs to zoom in around a fixed ξ , which is why we consider the process $\{\hat{\psi}_\varepsilon(t, \xi + \varepsilon^2 \eta)\}_{(t, \eta) \in \bar{\mathbb{R}}_+ \times \mathbb{R}^d}$.

In what follows we shall use the following notation: for any set A and functions $f, g : A \rightarrow \bar{\mathbb{R}}_+$ we say that

$$f(a) \lesssim g(a), \quad a \in A,$$

if there exists $C > 0$ independent of ε such that $f(a) \leq Cg(a)$, $a \in A$.

Lemma 5.3. *We have*

$$(5.9) \quad Q_\xi^\varepsilon(t) \leq \frac{\|R\|_{L^1(\mathbb{R}^d)}}{\|\hat{R}\|_{L^1(\mathbb{R}^d)}} e^{R(0)t} \|\hat{\phi}_0\|_{L^2(\mathbb{R}^d)}^2, \quad (t, \xi) \in \bar{\mathbb{R}}_+ \times \mathbb{R}^d, \quad \varepsilon \in (0, 1], \quad \mathbb{P} \text{ a.s.}$$

In consequence, for any integer $n \geq 0$ and $T > 0$ that

$$(5.10) \quad \sup_{t \in [0, T], \xi \in \mathbb{R}^d, \varepsilon \in (0, 1]} \mathbb{E}[|M_\varepsilon(t, \xi)|^{2n}] < +\infty.$$

Proof. Using the Burkholder-Davis-Gundy inequality we conclude that

$$\mathbb{E}[|M_\varepsilon(t, \xi)|^{2n}] \leq C \mathbb{E}[Q_\xi^\varepsilon(t)^n], \quad (t, \xi) \in \bar{\mathbb{R}}_+ \times \mathbb{R}^d, \quad \varepsilon \in (0, 1],$$

where the positive constant C is independent of ε, t, ξ . Estimate (5.10) follows then directly from (5.9).

To show (5.9), we use the fact that $R \in L^1(\mathbb{R}^d)$ and obtain

$$(5.11) \quad \begin{aligned} & \int_{\mathbb{R}^d} \frac{\hat{R}(p)}{(2\pi)^d} |\hat{\psi}_\varepsilon(s, \xi - p)|^2 dp \leq \frac{\|R\|_{L^1(\mathbb{R}^d)}}{(2\pi)^d} \int_{\mathbb{R}^d} |\hat{\psi}_\varepsilon(s, \xi - p)|^2 dp \\ & = \frac{\|R\|_{L^1(\mathbb{R}^d)} e^{R(0)s}}{(2\pi)^d} \int_{\mathbb{R}^d} |\hat{\phi}_\varepsilon(s, p)|^2 dp = \frac{\|R\|_{L^1(\mathbb{R}^d)} e^{R(0)s}}{(2\pi)^d} \|\hat{\phi}_0\|_{L^2(\mathbb{R}^d)}^2, \quad (s, \xi) \in \mathbb{R}_+ \times \mathbb{R}^d, \end{aligned}$$

where in the last step we used the conservation of the L^2 norm of the solution of the Schrödinger equation. Estimate (5.9) is then a direct consequence of the first formula of (5.7). \square

From (5.5) and (5.10) we immediately conclude that following.

Corollary 5.4. *For any integer $n \geq 0$ and $T > 0$ we have that*

$$\sup_{t \in [0, T], \xi \in \mathbb{R}^d, \varepsilon \in (0, 1]} \mathbb{E}[|\hat{\psi}_\varepsilon(t, \xi)|^{2n}] < +\infty.$$

For $\xi \in \mathbb{R}^d$ fixed, we use the notation:

$$\mathcal{M}_\varepsilon(t, \eta) := M_\varepsilon(t, \xi + \varepsilon^2 \eta), \quad (t, \eta) \in \bar{\mathbb{R}}_+ \times \mathbb{R}^d.$$

The proof of Theorem 2.2 reduces to showing the convergence in the law of $\{\mathcal{M}_\varepsilon(t, \eta)\}_{(t, \eta) \in \bar{\mathbb{R}}_+ \times \mathbb{R}^d}$ over $C(\bar{\mathbb{R}}_+ \times \mathbb{R}^d)$.

5.2. Tightness. The main result of the present section is the following estimate.

Proposition 5.5. *For any $T > 0$, $\xi \in \mathbb{R}^d$ and an integer $n \geq 1$, there exists a constant $C(T, n) > 0$ such that*

$$(5.12) \quad \mathbb{E}[|\mathcal{M}_\varepsilon(t_1, \eta_1) - \mathcal{M}_\varepsilon(t_2, \eta_2)|^{2n}] \leq C(T, n)(|t_1 - t_2|^n + |\eta_1 - \eta_2|^{2n})$$

for all $t_1, t_2 \in [0, T]$, $\eta_1, \eta_2 \in \mathbb{R}^d$, $\varepsilon \in (0, 1]$.

From (5.10) for any $T > 0$, $\xi \in \mathbb{R}^d$ and $n \geq 1$ we have

$$(5.13) \quad \sup_{t \in [0, T], \eta \in \mathbb{R}^d, \varepsilon \in (0, 1]} \mathbb{E}[|\mathcal{M}_\varepsilon(t, \eta)|^{2n}] < +\infty.$$

As a consequence of the Kolmogorov tightness criterion for continuous random fields, see [13, Theorem 1.4.7, p. 38], (5.12) and (5.13) imply tightness of the laws

of $\{\mathcal{M}_\varepsilon(t, \eta)\}_{(t, \eta) \in \bar{\mathbb{R}}_+ \times \mathbb{R}^d}$, $\varepsilon \in (0, 1]$ over $C(\bar{\mathbb{R}}_+ \times \mathbb{R}^d)$, equipped with the standard Fréchet topology.

To show the proposition we shall need the following.

Lemma 5.6. *For any $T > 0$, $\xi \in \mathbb{R}^d$ and an integer $n \geq 1$, there exists a constant $C(T, n) > 0$ such that*

$$(5.14) \quad \mathbb{E}[|\mathcal{M}_\varepsilon(t, \eta_1) - \mathcal{M}_\varepsilon(t, \eta_2)|^{2n}] \leq C(n, T)|\eta_1 - \eta_2|^{2n}$$

for all $t \in [0, T]$, $\eta_1, \eta_2 \in \mathbb{R}^d$, $\varepsilon \in (0, 1]$ and

$$(5.15) \quad \mathbb{E}[|\mathcal{M}_\varepsilon(t_1, \eta) - \mathcal{M}_\varepsilon(t_2, \eta)|^{2n}] \leq C(n, T)|t_2 - t_1|^n$$

for all $t_1, t_2 \in [0, T]$, $\eta \in \mathbb{R}^d$, $\varepsilon \in (0, 1]$.

Proof. We prove (5.14). The argument for (5.15) is analogous. The difference is written as

$$(5.16) \quad \begin{aligned} f_\varepsilon(t, \xi, \eta_1, \eta_2) &:= M_\varepsilon(t, \xi + \varepsilon^2 \eta_1) - M_\varepsilon(t, \xi + \varepsilon^2 \eta_2) \\ &= \int_0^t \int_{\mathbb{R}^d} \frac{\hat{B}(ds, dp)}{i(2\pi)^d} \left[\exp \left\{ i \left[2(\xi + \varepsilon^2 \eta_1) \cdot p - |p|^2 \right] \frac{s}{2\varepsilon^2} \right\} - \exp \left\{ i \left[2(\xi + \varepsilon^2 \eta_2) \cdot p - |p|^2 \right] \frac{s}{2\varepsilon^2} \right\} \right] \\ &\quad \times \hat{\psi}_\varepsilon(s, \xi + \varepsilon^2 \eta_1 - p) \\ &+ \int_0^t \int_{\mathbb{R}^d} \frac{\hat{B}(ds, dp)}{i(2\pi)^d} \exp \left\{ i \left[2(\xi + \varepsilon^2 \eta_2) \cdot p - |p|^2 \right] \frac{s}{2\varepsilon^2} \right\} [\hat{\psi}_\varepsilon(s, \xi + \varepsilon^2 \eta_1 - p) - \hat{\psi}_\varepsilon(s, \xi + \varepsilon^2 \eta_2 - p)] \\ &= A_1 + A_2, \end{aligned}$$

where A_1, A_2 denote the two integral terms appearing in (5.16).

Estimates of A_1 . Using the Burkholder-Davis-Gundy inequality and an elementary inequality $1 - \cos x \lesssim x^2$, we can write

$$(5.17) \quad \begin{aligned} \mathbb{E}[|A_1|^{2n}] &\lesssim \mathbb{E} \left[\left(\int_0^t \int_{\mathbb{R}^d} \hat{R}(p) [1 - \cos((\eta_1 - \eta_2) \cdot ps)] |\hat{\psi}_\varepsilon(s, \xi + \varepsilon \eta_1 - p)|^2 dp ds \right)^n \right] \\ &\lesssim |\eta_1 - \eta_2|^{2n} \mathbb{E} \left[\left(\int_0^t \int_{\mathbb{R}^d} |p|^2 s^2 \hat{R}(p) |\hat{\psi}_\varepsilon(s, \xi + \varepsilon \eta_1 - p)|^2 dp ds \right)^n \right]. \end{aligned}$$

Thanks to the fact that $\sup_{p \in \mathbb{R}^d} |p|^2 \hat{R}(p) < +\infty$ and the conservation of the L^2 norm of the solution of the Schrödinger equation, the right hand side of (5.17) can be estimated by an expression of the order $|\eta_1 - \eta_2|^{2n}$.

To abbreviate the notation we shall write $\|X\|_p := (\mathbb{E}|X|^p)^{1/p}$. Concerning A_2 , again by the Burkholder-Davis-Gundy inequality, we have

$$\mathbb{E}[|A_2|^{2n}] \lesssim \left\| \left(\int_0^t \int_{\mathbb{R}^d} \hat{R}(p) |\hat{\psi}_\varepsilon(s, \xi + \varepsilon \eta_1 - p) - \hat{\psi}_\varepsilon(s, \xi + \varepsilon \eta_2 - p)|^2 dp ds \right) \right\|_n^n.$$

By the triangle inequality the right hand side is estimated by

$$\left\{ \int_0^t \int_{\mathbb{R}^d} \hat{R}(p) \|\hat{\psi}_\varepsilon(s, \xi + \varepsilon \eta_1 - p) - \hat{\psi}_\varepsilon(s, \xi + \varepsilon \eta_2 - p)\|_{2n}^2 dp ds \right\}^n.$$

Invoking (5.5) we can further estimate this term by an expression

$$\begin{aligned} &\left(\int_0^t \int_{\mathbb{R}^d} \hat{R}(p) [\|\hat{\phi}_0(\xi + \varepsilon \eta_1 - p) - \hat{\phi}_0(\xi + \varepsilon \eta_2 - p)\|^2 + \|f_\varepsilon(s, \xi - p, \eta_1, \eta_2)\|_{2n}^2] dp ds \right)^n \\ &\lesssim \varepsilon^{2n} |\eta_1 - \eta_2|^{2n} + \int_0^t \int_{\mathbb{R}^d} \hat{R}(p) \|f_\varepsilon(s, \xi - p, \eta_1, \eta_2)\|_{2n}^{2n} dp ds, \end{aligned}$$

cf (5.16) for the definition of $f_\varepsilon(\cdot)$. In the last inequality we have used the Lipschitz regularity of the initial data and the Jensen inequality.

Combining the estimates for A_1, A_2 , we reach the integral inequality

$$\|f_\varepsilon(t, \xi, \eta_1, \eta_2)\|_{2^n}^{2n} \leq C|\eta_1 - \eta_2|^{2n} + C \int_0^t \int_{\mathbb{R}^d} \hat{R}(p) \|f_\varepsilon(s, \xi - p, \eta_1, \eta_2)\|_{2^n}^{2n} dp ds$$

for all $t \in [0, T], \xi, \eta_1, \eta_2 \in \mathbb{R}^d$, where the constant C only depends on T, n . Taking the supremum over ξ in both sides of the inequality and invoking the Gronwall inequality we conclude the lemma. \square

Proof of Proposition 5.5. Note that

$$|\mathcal{M}_\varepsilon(t_1, \eta_1) - \mathcal{M}_\varepsilon(t_2, \eta_2)|^{2n} \lesssim |\mathcal{M}_\varepsilon(t_1, \eta_1) - \mathcal{M}_\varepsilon(t_2, \eta_1)|^{2n} + |\mathcal{M}_\varepsilon(t_2, \eta_1) - \mathcal{M}_\varepsilon(t_2, \eta_2)|^{2n}.$$

The conclusion of the proposition is then a straightforward consequence of Lemma 5.6. \square

5.3. Convergence of finite dimensional distributions - limit identification.

Using (5.3) and (5.5), we can write the compensated wave function, see (2.6) and (5.4), as

$$(5.18) \quad X_\xi^\varepsilon(t, \eta) = e^{-\frac{1}{2}R(0)t} \hat{\phi}_0(\xi + \varepsilon^2 \eta) + e^{-\frac{1}{2}R(0)t} \mathcal{M}_\varepsilon(t, \eta).$$

Recall that $\mathcal{M}_\varepsilon(t, \eta) = M_\varepsilon(t, \xi + \varepsilon^2 \eta)$, with ξ fixed, and $M_\varepsilon(t, \xi)$ defined in (5.6). To complete the proof of Theorem 2.2, given the tightness proved in the previous section, we only need to show the convergence of finite dimensional distributions of the above field. For any integer $N \geq 1$ and $\{\eta_j\}_{j=1}^N$, we will show in this section that

$$(5.19) \quad (\mathcal{M}_\varepsilon(t, \eta_1), \dots, \mathcal{M}_\varepsilon(t, \eta_N)) \Rightarrow (Y(t, \eta_1), \dots, Y(t, \eta_N)), \quad \text{as } \varepsilon \rightarrow 0,$$

in distribution in $C([0, \infty); \mathbb{C}^N)$, with

$$Y(t, \eta) := \int_0^t e^{\frac{1}{2}R(0)s} B_\xi(ds, \eta),$$

where $B_\xi(\cdot)$ is defined in (2.8). The conclusion of the theorem is then a consequence of formula (2.7).

For $j, k = 1, \dots, N$, we define

$$(5.20) \quad Q_{\varepsilon, j, k}(t) := \langle \mathcal{M}_\varepsilon(\cdot, \eta_j), \mathcal{M}_\varepsilon^*(\cdot, \eta_k) \rangle(t) = \int_0^t \int_{\mathbb{R}^d} \frac{\hat{R}(p)}{(2\pi)^d} e^{i(\eta_j - \eta_k) \cdot ps} h_\varepsilon^{j, k}(s, \xi - p) dp ds$$

and,

$$(5.21) \quad \begin{aligned} \mathcal{Q}_{\varepsilon, j, k}(t) &:= \langle \mathcal{M}_\varepsilon(\cdot, \eta_j), \mathcal{M}_\varepsilon(\cdot, \eta_k) \rangle(t) \\ &= - \int_0^t \int_{\mathbb{R}^d} \frac{\hat{R}(p)}{(2\pi)^d} \exp \left\{ i \left(\eta_j - \eta_k - \frac{p}{\varepsilon^2} \right) \cdot ps \right\} g_\varepsilon^{j, k}(s, \xi, p) dp ds, \end{aligned}$$

where

$$(5.22) \quad h_\varepsilon^{j, k}(t, \xi) := \hat{\psi}_\varepsilon(t, \xi + \varepsilon^2 \eta_j) \hat{\psi}_\varepsilon^*(t, \xi + \varepsilon^2 \eta_k),$$

$$(5.23) \quad g_\varepsilon^{j, k}(t, \xi, p) := \hat{\psi}_\varepsilon(t, \xi - p + \varepsilon^2 \eta_j) \hat{\psi}_\varepsilon(t, \xi + p + \varepsilon^2 \eta_k).$$

From (3.19) we also know that $\langle Y(\cdot, \eta_j), Y(\cdot, \eta_k) \rangle(t) = 0$ and

$$(5.24) \quad \begin{aligned} \langle Y(\cdot, \eta_j), Y^*(\cdot, \eta_k) \rangle(t) &= \int_0^t e^{R(0)s} d \langle B_\xi(\cdot, \eta_j), B_\xi^*(\cdot, \eta_k) \rangle(s) \\ &= \int_0^t \int_{\mathbb{R}^d} e^{R(0)s} e^{-i(\eta_j - \eta_k) \cdot y} \mathbf{U}(s, y + \xi s, \xi) dy ds. \end{aligned}$$

Using (3.27) and (3.28) we can further write

$$(5.25) \quad \begin{aligned} & \langle Y(\cdot, \eta_j), Y^*(\cdot, \eta_k) \rangle(t) \\ &= \sum_{n \geq 1} \int_{[0, t]_x^n} \int_{\mathbb{R}^{nd}} \prod_{\ell=1}^n \frac{\hat{R}(p_\ell) e^{i(\eta_j - \eta_k) \cdot p_\ell s_\ell}}{(2\pi)^d} |\hat{\phi}_0(\xi - p_1 - \dots - p_n)|^2 d\mathbf{p}_{1,n} ds_{1,n}. \end{aligned}$$

By Theorem IX.3.21 of [11], the proof of (5.19) reduces to the following proposition.

Proposition 5.7. *For any η_1, \dots, η_N and $j, k = 1, \dots, N$, the processes*

$$(5.26) \quad Q_{\varepsilon, j, k}(\cdot) \rightarrow \langle Y(\cdot, \eta_j), Y^*(\cdot, \eta_k) \rangle$$

and

$$(5.27) \quad \mathcal{Q}_{\varepsilon, j, k}(\cdot) \rightarrow 0,$$

in probability in $C[0, \infty)$, as $\varepsilon \rightarrow 0$.

Establishing this result finishes the proof of Theorem 2.2.

5.4. Proof of Proposition 5.7. The result will be concluded at the end of a series of lemmas. First we establish tightness property for the respective families.

Lemma 5.8. *The families of processes $\{Q_{\varepsilon, j, k}(\cdot)\}_{\varepsilon > 0}$ and $\{\mathcal{Q}_{\varepsilon, j, k}(\cdot)\}_{\varepsilon > 0}$ are tight in $C[0, \infty)$.*

Proof. We use the Kolmogorov tightness criterion, see [3, Theorem 12.3, p. 95]. From the definition of $Q_{\varepsilon, j, k}(\cdot)$, we have

$$|Q_{\varepsilon, j, k}(t_2) - Q_{\varepsilon, j, k}(t_1)| \leq \int_{t_1}^{t_2} \int_{\mathbb{R}^d} \frac{\hat{R}(p)}{(2\pi)^d} |h_\varepsilon^{j, k}(s, \xi - p)| dp ds$$

for any $0 \leq t_1 < t_2$. Applying the triangle inequality and Corollary 5.4, we conclude that for any $T > 0$

$$\begin{aligned} \|Q_{\varepsilon, j, k}(t_2) - Q_{\varepsilon, j, k}(t_1)\|_n &\leq \int_{t_1}^{t_2} \int_{\mathbb{R}^d} \frac{\hat{R}(p)}{(2\pi)^d} \|h_\varepsilon^{j, k}(s, \xi - p)\|_n dp ds \\ &\leq C|t_2 - t_1| \end{aligned}$$

for all $t_1, t_2 \in [0, T]$, with the constant C independent of ε . The proof for $\{\mathcal{Q}_{\varepsilon, j, k}(\cdot)\}_{\varepsilon > 0}$ is similar so we omit it here. \square

With the tightness property established, we only need to show (5.26) and (5.27) for fixed $t > 0$. We will study $Q_{\varepsilon, j, k}$ and $\mathcal{Q}_{\varepsilon, j, k}$ separately in the following sections.

5.4.1. Convergence of $Q_{\varepsilon, j, k}(t)$. For fixed $t > 0$, to show the convergence of $Q_{\varepsilon, j, k}(t)$ in probability, we first consider the expectation and prove the following.

Lemma 5.9. *For any $t > 0$ and $j, k = 1, \dots, N$, we have*

$$(5.28) \quad \lim_{\varepsilon \rightarrow 0} \mathbb{E}[Q_{\varepsilon, j, k}(t)] = \langle Y(\cdot, \eta_j), Y^*(\cdot, \eta_k) \rangle(t).$$

Proof. Fix j, k , define $\bar{h}_\varepsilon^{j, k}(t, \xi) = \mathbb{E}h_\varepsilon^{j, k}(s, \xi)$. By (5.5) we obtain the integral equation for $\bar{h}_\varepsilon^{j, k}$:

$$\bar{h}_\varepsilon^{j, k}(t, \xi) = \hat{\phi}_0(\xi + \varepsilon^2 \eta_j) \hat{\phi}_0^*(\xi + \varepsilon^2 \eta_k) + \int_0^t \int_{\mathbb{R}^d} \frac{\hat{R}(p)}{(2\pi)^d} e^{i(\eta_j - \eta_k) \cdot ps} \bar{h}_\varepsilon^{j, k}(t, \xi - p) dp ds.$$

Iterating the above integral equation, we obtain

$$(5.29) \quad \bar{h}_\varepsilon^{j, k}(t, \xi) = \hat{\phi}_0(\xi + \varepsilon^2 \eta_j) \hat{\phi}_0^*(\xi + \varepsilon^2 \eta_k) + \sum_{n=1}^{+\infty} \bar{h}_{\varepsilon, n}^{j, k}(t, \xi),$$

with

$$\begin{aligned} \bar{h}_{\varepsilon,n}(t, \xi) &:= \int_{[0,t]_z^n} \int_{\mathbb{R}^{nd}} \prod_{\ell=1}^n \frac{\hat{R}(p_\ell) e^{i(\eta_j - \eta_\ell) \cdot p_\ell s_\ell}}{(2\pi)^d} \\ &\quad \times \hat{\phi}_0(\xi + \varepsilon^2 \eta_j - p_1 - \dots - p_n) \hat{\phi}_0^*(\xi + \varepsilon^2 \eta_k - p_1 - \dots - p_n) d\mathbf{p}_{1,n} ds_{1,n} \end{aligned}$$

Passing to the limit in the series in the right hand side of (5.29) on the term by term basis and computing $\lim_{\varepsilon \rightarrow 0} \bar{h}_{\varepsilon,n}^{j,k}(t, \xi)$, we conclude (5.28). \square

5.4.2. *Convergence of $\mathbb{E}|Q_{\varepsilon,j,k}(t)|^2$.* Next we analyze $\mathbb{E}|Q_{\varepsilon,j,k}(t)|^2$, which is the main technical part of the paper. The goal is to show that

Lemma 5.10. *For any $t > 0$ and $j, k = 1, \dots, N$, we have*

$$(5.30) \quad \mathbb{E}|Q_{\varepsilon,j,k}(t)|^2 \rightarrow |\langle Y(\cdot, \eta_j), Y(\cdot, \eta_k)^* \rangle(t)|^2, \quad \text{as } \varepsilon \rightarrow 0.$$

Combining Lemmas 5.8, 5.9 and 5.10, we complete the proof of (5.26).

The expression of $\mathbb{E}|Q_{\varepsilon,j,k}(t)|^2$ involves the fourth moments of $\hat{\psi}_\varepsilon$, which we will analyze through a rather standard diagram expansion. Before entering the details of the proof, we introduce the notation that will be used and prove some preliminary results.

Diagram expansion and moments calculation. Starting from the integral equation (5.5), for fixed (t, ξ) , we can write the random variable $\hat{\psi}_\varepsilon(t, \xi)$ as an infinite Wiener chaos expansion:

$$(5.31) \quad \hat{\psi}_\varepsilon(t, \xi) = \sum_{n=0}^{+\infty} \hat{\psi}_{n,\varepsilon}(t, \xi),$$

where $\hat{\psi}_{0,\varepsilon}(t, \xi) = \hat{\phi}_0(\xi)$ and, cf (5.6),

$$(5.32) \quad \hat{\psi}_{n,\varepsilon}(t, \xi) := \int_{[0,t]_z^n} \int_{\mathbb{R}^{nd}} \prod_{j=1}^n \frac{\hat{B}(ds_j, dp_j)}{i(2\pi)^d} e^{i\Theta_n(\xi, \mathbf{p}, \mathbf{s})\varepsilon^{-2}} \hat{\phi}_0(\xi - p_1 - \dots - p_n), \quad n \geq 1.$$

For each $n \geq 1$, the phase factor is

$$(5.33) \quad \begin{aligned} 2\Theta_n(\xi, \mathbf{p}, \mathbf{s}) &= (|\xi|^2 - |\xi - p_1|^2) s_1 + (|\xi - p_1|^2 - |\xi - p_1 - p_2|^2) s_2 \\ &\quad + \dots + (|\xi - \dots - p_{n-1}|^2 - |\xi - \dots - p_n|^2) s_n. \end{aligned}$$

In what follows we will need to estimate moments of the form

$$(5.34) \quad \mathcal{N}_\varepsilon(\mathbf{n}, \mathbf{n}', \mathbf{t}, \mathbf{t}', \boldsymbol{\xi}, \boldsymbol{\xi}') := \mathbb{E} \left[\prod_{j=1}^{N_1} \hat{\psi}_{n_j, \varepsilon}(t_j, \xi_j) \prod_{j'=1}^{N_2} \hat{\psi}_{n'_{j'}, \varepsilon}^*(t'_{j'}, \xi'_{j'}) \right].$$

Here N_1, N_2 are positive integers and since for our purpose it is enough to consider the 4-th order moments we have $N_1 + N_2 = 4$. We use the boldface notation, e.g. $\mathbf{n}, \boldsymbol{\xi}$, to denote the vectors formed by the respective elements, e.g. $\{n_j\}, \{\xi_j\}$. Let $|\mathbf{n}|_1 = \sum_{j=1}^{N_1} n_j$ and $|\mathbf{n}'|_1 = \sum_{j'=1}^{N_2} n'_{j'}$ be the ℓ_1 norm of \mathbf{n}, \mathbf{n}' , and $K = |\mathbf{n}|_1 + |\mathbf{n}'|_1$. From the property of multiple moments of Gaussians, in order for the expression in (5.34) to be non-zero, the integer $K > 0$ has to be even.

The expression for $\hat{\psi}_{n,\varepsilon}(t, \xi)$ in (5.32) involves an n -fold stochastic time integral in the s -variable and an n -fold integral in the momentum variable p . For $\hat{\psi}_{n_j, \varepsilon}(t_j, \xi_j)$, we will use $\mathbf{s}_j = (s_{j,1}, \dots, s_{j,n_j})$ as the “ s -variable” ensemble corresponding to the index j and, similarly, $\mathbf{p}_j = (p_{j,1}, \dots, p_{j,n_j})$ as the “ p -variable”. Similarly, we will use an analogous notation for the primed variables.

With the above convention, we can write

$$(5.35) \quad \prod_{j=1}^{N_1} \hat{\psi}_{n_j, \varepsilon}(t_j, \xi_j) = \prod_{j=1}^{N_1} \left\{ \int_{[0, t_j]_{<}}^{n_j} \int_{\mathbb{R}^{n_j d}} \prod_{k=1}^{n_j} \frac{\hat{B}(ds_{j,k}, dp_{j,k})}{i(2\pi)^d} \right. \\ \left. \times \exp \left\{ i\Theta_{n_j}(\xi_j, \mathbf{p}_j, \mathbf{s}_j) \varepsilon^{-2} \right\} \hat{\phi}_0(\xi_j - p_{j,1} - \dots - p_{j,n_j}) \right\}$$

and

$$(5.36) \quad \prod_{j'=1}^{N_2} \hat{\psi}_{n_{j'}, \varepsilon}^*(t_{j'}, \xi_{j'}) = \prod_{j'=1}^{N_2} \left\{ \int_{[0, t_{j'}]_{<}}^{n_{j'}} \int_{\mathbb{R}^{n_{j'} d}} \prod_{k'=1}^{n_{j'}} \frac{\hat{B}^*(ds'_{j',k'}, dp'_{j',k'})}{-i(2\pi)^d} \right. \\ \left. \exp \left\{ -i\Theta_{n_{j'}}(\xi_{j'}, \mathbf{p}'_{j'}, \mathbf{s}'_{j'}) \varepsilon^{-2} \right\} \hat{\phi}_0^*(\xi_{j'} - p'_{j',1} - \dots - p'_{j',n_{j'}}) \right\}.$$

Pairing. To compute $\mathcal{N}_\varepsilon(\mathbf{n}, \mathbf{n}', \mathbf{t}, \mathbf{t}', \boldsymbol{\xi}, \boldsymbol{\xi}')$, we need to evaluate the expectation of the K -th moment of the Gaussian element

$$(5.37) \quad \mathbb{E} \left[\prod_{j=1}^{N_1} \left(\prod_{k=1}^{n_j} \hat{B}(ds_{j,k}, dp_{j,k}) \right) \prod_{j'=1}^{N_2} \left(\prod_{k'=1}^{n_{j'}} \hat{B}^*(ds'_{j',k'}, dp'_{j',k'}) \right) \right]$$

that we handle using the Wick theorem, see e.g. [12, Theorem 1.36]. To apply it we introduce some further notations.

Suppose that A is a finite subset of even cardinality. By a pairing \mathcal{P} over the elements of the set, we mean any partition of A into two element disjoint subsets. Consider the set of all pairs (λ, w) belonging to the set

$$\mathcal{Z} := \{(s_{j,k}, p_{j,k}), (s'_{j',k'}, p'_{j',k'})\}_{j,k,j',k'}$$

ordered by the lexicographical order, i.e. (λ_1, w_1) precedes (λ_2, w_2) and we write $(\lambda_1, w_1) \leq (\lambda_2, w_2)$, if any of the following happens:

- 1) $(\lambda_1, w_1) = (s_{j,k}, p_{j,k})$ and $(\lambda_2, w_2) = (s'_{j',k'}, p'_{j',k'})$,
- 2) $(\lambda_\ell, w_\ell) = (s_{j_\ell, k_\ell}, p_{j_\ell, k_\ell})$, $\ell = 1, 2$ and $j_1 < j_2$,
- 3) $(\lambda_\ell, w_\ell) = (s'_{j'_\ell, k'_\ell}, p'_{j'_\ell, k'_\ell})$, $\ell = 1, 2$ and $j'_1 < j'_2$.

Consider all ordered pairings formed over the set $\{(s_{j,k}, p_{j,k}), (s'_{j',k'}, p'_{j',k'})\}_{j,k,j',k'}$ such that two elements with the same j -s, or j' -s cannot be paired. In other words, there is no pair formed inside any vector

$$[(s_{j,1}, p_{j,1}), \dots, (s_{j,n_j}, p_{j,n_j})], \quad \text{or} \quad [(s'_{j',1}, p'_{j',1}), \dots, (s'_{j',n_{j'}}, p'_{j',n_{j'}})].$$

Denote $((\lambda, w), (\lambda', w'))$ a typical pair. Assume that (λ, w) is the left element of the pair in the sense that $(\lambda, w) \leq (\lambda', w')$. Each pairing also can be easily ordered with the order inherited from the order of the set \mathcal{Z} on its left vertices.

Denote by Π the set of all ordered pairings as described above. Let

$$\iota_{w,w'} = \begin{cases} 1, & \text{if } (\lambda, w) = (s_{j,k}, p_{j,k}) \text{ and } (\lambda', w') = (s'_{j',k'}, p'_{j',k'}), \\ 0, & \text{if otherwise,} \end{cases}$$

and

$$S_{\mathbf{n}, \mathbf{n}', \mathbf{t}, \mathbf{t}'} := [0, t_1]_{<}^{n_1} \times \dots \times [0, t_{N_1}]_{<}^{n_{N_1}} \times [0, t'_1]_{<}^{n'_1} \times \dots \times [0, t'_{N_2}]_{<}^{n'_{N_2}}.$$

Using the Wick theorem we can write that

$$(5.38) \quad \mathcal{N}_\varepsilon(\mathbf{n}, \mathbf{n}', \mathbf{t}, \mathbf{t}', \boldsymbol{\xi}, \boldsymbol{\xi}') = \sum_{\mathcal{P} \in \Pi} \int_{\mathbb{R}^{Kd}} \int_{S_{\mathbf{n}, \mathbf{n}', \mathbf{t}, \mathbf{t}'}} \\ \left\{ \prod_{((\lambda, w), (\lambda', w')) \in \mathcal{P}} \frac{\hat{R}(w)}{(2\pi)^d} \delta(\lambda - \lambda') \delta(w + (-1)^{\iota_{w,w'}} w') \right\} e^{i\varepsilon^{-2} \Theta} \Phi d\lambda d\lambda' dw dw',$$

where to ease the notation, we write

$$d\boldsymbol{\lambda}d\boldsymbol{\lambda}' = \prod_{j=1}^{N_1} \prod_{k=1}^{n_j} ds_{j,k} \prod_{j'=1}^{N_2} \prod_{k'=1}^{n'_{j'}} ds'_{j',k'} \quad \text{and} \quad d\mathbf{w}d\mathbf{w}' = \prod_{j=1}^{N_1} \prod_{k=1}^{n_j} dp_{j,k} \prod_{j'=1}^{N_2} \prod_{k'=1}^{n'_{j'}} dp'_{j',k'}.$$

Performing the integration over the (λ', w') -variables, we conclude that

$$(5.39) \quad \mathcal{N}_\varepsilon(\mathbf{n}, \mathbf{n}', \mathbf{t}, \mathbf{t}', \boldsymbol{\xi}, \boldsymbol{\xi}') = \sum_{\mathcal{P} \in \Pi} \int_{\mathbb{R}^{Kd/2}} \int_{S_{\mathcal{P}}} \left\{ \prod_{\ell=1}^{K/2} \frac{\hat{R}(w_\ell)}{(2\pi)^d} \right\} e^{i\varepsilon^{-2}\Theta_{\mathcal{P}}} \Phi_{\mathcal{P}} d\boldsymbol{\lambda}d\mathbf{w},$$

where the domain of integration $S_{\mathcal{P}}$ - that is a convex set, the phase $\Theta_{\mathcal{P}}$ and the factor $\Phi_{\mathcal{P}}$ in (5.39) are induced from $S_{\mathbf{n}, \mathbf{m}, \mathbf{t}, \mathbf{s}}$, Θ and Φ by the collapse of λ' and w' - variables. We have also denoted all $\boldsymbol{\lambda}$ -variables by $\{\lambda_\ell\}$ and all \mathbf{w} -variables by $\{w_\ell\}$, and assume that $\{\lambda_1, \dots, \lambda_{K/2}\}$ is ordered according to the pairing order. Note that the components of $\boldsymbol{\lambda}$ depend on the partition \mathcal{P} . However they obey the ordering inherited from each individual simplex. It is quite possible that $S_{\mathcal{P}} = \emptyset$ for some pairings, e.g. if we have the domain $[0, t_1]_{<}^2 \times [0, t'_1]_{<}^2$, and the pairing

$$\{(s_{1,1}, p_{1,1}), (s'_{1,2}, p'_{1,2}), ((s_{1,2}, p_{1,2}), (s'_{1,1}, p'_{1,1}))\},$$

then the set $\{s_{1,1} > s_{1,2}, s'_{1,1} > s'_{1,2}\}$ does not intersect with $\{s_{1,1} = s'_{1,2}, s_{1,2} = s'_{1,1}\}$ and, as a result, $S_{\mathcal{P}} = \emptyset$.

For each \mathcal{P} , we can partition $S_{\mathcal{P}}$, up to a null Lebesgue measure set, into subsets depending on the ordering of $\{\lambda_1, \dots, \lambda_{K/2}\}$. More precisely, given a permutation σ of the set $\{1, \dots, \frac{K}{2}\}$, we define

$$S_{\mathcal{P}, \sigma} := S_{\mathcal{P}} \cap \{\lambda_{\sigma(1)} > \lambda_{\sigma(2)} > \dots > \lambda_{\sigma(\frac{K}{2})}\}.$$

Obviously the sets are disjoint for different permutations and $\mathbf{m}(S_{\mathcal{P}} \setminus \cup_{\sigma} S_{\mathcal{P}, \sigma}) = 0$, where the union \cup_{σ} is taken over all the permutations and $\mathbf{m}(\cdot)$ denotes the Lebesgue measure on $\mathbb{R}^{K/2}$.

As in the case of $S_{\mathcal{P}}$, it is entirely possible that some $S_{\mathcal{P}, \sigma}$ can be empty sets - we will take care of them later in the proof, and for the moment, we do not distinguish between them to keep the notation simple. With the above notation, we write:

$$(5.40) \quad \mathcal{N}_\varepsilon(\mathbf{n}, \mathbf{n}', \mathbf{t}, \mathbf{t}', \boldsymbol{\xi}, \boldsymbol{\xi}') = \sum_{\mathcal{P}} \sum_{\sigma} \mathcal{I}_\varepsilon(\mathcal{P}, \sigma)$$

with

$$(5.41) \quad \mathcal{I}_\varepsilon(\mathcal{P}, \sigma) := \int_{\mathbb{R}^{Kd/2}} \int_{S_{\mathcal{P}, \sigma}} \left\{ \prod_{\ell=1}^{K/2} \frac{\hat{R}(w_\ell)}{(2\pi)^d} \right\} e^{i\varepsilon^{-2}\Theta_{\mathcal{P}}} \Phi_{\mathcal{P}} d\boldsymbol{\lambda}d\mathbf{w}.$$

Bounds on $\mathcal{N}_\varepsilon(\mathbf{n}, \mathbf{n}', \mathbf{t}, \mathbf{t}', \boldsymbol{\xi}, \boldsymbol{\xi}')$. Let $\{W_t\}_{t \geq 0}$ be a standard real-valued Brownian motion. Define the processes

$$(5.42) \quad H_0(t) \equiv 1, \\ H_n(t) = R(0)^{\frac{n}{2}} \int_{[0, t]_{<}^2} dW_{s_n} \dots dW_{s_1} = \frac{1}{n!} (R(0)t)^{n/2} h_n \left(\frac{W_t}{\sqrt{t}} \right), \quad n \geq 1,$$

where $h_n(x) := (-1)^n e^{x^2/2} (e^{-x^2/2})^{(n)}$ is the n -th degree Hermite polynomial, cf (3.3.8) of [16].

Lemma 5.11. *The following estimate holds*

$$(5.43) \quad |\mathcal{N}_\varepsilon(\mathbf{n}, \mathbf{n}', \mathbf{t}, \mathbf{t}', \boldsymbol{\xi}, \boldsymbol{\xi}')| \leq \|\hat{\phi}_0\|_{L^\infty(\mathbb{R}^d)}^4 \mathbb{E} \left[\prod_{j=1}^{N_1} H_{n_j}(t_j) \prod_{j'=1}^{N_2} H_{n'_{j'}}(t'_{j'}) \right]$$

for all $(\mathbf{n}, \mathbf{n}', \mathbf{t}, \mathbf{t}', \boldsymbol{\xi}, \boldsymbol{\xi}')$. In consequence,

$$(5.44) \quad \sum_{\mathbf{n}, \mathbf{n}'} \left| \mathbb{E} \left[\prod_{j=1}^{N_1} \hat{\psi}_{n_j, \varepsilon}(t_j, \xi_j) \prod_{j'=1}^{N_2} \hat{\psi}_{n_{j'}, \varepsilon}^*(t_{j'}, \xi_{j'}) \right] \right| \leq \|\hat{\phi}_0\|_{L^\infty(\mathbb{R}^d)}^4 \exp \left\{ 6R(0) \max_{j, j'}(t_j, t_{j'}) \right\}$$

for all $(\mathbf{t}, \mathbf{t}', \boldsymbol{\xi}, \boldsymbol{\xi}')$.

Proof. We use expression (5.39) to represent $\mathcal{N}_\varepsilon(\mathbf{n}, \mathbf{n}', \mathbf{t}, \mathbf{t}', \boldsymbol{\xi}, \boldsymbol{\xi}')$. Recall that $N_1 + N_2 = 4$. Thanks to the obvious bounds $|e^{i\varepsilon^{-2}\Theta_{\mathcal{P}}}| \leq 1$ and $|\Phi_{\mathcal{P}}| \leq \|\hat{\phi}_0\|_{L^\infty(\mathbb{R}^d)}^4$, we get

$$\begin{aligned} & |\mathcal{N}_\varepsilon(\mathbf{n}, \mathbf{n}', \mathbf{t}, \mathbf{t}', \boldsymbol{\xi}, \boldsymbol{\xi}')}| \\ & \leq \|\hat{\phi}_0\|_{L^\infty(\mathbb{R}^d)}^4 \sum_{\mathcal{P} \in \Pi} \int_{\mathbb{R}^{Kd/2}} \int_{S_{\mathcal{P}}} \left\{ \prod_{\ell=1}^{K/2} \frac{\hat{R}(w_\ell)}{(2\pi)^d} \right\} d\lambda d\mathbf{w} \\ & = \|\hat{\phi}_0\|_{L^\infty(\mathbb{R}^d)}^4 [R(0)]^{K/2} \sum_{\mathcal{P} \in \Pi} \mathfrak{m}(S_{\mathcal{P}}). \end{aligned}$$

Applying the Wick formula, we also conclude that

$$[R(0)]^{K/2} \sum_{\mathcal{P} \in \Pi} \mathfrak{m}(S_{\mathcal{P}}) = \mathbb{E} \left[\prod_{j=1}^{N_1} H_{n_j}(t_j) \prod_{j'=1}^{N_2} H_{n_{j'}}(t_{j'}) \right].$$

As a result (5.43) follows. To prove (5.44) we use the well-known formula, see e.g. [15, formula (1.1), p. 4]

$$\sum_{n=0}^{+\infty} H_n(t) = e^{\sqrt{R(0)}W_t - \frac{1}{2}R(0)t} =: \mathcal{E}_t,$$

where the convergence holds both a.s. and in the L^p -sense for any $p \in [1, +\infty)$. Thus,

$$\begin{aligned} & \sum_{\mathbf{n}, \mathbf{n}'} \mathbb{E} \left[\prod_{j=1}^{N_1} H_{n_j}(t_j) \prod_{j'=1}^{N_2} H_{n_{j'}}(t_{j'}) \right] = \mathbb{E} \left[\prod_{j=1}^{N_1} \mathcal{E}_{t_j} \prod_{j'=1}^{N_2} \mathcal{E}_{t_{j'}} \right] \\ & \leq \exp \left\{ \frac{(N_1 + N_2)(N_1 + N_2 - 1)}{2} R(0) \max_{j, j'}(t_j, t_{j'}) \right\} = \exp \left\{ 6R(0) \max_{j, j'}(t_j, t_{j'}) \right\}, \end{aligned}$$

which completes the proof. \square

5.4.3. *Proof of Lemma 5.10.* Fix $t > 0, j, k = 1, \dots, N$. Recall that

$$Q_{\varepsilon, j, k}(t) = \int_0^t \int_{\mathbb{R}^d} \frac{\hat{R}(p)}{(2\pi)^d} e^{i(\eta_j - \eta_k)ps} \hat{\psi}_\varepsilon(s, \xi + \varepsilon^2 \eta_j - p) \hat{\psi}_\varepsilon^*(s, \xi + \varepsilon^2 \eta_k - p) dp ds.$$

We write its second moment as

$$(5.45) \quad \mathbb{E}[|Q_{\varepsilon, j, k}(t)|^2] = \int_{[0, t]^2} \int_{\mathbb{R}^{2d}} \frac{\hat{R}(p)\hat{R}(p')}{(2\pi)^{2d}} e^{i(\eta_j - \eta_k) \cdot (ps - p's')} \mathbf{M}_{4, \varepsilon}(s, s', p, p') dp dp' ds ds',$$

with

$$\begin{aligned} \mathbf{M}_{4, \varepsilon}(s, s', p, p') &= \mathbb{E}[\hat{\psi}_\varepsilon(s, \xi + \varepsilon^2 \eta_j - p) \hat{\psi}_\varepsilon^*(s, \xi + \varepsilon^2 \eta_k - p) \\ & \quad \times \hat{\psi}_\varepsilon^*(s', \xi + \varepsilon^2 \eta_j - p') \hat{\psi}_\varepsilon(s', \xi + \varepsilon^2 \eta_k - p')]. \end{aligned}$$

By Corollary 5.4, we know that

$$|\mathbf{M}_{4, \varepsilon}(s, s', p, p')| \leq C(t),$$

therefore, to study the limit of $\mathbb{E}[|Q_{\varepsilon, j, k}(t)|^2]$, by the dominated convergence theorem, we only need to analyze the limit of $\mathbf{M}_{4, \varepsilon}(s, s', p, p')$ as $\varepsilon \rightarrow 0$, for a.e. s, s', p, p' in the respective Lebesgue measure. Define

$$\mathbf{M}_{2, \varepsilon}(s, p) = \mathbb{E}[\hat{\psi}_\varepsilon(s, \xi + \varepsilon^2 \eta_j - p) \hat{\psi}_\varepsilon^*(s, \xi + \varepsilon^2 \eta_k - p)].$$

Using (5.29) with $\hat{\phi}_0(\xi - p)$ in place of $\hat{\phi}_0(\xi)$, we conclude that

$$\lim_{\varepsilon \rightarrow 0} \mathbf{M}_{2,\varepsilon}(s, p) = |\hat{\phi}_0(\xi - p)|^2 + \sum_{n=1}^{+\infty} \int_{[0,s]^2} \int_{\mathbb{R}^{nd}} \prod_{\ell=1}^n \frac{\hat{R}(p_\ell) e^{i(\eta_j - \eta_k) \cdot p_\ell s_\ell}}{(2\pi)^d} \times |\hat{\phi}_0(\xi - p - p_1 - \dots - p_n)|^2 d\mathbf{p}_{1,n} ds_{1,n}$$

for each s, p . The conclusion of Lemma 5.10 then is a consequence of Lemma 5.9 and the following.

Lemma 5.12. *For any s, s', p, p' such that $p \neq p'$, we have*

$$(5.46) \quad \mathbf{M}_{4,\varepsilon}(s, s', p, p') - \mathbf{M}_{2,\varepsilon}(s, p) \mathbf{M}_{2,\varepsilon}^*(s', p') \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0.$$

Proof. We let

$$(5.47) \quad \begin{aligned} t_1 = t'_1 = s, \quad t_2 = t'_2 = s', \\ \xi_1 = \xi + \varepsilon^2 \eta_j - p, \quad \xi_2 = \xi + \varepsilon^2 \eta_k - p', \quad \xi'_1 = \xi + \varepsilon^2 \eta_k - p, \quad \xi'_2 = \xi + \varepsilon^2 \eta_j - p', \end{aligned}$$

in the diagram expansion (5.38) and (5.40). Thus,

$$(5.48) \quad \begin{aligned} \mathbf{M}_{4,\varepsilon}(s, s', p, p') &= \mathbb{E}[\hat{\psi}_\varepsilon(t_1, \xi_1) \hat{\psi}_\varepsilon(t_2, \xi_2) \hat{\psi}_\varepsilon^*(t'_1, \xi'_1) \hat{\psi}_\varepsilon^*(t'_2, \xi'_2)] \\ &= \sum_{\mathbf{n}, \mathbf{n}'} \mathcal{N}_\varepsilon(\mathbf{n}, \mathbf{n}', \mathbf{t}, \mathbf{t}', \boldsymbol{\xi}, \boldsymbol{\xi}'). \end{aligned}$$

Here $\mathcal{N}_\varepsilon(\mathbf{n}, \mathbf{n}', \mathbf{t}, \mathbf{t}', \boldsymbol{\xi}, \boldsymbol{\xi}')$ is given by (5.34):

$$\mathcal{N}_\varepsilon(\mathbf{n}, \mathbf{n}', \mathbf{t}, \mathbf{t}', \boldsymbol{\xi}, \boldsymbol{\xi}') = \mathbb{E} \left[\prod_{j=1}^{N_1} \hat{\psi}_{n_j, \varepsilon}(t_j, \xi_j) \prod_{j'=1}^{N_2} \hat{\psi}_{n_{j'}, \varepsilon}^*(t_{j'}, \xi_{j}') \right].$$

with $N_1 = N_2 = 2$ and $\mathbf{t}, \mathbf{t}', \boldsymbol{\xi}, \boldsymbol{\xi}'$ determined from (5.47).

By Lemma 5.11, while computing the limit of $\mathbf{M}_{4,\varepsilon}(s, s', p, p')$, as $\varepsilon \rightarrow 0$, we can enter with the limit under the series in the right hand side of (5.48). So the question reduces to the computation of the limits of each $\mathcal{N}_\varepsilon(\mathbf{n}, \mathbf{n}', \mathbf{t}, \mathbf{t}', \boldsymbol{\xi}, \boldsymbol{\xi}')$.

We have

$$\mathbf{M}_{2,\varepsilon}(s, p) \mathbf{M}_{2,\varepsilon}^*(s', p') = \prod_{j=1}^2 \sum_{n=0}^{+\infty} \mathbb{E}[\hat{\psi}_{n,\varepsilon}(t_j, \xi_j) \hat{\psi}_{n,\varepsilon}^*(t'_j, \xi'_j)],$$

thus, to complete the proof of (5.46), it suffices to show that

$$(5.49) \quad \lim_{\varepsilon \rightarrow 0} \left\{ \mathcal{N}_\varepsilon(\mathbf{n}, \mathbf{n}', \mathbf{t}, \mathbf{t}', \boldsymbol{\xi}, \boldsymbol{\xi}') - \delta_{n_1, n'_1} \delta_{n_2, n'_2} \prod_{j=1}^2 \mathbb{E}[\hat{\psi}_{n_j, \varepsilon}(t_j, \xi_j) \hat{\psi}_{n_j, \varepsilon}^*(t'_j, \xi'_j)] \right\} = 0,$$

where $\delta_{n,m}$ is the Kronecker symbol.

Consider the diagram expansion (5.40) for $\mathcal{N}_\varepsilon(\mathbf{n}, \mathbf{n}', \mathbf{t}, \mathbf{t}', \boldsymbol{\xi}, \boldsymbol{\xi}')$, which we write as a *finite* sum:

$$\mathcal{N}_\varepsilon(\mathbf{n}, \mathbf{n}', \mathbf{t}, \mathbf{t}', \boldsymbol{\xi}, \boldsymbol{\xi}') = \sum_{\mathcal{P}} \sum_{\sigma} \mathcal{I}_\varepsilon(\mathcal{P}, \sigma).$$

Among all the ordered pairings, we distinguish one special called the *ladder pairing*, defined in the case of $n_j = n'_j$ for $j = 1, 2$, as follows

$$\begin{aligned} \mathcal{P}_{\text{lad}} := & \left\{ ((s_{1,1}, p_{1,1}), (s'_{1,1}, p'_{1,1})), \dots, ((s_{1,n_1}, p_{1,n_1}), (s'_{1,n_1}, p'_{1,n_1})), \right. \\ & \left. ((s_{2,1}, p_{2,1}), (s'_{2,1}, p'_{2,1})), \dots, ((s_{2,n_2}, p_{2,n_2}), (s'_{2,n_2}, p'_{2,n_2})) \right\}. \end{aligned}$$

Using the diagram expansion to represent $\mathbb{E}[\hat{\psi}_{n_j, \varepsilon}(t_j, \xi_j) \hat{\psi}_{n_j, \varepsilon}^*(t'_j, \xi'_j)]$, $j = 1, 2$ we conclude that, cf (5.40),

$$\sum_{\sigma} \mathcal{I}_{\varepsilon}(\mathcal{P}_{\text{lad}}, \sigma) = \prod_{j=1}^2 \mathbb{E}[\hat{\psi}_{n_j, \varepsilon}(t_j, \xi_j) \hat{\psi}_{n_j, \varepsilon}^*(t'_j, \xi'_j)].$$

Next we show that

$$(5.50) \quad \lim_{\varepsilon \rightarrow 0} \mathcal{I}_{\varepsilon}(\mathcal{P}, \sigma) = 0, \quad \text{if } \mathcal{P} \neq \mathcal{P}_{\text{lad}}.$$

This would end the proof of (5.49), finishing in this way the proof of Lemma 5.12.

Proof of (5.50). Since in our argument only the time components of the paired elements (λ, w) (see the definition of a pairing) play a role, to simplify the notation, when speaking about \mathcal{P} we shall refer to the pairing between the λ (temporal) components only. Let us consider a pairing

$$(5.51) \quad \mathcal{P} \neq \mathcal{P}_{\text{lad}}$$

and suppose that

$$(5.52) \quad \lim_{\varepsilon \rightarrow 0} \mathcal{I}_{\varepsilon}(\mathcal{P}, \sigma) \neq 0.$$

All the paired λ -s come from the set of variables

$$\{s_{1,1}, \dots, s_{1,n_1}, s_{2,1}, \dots, s_{2,n_2}, s'_{1,1}, \dots, s'_{1,n'_1}, s'_{2,1}, \dots, s'_{2,n'_2}\},$$

and the partition is $\mathcal{P} = \{(\lambda_1, \lambda'_1), \dots, (\lambda_{K/2}, \lambda'_{K/2})\}$. Recall that $\{\lambda_1, \dots, \lambda_{K/2}\}$ are ordered according to the pairing ordering, and $\{w_1, \dots, w_{K/2}\}$ are the corresponding momentum variables. For any permutation σ , we defined

$$S_{\mathcal{P}, \sigma} = S_{\mathcal{P}} \cap \{\lambda_{\sigma(1)} > \lambda_{\sigma(2)} > \dots > \lambda_{\sigma(\frac{K}{2})}\},$$

and

$$\mathcal{I}_{\varepsilon}(\mathcal{P}, \sigma) = \int_{\mathbb{R}^{Kd/2}} \int_{S_{\mathcal{P}, \sigma}} \left\{ \prod_{\ell=1}^{K/2} \frac{\hat{R}(w_{\ell})}{(2\pi)^d} \right\} e^{i\varepsilon^{-2}\Theta_{\mathcal{P}}} \Phi_{\mathcal{P}} d\lambda d\mathbf{w}.$$

As we will integrate $\lambda_{\sigma(1)}, \lambda_{\sigma(2)}, \dots$ in order, to ease the notation, we perform a change of variable so that after the change we have $\lambda_1 > \dots > \lambda_{K/2}$. More precisely, we change

$$(5.53) \quad \lambda_{\ell}, w_{\ell} \mapsto \lambda_{\sigma^{-1}(\ell)}, w_{\sigma^{-1}(\ell)}, \quad \ell = 1, \dots, \frac{K}{2},$$

and let

$$\tilde{S}_{\mathcal{P}, \sigma} := \{(\lambda_1, \dots, \lambda_{K/2}) : (\lambda_{\sigma^{-1}(1)}, \dots, \lambda_{\sigma^{-1}(K/2)}) \in S_{\mathcal{P}, \sigma}\}.$$

It is clear that $\lambda_1 > \dots > \lambda_{K/2}$ for any $(\lambda_1, \dots, \lambda_{K/2}) \in \tilde{S}_{\mathcal{P}, \sigma}$. Now we can write $\mathcal{I}_{\varepsilon}(\mathcal{P}, \sigma)$ as

$$(5.54) \quad \mathcal{I}_{\varepsilon}(\mathcal{P}, \sigma) = \int_{\mathbb{R}^{Kd/2}} \int_{\tilde{S}_{\mathcal{P}, \sigma}} \left\{ \prod_{\ell=1}^{K/2} \frac{\hat{R}(w_{\ell})}{(2\pi)^d} \right\} e^{i\varepsilon^{-2}\Theta_{\mathcal{P}}} \Phi_{\mathcal{P}} d\lambda d\mathbf{w},$$

where we have changed variables in $\Theta_{\mathcal{P}}, \Phi_{\mathcal{P}}$ according to (5.53).

Let us first present a rough sketch of the proof. In the expression (5.54), the phase factor $\Theta_{\mathcal{P}}$ is a linear combination of $\lambda_1, \dots, \lambda_{K/2}$. As we will see later, the fact that $\mathcal{P} \neq \mathcal{P}_{\text{lad}}$ induces a nonzero order $O(1)$ coefficient associated with some λ_{ℓ} , so we write $\Theta_{\mathcal{P}} = \theta_{\mathcal{P}} \lambda_{\ell} + \tilde{\Theta}_{\mathcal{P}}$ for some $\theta_{\mathcal{P}} \neq 0$. Using the elementary fact that

$$\sup_{a, b \in [0, T]} \int_a^b e^{i\varepsilon^{-2}\theta_{\mathcal{P}}\lambda_{\ell}} d\lambda_{\ell} \rightarrow 0, \quad \text{for any } T > 0, \text{ provided that } \theta_{\mathcal{P}} \neq 0,$$

we derive that

$$\int_{\tilde{S}_{\mathcal{P},\sigma}} e^{i\varepsilon^{-2}\Theta_{\mathcal{P}}} d\lambda = \int_{\tilde{S}_{\mathcal{P},\sigma}} e^{i\varepsilon^{-2}\theta_{\mathcal{P}}\lambda_{\ell}} e^{i\varepsilon^{-2}\tilde{\Theta}_{\mathcal{P}}} d\lambda \rightarrow 0.$$

Since the above integral is uniformly bounded by $\mathfrak{m}(S_{\mathcal{P},\sigma})$, an application of the dominated convergence theorem proves $\mathcal{I}_{\varepsilon}(\mathcal{P},\sigma) \rightarrow 0$.

Now let us enter the details of the discussion. Obviously, $\lambda_1 \in \{s_{1,1}, s_{2,1}, s'_{1,1}, s'_{2,1}\}$. Suppose that $\lambda_1 = s_{1,1}$ (so $w_1 = p_{1,1}$), we claim that (5.50) holds for $\lambda'_1 \neq s'_{1,1}$. Indeed, if the latter holds, then either $\mathfrak{m}(S_{\mathcal{P},\sigma}) = 0$, or we have $\lambda'_1 \in \{s_{2,1}, s'_{2,1}\}$. Assume that $\lambda'_1 = s'_{2,1}$. As we shall see from the argument below, the case $\lambda'_1 = s_{2,1}$ can be treated similarly.

Since $s_{1,1}$ is paired with $s'_{2,1}$, we have $p_{1,1}$ paired with $p'_{2,1}$, so the associated phase factor $\Theta_{\mathcal{P}}$ equals

$$\begin{aligned} \Theta_{\mathcal{P}} &= \frac{s_{1,1}}{2} (|\xi_1|^2 - |\xi_1 - p_{1,1}|^2) - \frac{s'_{2,1}}{2} (|\xi'_2|^2 - |\xi'_2 - p'_{2,1}|^2) + \tilde{\Theta}_{\mathcal{P}} \\ &= \frac{s_{1,1}}{2} \left[(|\xi_1|^2 - |\xi_1 - p_{1,1}|^2) - (|\xi'_2|^2 - |\xi'_2 - p_{1,1}|^2) \right] + \tilde{\Theta}_{\mathcal{P}} \\ &= (\xi_1 - \xi'_2) \cdot p_{1,1} s_{1,1} + \tilde{\Theta}_{\mathcal{P}}, \end{aligned}$$

where $\tilde{\Theta}_{\mathcal{P}}$ involves the temporal variables $\lambda_2, \dots, \lambda_{K/2}$ and the momentum variables $w_1, \dots, w_{K/2}$ that we do not track. By the definition of ξ_1, ξ'_2 in (5.47), we have $\xi_1 - \xi'_2 = p' - p$, which yields

$$(5.55) \quad \mathcal{I}_{\varepsilon}(\mathcal{P},\sigma) = \int_{\mathbb{R}^{Kd/2}} \int_{S_{\mathcal{P},\sigma}} \left\{ \prod_{\ell=1}^{K/2} \frac{\hat{R}(w_{\ell})}{(2\pi)^d} \right\} \exp \{ i\varepsilon^{-2}(p' - p) \cdot w_1 \lambda_1 \} \\ \times e^{i\varepsilon^{-2}\tilde{\Theta}_{\mathcal{P}}} \Phi_{\mathcal{P}} d\lambda d\mathbf{w}.$$

Let $\tilde{S}_{\mathcal{P},\sigma}^2 := \pi(\tilde{S}_{\mathcal{P},\sigma})$, where $\pi: \mathbb{R}^{K/2} \rightarrow \mathbb{R}^{K/2-1}$ is the coordinate projection:

$$\pi(\lambda_1, \lambda_2, \dots, \lambda_{K/2}) := (\lambda_2, \dots, \lambda_n), \quad (\lambda_1, \lambda_2, \dots, \lambda_{K/2}) \in \mathbb{R}^{K/2}.$$

Note that

$$\tilde{S}_{\mathcal{P},\sigma} = \{ (\lambda_1, \dots, \lambda_{K/2}) : (\lambda_2, \dots, \lambda_{K/2}) \in \tilde{S}_{\mathcal{P},\sigma}^2, \lambda_1 \in (\lambda_2, t_1 \wedge t'_2) \}.$$

Then, we can rewrite (5.55) as

$$(5.56) \quad \mathcal{I}_{\varepsilon}(\mathcal{P},\sigma) = \int_{\mathbb{R}^{Kd/2}} \Phi_{\mathcal{P}} \left\{ \prod_{\ell=1}^{K/2} \frac{\hat{R}(w_{\ell})}{(2\pi)^d} \right\} d\mathbf{w} \int_{S_{\mathcal{P},\sigma}^2} e^{i\varepsilon^{-2}\tilde{\Theta}_{\mathcal{P}}} d\lambda_2 \dots d\lambda_{K/2} \\ \times \int_{\lambda_2}^{t_1 \wedge t'_2} \exp \{ i\varepsilon^{-2}(p' - p) \cdot w_1 \lambda_1 \} d\lambda_1.$$

Note that both $\Phi_{\mathcal{P}}$ and the integral

$$\int_{S_{\mathcal{P},\sigma}^2} e^{i\varepsilon^{-2}\tilde{\Theta}_{\mathcal{P}}} d\lambda_2 \dots d\lambda_{K/2}$$

are bounded. Then we can argue that $\lim_{\varepsilon \rightarrow 0} \mathcal{I}_{\varepsilon}(\mathcal{P},\sigma) = 0$, by the dominated convergence theorem and the following fact

$$\lim_{\varepsilon \rightarrow 0} \sup_{a, b \in [0, t_1 \wedge t'_2]} \left| \int_a^b \exp \{ i\varepsilon^{-2}(p' - p) \cdot w_1 \lambda_1 \} d\lambda_1 \right| = 0$$

that holds for all w_1 such that $(p' - p) \cdot w_1 \neq 0$. This is where we rely on the assumption of $p' \neq p$. Hence, in order for (5.52) to be true we need to pair $s_{1,1}$ with $s'_{1,1}$.

Next, we note that $\lambda_2 \in \{s_{1,2}, s'_{1,2}, s_{2,1}, s'_{2,1}\}$. We argue, in exactly the same fashion as in the case of λ_1 that if $\lambda_2 = s_{i,j}$ and (5.52) is in force, then

$$(5.57) \quad s_{i,j} \text{ is paired with } s'_{i,j}.$$

The above argument holds also for other choices of λ_2 , which propagates down to all other λ_ℓ . Finally, we conclude that (5.52) forces condition (5.57) for all $i = 1, 2$ and $j = 1, \dots, n_i$. But the latter means that \mathcal{P} is the ladder diagram, which stands in a contradiction to our assumption (5.51). This ends the proof of (5.50), which in turn finishes the proof of Lemma 5.12. \square

5.4.4. *Convergence of $\mathcal{Q}_{\varepsilon,j,k}(t)$.* Recall that $\mathcal{Q}_{\varepsilon,j,k}(t)$ is given by (5.21). The limit in (5.27) is a consequence of the following lemma.

Lemma 5.13. *For any $t > 0$, $j, k = 1, \dots, N$, we have*

$$(5.58) \quad \lim_{\varepsilon \rightarrow 0} \mathbb{E}[|\mathcal{Q}_{\varepsilon,j,k}(t)|^2] = 0.$$

Proof. Recall that

$$\begin{aligned} \mathcal{Q}_{\varepsilon,j,k}(t) &= (\mathcal{M}_\varepsilon(\cdot, \eta_j), \mathcal{M}_\varepsilon(\cdot, \eta_k))(t) \\ &= - \int_0^t \int_{\mathbb{R}^d} \frac{\hat{R}(p)}{(2\pi)^d} \exp\left\{i\left(\eta_j - \eta_k - \frac{p}{\varepsilon^2}\right) \cdot ps\right\} g_\varepsilon^{j,k}(s, \xi, p) dp ds, \end{aligned}$$

with $g_\varepsilon^{j,k}$ defined in (5.23). We have

$$(5.59) \quad \mathbb{E}[|\mathcal{Q}_{\varepsilon,j,k}(t)|^2] = \int_{[0,t]^2} \int_{\mathbb{R}^{2d}} \frac{\hat{R}(p)\hat{R}(p')}{(2\pi)^{2d}} e^{i\varepsilon^{-2}\theta_\varepsilon(s, s', p, p')} \tilde{\mathbf{M}}_{4,\varepsilon}(s, s', p, p') dp dp' ds ds',$$

with

$$\theta_\varepsilon(s, s', p, p') = \varepsilon^2(\eta_j - \eta_k) \cdot (ps - p's') - (|p|^2 s - |p'|^2 s')$$

and

$$\begin{aligned} \tilde{\mathbf{M}}_{4,\varepsilon}(s, s', p, p') &= \mathbb{E}[\hat{\psi}_\varepsilon(s, \xi + \varepsilon^2\eta_j - p)\hat{\psi}_\varepsilon(s, \xi + \varepsilon^2\eta_k + p) \\ &\quad \times \hat{\psi}_\varepsilon^*(s', \xi + \varepsilon^2\eta_j - p')\hat{\psi}_\varepsilon^*(s', \xi + \varepsilon^2\eta_k + p')]. \end{aligned}$$

By Corollary 5.4, we have

$$\sup_{s, s' \in [0,t], p, p' \in \mathbb{R}^d} |\tilde{\mathbf{M}}_{4,\varepsilon}(s, s', p, p')| \leq C(t).$$

The integral in $ds ds'$ involves a large phase factor $\varepsilon^{-2}\theta_\varepsilon$, which should be contrasted with our calculation in case of $Q_{\varepsilon,j,k}(t)$, see (5.20), where this situation happened only for the non-ladder pairings. The presence of such a factor explains why the expression in the left hand side of (5.59) vanishes, as $\varepsilon \rightarrow 0$. To prove this fact rigorously, we write $\tilde{\mathbf{M}}_{4,\varepsilon}(s, s', p, p')$ in terms of the diagram expansion and proceed, as in Section 5.4.2, integrating first the largest temporal variables in s, s' , analogous to what has been done in (5.56). As the argument is very similar, we do not provide all details but only the sketch.

First, we have

$$\tilde{\mathbf{M}}_{4,\varepsilon}(s, s', p, p') = \sum_{\mathbf{n}, \mathbf{n}'} \tilde{\mathcal{N}}_\varepsilon(\mathbf{n}, \mathbf{n}', \mathbf{t}, \mathbf{t}', \boldsymbol{\xi}, \boldsymbol{\xi}'),$$

where, cf (5.40), $\mathbf{t}, \mathbf{t}', \boldsymbol{\xi}, \boldsymbol{\xi}'$ are given by

$$(5.60) \quad \begin{aligned} t_1 = t_2 = s, \quad t'_1 = t'_2 = s', \\ \xi_1 = \xi + \varepsilon^2\eta_j - p, \quad \xi_2 = \xi + \varepsilon^2\eta_k + p, \quad \xi'_1 = \xi + \varepsilon^2\eta_j - p', \quad \xi'_2 = \xi + \varepsilon^2\eta_k + p', \end{aligned}$$

and

$$(5.61) \quad \tilde{\mathcal{N}}_\varepsilon(\mathbf{n}, \mathbf{n}', \mathbf{t}, \mathbf{t}', \boldsymbol{\xi}, \boldsymbol{\xi}') := \sum_{\mathcal{P}} \sum_{\sigma} \int_{\mathbb{R}^{Kd/2}} \int_{S_{\mathcal{P}, \sigma}} \left\{ \prod_{\ell=1}^{K/2} \frac{\hat{R}(w_\ell)}{(2\pi)^d} \right\} e^{i\varepsilon^{-2}\hat{\Theta}_{\mathcal{P}}} \hat{\Phi}_{\mathcal{P}} d\boldsymbol{\lambda} d\mathbf{w},$$

where $\hat{\Theta}_{\mathcal{P}}$ is some, appropriately defined phase factor, involving only the variables $\boldsymbol{\lambda}$ and \mathbf{w} , and $\hat{\Phi}_{\mathcal{P}}$ is the expression corresponding to the product of the initial data. Here we use the same notation for variables, pairing \mathcal{P} , permutation σ and the domain of integration as in Section 5.4.2. We can write then

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E}[|\mathcal{Q}_{\varepsilon, j, k}(t)|^2] = \sum_{\mathbf{n}, \mathbf{n}'} \sum_{\mathcal{P}, \sigma} \lim_{\varepsilon \rightarrow 0} \mathcal{J}_\varepsilon(\mathcal{P}, \sigma),$$

where

$$\begin{aligned} \mathcal{J}_\varepsilon(\mathcal{P}, \sigma) := & \int_{\mathcal{T}_{\mathcal{P}, \sigma}} ds ds' d\boldsymbol{\lambda} \int_{\mathbb{R}^{2d+Kd/2}} d\mathbf{w} dp dp' \\ & \times e^{i\varepsilon^{-2}\theta_\varepsilon(s, s', p, p')} \frac{\hat{R}(p)\hat{R}(p')}{(2\pi)^{2d}} \prod_{\ell=1}^{K/2} \frac{\hat{R}(w_\ell)}{(2\pi)^d} e^{i\varepsilon^{-2}\hat{\Theta}_{\mathcal{P}}} \hat{\Phi}_{\mathcal{P}} \end{aligned}$$

and

$$\mathcal{T}_{\mathcal{P}, \sigma} := \{(s, s', \boldsymbol{\lambda}) : (s, s') \in [0, t]^2, \boldsymbol{\lambda} \in S_{\mathcal{P}, \sigma}\}.$$

We emphasize that $S_{\mathcal{P}, \sigma}$ in fact depends on s and s' , through the dependence of $\tilde{\mathcal{N}}_\varepsilon$ on \mathbf{t}, \mathbf{t}' . Without loss of generality, consider the region of $s > s'$. Given the partition \mathcal{P} and the permutation σ , the largest $\boldsymbol{\lambda}$ variable is $\lambda_{\sigma(1)}$, thus for fixed s' and $\boldsymbol{\lambda}$, the domain of integration for s is $[\lambda_{\sigma(1)} \vee s', t]$. Using the fact that

$$\sup_{s', \lambda_{\sigma(1)} \in [0, t]} \left| \int_{\lambda_{\sigma(1)} \vee s'}^t \exp\{i(\eta_j - \eta_k) \cdot ps - \varepsilon^{-2}|p|^2 s\} ds \right| \rightarrow 0, \quad \text{for } p \neq 0,$$

and applying dominated convergence theorem, we conclude the proof of $\mathcal{J}_\varepsilon(\mathcal{P}, \sigma) \rightarrow 0$ as $\varepsilon \rightarrow 0$. \square

6. PROOFS OF THE RESULTS FROM SECTION 3.1

We show only how to prove Proposition 3.1. The other results from Section 3.1 can be argued similarly.

6.1. Proof of (3.8). Thanks to the moment estimate proved in Corollary 5.4 we conclude that

$$(6.1) \quad \sup_{t \in [0, T], \boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^d, \varepsilon \in (0, 1]} \mathbb{E}[|X_\xi^\varepsilon(t, \boldsymbol{\eta})|^{2n}] < +\infty,$$

for any integer $n \geq 0$ and $T > 0$. Therefore, it suffices only to show (3.8) for $J(x, \boldsymbol{\xi})$ whose Fourier transform in the x -variable - $\hat{J}(\boldsymbol{\eta}, \boldsymbol{\xi})$ - is compactly supported.

Fix $\boldsymbol{\xi} \in \mathbb{R}^d$ and $t \geq 0$. Suppose also that $\varepsilon_n \rightarrow 0+$. Thanks to the Skorokhod embedding theorem, see e.g. Theorem I.6.7 of [4] and Theorem 2.2, we can assume that there exist a sequence of the fields $(\tilde{X}_\xi^{\varepsilon_n}(t, \cdot))_{n \geq 1}$ and a field $\tilde{X}_\xi(t, \cdot)$ such that

- i) the law of $\tilde{X}_\xi^{\varepsilon_n}(t, \cdot)$ coincides with that of $X_\xi^{\varepsilon_n}(t, \cdot)$ for each $n \geq 1$. Likewise the laws of $\tilde{X}_\xi(t, \cdot)$ and that of $X_\xi(t, \cdot)$ are equal,
- ii) $\tilde{X}_\xi^{\varepsilon_n}(t, \cdot)$ converge to $\tilde{X}_\xi(t, \cdot)$, as $n \rightarrow +\infty$, uniformly on compact subsets of \mathbb{R}^d , for a.s. realization of the fields.

In consequence, the law of $\int_{\mathbb{R}^d} \mathcal{X}_\varepsilon(t, x, \xi) J^*(x, \xi) dx$ coincides with that of

$$(6.2) \quad \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \tilde{X}_\xi^{\varepsilon_n}(t, \eta) \hat{J}^*(\eta, \xi) d\eta.$$

It follows from ii) that the expressions in (6.3) converge a.s. (thus also in law), as $n \rightarrow +\infty$, to

$$(6.3) \quad \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \tilde{X}_\xi(t, \eta) \hat{J}^*(\eta, \xi) d\eta.$$

Thus, (3.8) follows.

6.2. Proof of (3.9). Thanks to Theorem 2.2 and estimate (6.1) we infer that

$$(6.4) \quad \lim_{\varepsilon \rightarrow 0} \frac{1}{(2\pi)^d} \mathbb{E} \left[\int_{\mathbb{R}^{2d}} X_\xi^\varepsilon(t, \eta) \hat{J}^*(\eta, \xi) d\eta d\xi \right] = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} \mathbb{E} X_\xi(t, \eta) \hat{J}^*(\eta, \xi) d\eta d\xi.$$

The expression in the right hand side equals to the right hand side of (3.9), by virtue of (2.7).

Using Remark 3.10 we can also easily conclude that

$$(6.5) \quad \lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[\int_{\mathbb{R}^{2d}} X_\xi^\varepsilon(t, \eta) \hat{J}^*(\eta, \xi) d\eta d\xi \right]^2 = \left[\int_{\mathbb{R}^{2d}} \mathbb{E} X_\xi(t, \eta) \hat{J}^*(\eta, \xi) d\eta d\xi \right]^2,$$

thus (3.9) follows.

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