LONG-TIME BEHAVIOR FOR A NONLOCAL MODEL FROM DIRECTED POLYMERS

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Abstract. We consider the long time behavior of solutions to a nonlocal reaction diffusion equation that arises in the study of directed polymers. The model is characterized by convolution with a kernel R and an L^2 inner product. In one spatial dimension, we extend a previous result of the authors [arXiv:2002.02799], where only the case $R = \delta$ was considered; in particular, we show that solutions spread according to a 2/3 power law consistent with the KPZ scaling conjectured for direct polymers. In the special case when $R = \delta$, we find the exact profile of the solution in the rescaled coordinates. We also consider the behavior in higher dimensions. When the dimension is three or larger, we show that the long-time behavior is the same as the heat equation in the sense that the solution converges to a standard Gaussian. In contrast, when the dimension is two, we construct a non-Gaussian self-similar solution.

1. Introduction

In this paper, we investigate the following collection of models:

$$\begin{cases} \partial_t g = \frac{1}{2}\Delta g + g\left(\langle R * g, g\rangle - R * g\right) & \text{in } (0,\infty)\times\mathbb{R}^d, \\ g = g_0 & \text{on } \{0\}\times\mathbb{R}^d, \end{cases} \tag{1.1}$$

where $R * g$ refers to convolution in the spatial variables only and the brackets $\langle\cdot,\cdot\rangle$ denote the L^2 inner product in the spatial variables. We always assume that

$$\int g_0(x)\,dx = 1 \qquad \text{and} \qquad 0 \le g_0 \in C_c(\mathbb{R}^d), \tag{1.2}$$

which implies that, for all $t > 0$,

$$\int g(t,x)dx = 1; \tag{1.3}$$

thus, (1.1) describes the evolution of a probability density. Here, either R is the delta distribution δ or R is a continuous, nonnegative function such that $\int R\,dx = 1$. In the latter case, we assume that there is a continuous even function $\phi \ge 0$ such that

$$R(x) = \phi * \phi(x), \quad \text{and} \quad \int \phi(x)\,dx = 1. \tag{1.4}$$

The connection to directed polymers. The equation arises from our study of directed polymers in random environment, and we discuss the model below. For a Gaussian random field $\{V(t,x) : (t,x) \in \mathbb{R}^{d+1}\}$ and an independent Brownian motion $B = \{B_t : t \ge 0\}$, consider the Gibbs measure associated with the Hamiltonian $H(t,B) = \int_0^t V(s,B_s)ds$:

$$\mu_t(dx) = q(t,x)\,dx,$$
$$q(t,x) = Z_t^{-1}\mathbb{E}_B[\delta(B_t - x)\exp(H_t(B))], \qquad Z_t = \mathbb{E}_B[\exp(H_t(B))].$$

Here $\mathbb{E}_B$ is the expectation only with respect to the Brownian motion B, with the random Gaussian field fixed. Thus, $\mu_t(dx)$ can be viewed as the endpoint distribution of the Brownian motion "reweighted" by the random environment, through the factor $\exp(H_t(B))$, which is to model the polymer path in a heterogeneous environment. The properties of $q(t,\cdot)$, in particular how the x-variable scales with respect to time, is a notoriously difficult problem in probability and statistical physics. It is conjectured that, in $d=1$ and if V is sufficiently short-range correlated, we should have $\int |x|^p q(t,x)\,dx \approx t^{2p/3}$ for large t, i.e., the endpoint of the polymer path is superdiffusive with an exponent $2/3$, which falls into the KPZ universality class. So far the conjecture is only proved for a few specific models with certain integrable structures, and the proof is very much model-dependent. In $d\geq 2$, much less is conjectured and known, and even the correct superdiffusive exponent is unclear. We refer to the monograph [8] and the survey [10] for results and discussions in this direction.

Our interest is in the averaged density $Q_1(t,x) := \mathbb{E}[q(t,x)]$, where $\mathbb{E}$ is now taken with respect to the Gaussian field V, and the ultimate goal is to study the asymptotic behavior of Q_1 by a robust analytic approach that covers all possible correlation structures of V. For example, in $d=1$ one would like to prove a *universality* result saying that $\int |x|^p Q_1(t,x)\,dx \approx t^{2p/3}$ for $t \gg 1$. As Q_1 is the averaged density of the endpoint of a random path, it is tempting to try to derive a PDE for its evolution, similar to the Fokker-Planck equations associated with diffusion processes. This motivated the study in [14]. For a large class of Gaussian fields V, which has zero mean and is white in time and possibly colored in space, with the covariance function

$$\mathbb{E}[V(t,x)V(s,y)] = \delta(t-s)R(x-y)^{a},$$

we find that, instead of solving a single Fokker-Planck equation, what governs the evolution of Q_1 is a hierarchical system: define the n–point correlation function

$$Q_n(t,x_1,\ldots,x_n) := \mathbb{E}[\prod_{j=1}^n q(t,x_j)],$$

then Q_1 solves the equation

$$\begin{aligned}\partial_t Q_1(t,x) = \tfrac{1}{2}\Delta Q_1(t,x) &- \int Q_2(t,x,y)R(x-y)\,dy \\ &+ \int Q_3(t,x,y,z)R(y-z)\,dydz.\end{aligned} \tag{1.5}$$

In fact, for any $n\geq 1$, the equation of Q_n contains Q_{n+1} and Q_{n+2}. The nonlocal terms in (1.5) describe the mutual intersection of multiple polymer paths as they wander in the random environment to maximize the collected energy, and the kernel R corresponds to how the paths' intersection is measured. The hierarchical PDE system is similar to the BBGKY hierarchy in kinetic theory. Inspired by the molecular chaos assumption there, we assume that in large time Q_n can be approximately factorized: $Q_n(t,x_1,\ldots,x_n) \approx \prod_{j=1}^n Q_1(t,x_j)$, and this helps to reduce (1.5) to (1.1). Therefore, the equation we study in this paper, can be viewed as an approximation of the hierarchical system which describes the actual evolution of the polymer endpoint density. While it is unclear at the moment how to justify the factorization assumption used to link the true evolution (1.5) with the "approximate" evolution (1.1), the results in [14] already show an intriguing connection, which we discuss in greater detail below. Despite this simplification, we expect (1.1) to retain several key features of the original equation (1.5), and furthering our understanding of (1.1) may also help with the study of the hierarchy. This motivates the current study.

[a] Since R is the spatial covariance function, there exists a function ϕ so that (1.4) holds, and the construction can be found in [14, Page 2].

Rough description of the main results. In [14, Theorem 1.3], we examined the special case of (1.1) in which $R = \delta$ and $d = 1$, which corresponds to the situation when the Gaussian environment is also white in space. Here, we showed that the KPZ scaling, $x \sim t^{2/3}$ is exhibited in the sense that, $\|g(t)\|_\infty = \mathcal{O}(t^{-2/3})$ and, for any $p \geq 1$, the pth moment of g is bounded from above and below by $t^{2p/3}$, up to a constant, i.e.,

$$\int |x|^p g(t,x)dx \approx t^{\frac{2p}{3}}, \qquad \text{for } t \geq 1. \tag{1.6}$$

As a result, it follows that $g \approx t^{-2/3}$ for any $|x| \leq \mathcal{O}(t^{2/3})$ and $g \ll t^{-2/3}$ for any $|x| \gg t^{2/3}$

Our focus in this paper is in generalizing and refining the above result in one dimension and in investigating the behavior of g in higher dimensions. Roughly, we establish the following properties of (1.1).

(1) When R is continuous and $d = 1$, we show that the KPZ-scaling, $x \sim t^{2/3}$, conjectured for the full polymer model is exhibited by g. In particular, we prove that $\|g(t)\|_\infty = \mathcal{O}(t^{-2/3})$ and, for any $p \geq 0$, the pth moment of g is bounded above and below by $t^{2p/3}$, as in (1.6). This extends the results of [14, Theorem 1.3], and can be viewed as a universality result as the scaling exponent $2/3$ does not depend on the detailed expression of R. As discussed below, significant difficulties arise in generalizing the proof from the case $R = \delta$ to the present one.
(2) When $R = \delta$ and $d = 1$, we establish more precise estimates on the long-time behavior of g. Specifically, we identify the limit of $t^{2/3}g(t, xt^{2/3})$ as $t \to \infty$ to be, up to a multiplicative constant, the indicator function of an interval. In other words, if we let X_t be a random variable with the density $g(t,\cdot)$, with (1.6) we have $\mathbb{E}[|X_t|^p] \approx t^{2p/3}$, and here we further prove that $t^{-2/3}X_t$ converges to a uniform distribution, as $t \to \infty$.
(3) When $d \geq 2$, the behavior is quite different, as expected for the directed polymers. In all cases, the diffusive scaling $x \sim t^{1/2}$ and $g \sim t^{-d/2}$ holds. When $d \geq 3$, we show that g, under this scaling, converges to a standard Gaussian. This is exactly the behavior of the heat equation; in other words, the effect of the nonlinearity is negligible. The result is consistent with the diffusive behaviors of directed polymers in high temperatures in $d \geq 3$ [4, 16] [b]. On the other hand, when $d = 2$, there are solutions g of (1.1) that, in the same diffusive scaling, do not converge to a Gaussian. Thus, $d = 2$ is the critical dimension for (1.1), another feature of the polymer model.

1.1. **Main Results.** We now state our results more precisely.

1.1.1. *Decay consistent with the conjectured scaling for directed polymers.* Our first theorem is the extension of the results in [14, Theorem 1.3] on the growth of moments to the setting where R is continuous.

Theorem 1.1. *Let $d = 1$. Suppose that g solves (1.1) with g_0 satisfying (1.2) and with R satisfying (1.4) or $R = \delta$. Then, for any $p > 0$,*

$$\left(\int_{\mathbb{R}} |x|^p g(t,x)dx\right)^{1/p} \approx t^{\frac{2}{3}} \qquad \textit{for all } t \geq 1. \tag{1.7}$$

As a consequence, we have $\liminf_{t\to\infty} t^{2/3}\|g(t)\|_\infty, t^{2/3}\|g(t)\|_2^2 > 0$. The bounds here depend only on $\mathrm{supp}(g_0)$.

[b] The polymer model actually depends on a parameter β which is the inverse temperature, and it goes into our approximate model (1.1) only as a multiplicative constant of the nonlinear terms (see [14, Equation (1.10)]). As it does not play a role in our analysis, we do not specify it here.

As alluded to above, the proof of (1.7) when $R = \delta$ is exactly the content of [14, Theorem 1.3]; however, the lower bound on the rescaled L^∞ and L^2 norms of g are new. These are not difficult to prove after the other moment bounds have been established, but are stated here as these lower bounds are an essential ingredient in the establishing the precise behavior of g (see Theorem 1.3).

The main new content of Theorem 1.1 are the moment bounds in the case where R is continuous. The key step in proving this is the following L^∞ bound on g.

Proposition 1.2. *Under the assumptions of Theorem 1.1, for all $t \geq 0$,*

$$\|g(t)\|_\infty \lesssim \min\{t^{-1/2}, t^{-2/3}\}.$$

While the connection between Proposition 1.2 and Theorem 1.1 is analogous to that in the case $R = \delta$, the proof of Proposition 1.2 is significantly more difficult, which we discuss now.

The proof in the case $R = \delta$ contained in [14, Theorem 1.3] relies on several exact identities that are no longer available. In fact, the first step when $R = \delta$ is noticing that (1.1) yields

$$\frac{d}{dt}\|g(t)\|_\infty \leq \|g(t)\|_\infty(\langle R * g, g\rangle - R * g) = \|g(t)\|_\infty(\|g\|_2^2 - \|g(t)\|_\infty).$$

The inequality follows from the fact that Δg is nonpositive at a maximum, and the equality follows from using that $R = \delta$. By Hölder's inequality and the fact that $g(t)$ is a probability measure, the right hand side is clearly nonpositive. It is then a matter of quantifying this non-positivity (cf. [14, Lemma 4.3]). Unfortunately, when R is continuous, the above equality does not hold. In fact, it is not even clear if $\|g(t)\|_\infty$ is decreasing with respect to t.

To overcome this difficulty, we may try to work, instead, with the L^2-norm of $g(t, \cdot)$. When multiplying (1.1) by g, and integrating by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\|g(t)\|_2^2 + \frac{1}{2}\int |\nabla g|^2 dx = \|g\|_2^2\langle R * g, g\rangle - \int g^2 R * g \, dx. \tag{1.8}$$

When $R = \delta$, the right hand side has the form $\|g\|_2^4 - \|g\|_3^3$, which, using Hölder's inequality, one easily sees is nonpositive. However, when R is continuous, it is no longer clear that the right hand side is even nonpositive. Again, it may not be.

Given the convolution term $R * g$ appearing in (1.1), a better approach is to multiply the equation by $R * g$ and integrate by parts. Then (1.1) becomes

$$\frac{1}{2}\frac{d}{dt}\langle g, R * g\rangle + \frac{1}{2}\int \nabla g \cdot \nabla(R * g) dx = \langle R * g, g\rangle^2 - \int g(R * g)^2 dx.$$

The right hand side is once again nonpositive. The goal of the proof is to establish a lower bound on

$$\frac{1}{2}\int \nabla g \cdot \nabla(R * g) dx + \left[-\langle R * g, g\rangle^2 + \int g(R * g)^2 dx\right]. \tag{1.9}$$

After, we then must relate $\langle g, R * g\rangle$ back to $\|g\|_2$ and $\|g\|_\infty$.

The main tool in the analysis is the local-in-time Harnack inequality (see Proposition 2.5) that was first established in [5]. This inequality, which quantifies how spread out the level sets of $g(t, \cdot)$ are, allows us to show that, roughly, either g is flat, in which case the bracketed terms in (1.9) are large, or g is not flat and the gradient term in (1.9) is large.

Interestingly, while the functional inequalities used in [14] to bound (1.9) from below in the $R = \delta$ case held for *any* H^1 function, the lower bound on (1.9) in the case where R is continuous relies strongly on the regularizing effect of the heat equation, seen through the local-in-time Harnack inequality. In some sense we are showing that the key functional inequality [14, Lemma 4.3](see also [9, Lemma 2]) is stable with respect to convolutions *as long as* the function is suitably regular.

The proofs of Theorem 1.1 and Proposition 1.2 are contained in Section 2.

1.2. **Long time dynamics in one dimension.** Restricting to the case $R = \delta$, we investigate the behavior of g under the $t^{2/3}$ scaling. In particular, Theorem 1.1 suggests that we may see non-trivial limiting behavior of $t^{2/3}g(t, xt^{2/3})$ as $t \to \infty$. We establish that here.

In order to describe the long-time behavior, we define two important constants:

$$c_{\text{crit}} = \left(\frac{3}{2}\right)^{2/3} \approx 1.31037... \qquad \text{and} \qquad \theta_{\text{crit}} = \left(\frac{1}{18}\right)^{1/3} \approx .38157... \tag{1.10}$$

Notice that $2\theta_{\text{crit}}c_{\text{crit}} = 1$. We now state our main results.

Theorem 1.3. *Suppose that g solves (1.1) with initial data g_0 satisfying (1.2) and $R = \delta$. Suppose further that g_0 is even and radially decreasing. Then*

$$\lim_{t\to\infty} t^{2/3}\|g(t)\|_2^2 = \lim_{t\to\infty} t^{2/3}\|g(t)\|_\infty = \theta_{\text{crit}}.$$

Further, for all $x \neq \pm c_{\text{crit}}$,

$$\lim_{t\to\infty} t^{2/3}g(t, xt^{2/3}) = \theta_{\text{crit}}\mathbb{1}_{[-c_{\text{crit}}, c_{\text{crit}}]}(x),$$

where the limit holds uniformly for x away from $\pm c_{\text{crit}}$.

Informally, Theorem 1.3 implies that, for large t, we may write

$$g(t,x) \approx \frac{\theta_{\text{crit}}}{t^{2/3}}\mathbb{1}_{[-c_{\text{crit}}t^{2/3}, c_{\text{crit}}t^{2/3}]}(x) + o(t^{-2/3}).$$

It is clear that this is consistent with the results in Theorem 1.1.

We note that the convergence of the (rescaled) L^2- and L^∞-norms is *almost* equivalent to the convergence of the profile. Indeed, since g is a probability density, its L^2- and L^∞-norms can only be equal when the Hölder's inequality is an equality, which corresponds to functions that are a multiplicative constant of an indicator function. Using then the symmetry and monotonicity of g, it follows that g converges to the indicator function of an interval up to a multiplicative constant (though we note that this does not yield the exact constants c_{crit} and θ_{crit}). Of course, we never have exact equality and thus, the above heuristics hinge on understand the "stability" of Hölder's inequality.

We believe that the assumptions that g_0 is even and decreasing is purely technical and can be removed at the expense of a significantly more involved proof. As the proof is already quite involved even with these assumptions, we opt to use them. Throughout the proof of Theorem 1.3, we indicate where and how these assumptions are used.

A key step in the proof of Theorem 1.3 are the following bounds, which do not require these symmetry and monotonicity assumptions.

Proposition 1.4. *Suppose that g solves (1.1) with initial data g_0 satisfying (1.2). Then*

(i) $\liminf_{t\to\infty} t^{2/3}\|g(t)\|_2^2, \ \liminf_{t\to\infty} t^{2/3}\|g(t)\|_\infty \leq \theta_{\text{crit}}$;

(ii) $\theta_{\text{crit}} \leq \limsup_{t\to\infty} t^{2/3}\|g(t)\|_2^2, \ \limsup_{t\to\infty} t^{2/3}\|g(t)\|_\infty$.

The results in Proposition 1.4 show that $t^{2/3}\|g(t)\|_\infty$ and $t^{2/3}\|g(t)\|_2^2$ get arbitrarily close to $\theta_{\rm crit}$ infinitely often; however, they do not rule out the possibility that these quantities make non-trivial oscillations around $\theta_{\rm crit}$. We recall the discussion following Theorem 1.3 that indicates the importance of the L^2- and L^∞-norms in understanding the profile of g.

The intuition behind the constants $\theta_{\rm crit}$ and $c_{\rm crit}$ comes from various rescalings of the equation in which a non-local Fisher-KPP type equation arises. Finding and heuristically interpreting the correct rescalings is a subtle issue and so is covered in detail at the outset of Section 3. A discussion of the strategy of the proof and the major difficulties encountered is also contained there as it is best placed in the setting of the rescaled equations.

We note that a major difficulty in extending Theorem 1.3 to the case when $R \neq \delta$ is that the comparison principle no longer holds for (1.1) when $R \neq \delta$.

The proofs of Proposition 1.4 and Theorem 1.3 are contained in Section 3 and Section 4 respectively.

1.3. **Long time dynamics in higher dimensions.** We now discuss the behavior in higher dimensions. Our first result is about the time decay of the moments of g, as well as the L^2- and L^∞-norms of g.

Theorem 1.5. *Suppose $d \geq 2$, $R = \delta$ or R satisfies (1.4), and g solves (1.1) with initial data g_0 satisfying (1.2). Then, for any $p > 0$*

$$\left(\int_{\mathbb{R}} |x|^p g(t,x)dx\right)^{1/p} \approx t^{\frac{1}{2}} \qquad \text{for all } t \geq 1.$$

In addition, we have

$$\|g(t)\|_\infty, \|g(t)\|_2^2 \approx t^{-\frac{d}{2}} \qquad \text{for all } t \geq 1.$$

The proof of Theorem 1.5 uses classical techniques based on the Nash inequality, which, in its original form relates the L^1, L^2, and $\dot{H}^1$ norms of a function, in order to bound $\langle g, R * g\rangle$. In fact, we slightly extend the Nash inequality to apply to the quantities $\langle g, R * g\rangle$ and $\langle \nabla g, R * \nabla g\rangle$ in place of the L^2 and $\dot{H}^1$ norms, though the proof is analogous to the usual one. To understand why these convolved macroscopic quantities are more useful than the L^2 and $\dot{H}^1$ norms, we refer to the discussion around (1.8) and (1.9).

To bootstrap the bound on $\langle g, R * g\rangle$ to one on the moments of g, we construct a barrier function $\overline{g}$ such that, if g and $\overline{g}$ were to touch, at the touching point

$$\partial_t \overline{g} - \frac{1}{2}\Delta \overline{g} - \overline{g}\left(\langle g, R * g\rangle - R * g\right) > 0,$$

which rules out any touching points.

We now investigate the self-similar behavior of g for large times. We show that, when $d \geq 3$, the nonlinear terms are asymptotically negligible and g has the same Gaussian behavior as the usual heat equation. On the other hand, when $d = 2$, we show that this need not be true by constructing solutions of (1.1) that do not have a Gaussian profile in rescaled variables.

Theorem 1.6. *Suppose that $R = \delta$.*

(i) *If $d \geq 3$, g solves (1.1) with g_0 satisfying $0 \leq g_0(x) \leq A e^{-|x|^2/B}$ for some $A, B > 0$ and all $x \in \mathbb{R}^d$ and $\int g_0\, dx = 1$, then*

$$\limsup_{t\to\infty} t^{r_d} \left\| t^{d/2} g(t, x\sqrt{t}) - \frac{1}{(2\pi)^{d/2}} e^{-\frac{|x|^2}{2}} \right\|_\infty < \infty$$

where $r_d = 1$ if $d > 3$ and r_d is any element of $(0,1)$ if $d = 3$.

(ii) If $d = 2$, there exists $0 \le G \in C^\infty(\mathbb{R}^2)$ such that, letting

$$g(t,x) = \frac{1}{t} G\left(\frac{x}{\sqrt{t}}\right),$$

the following holds: G is not a Gaussian (that is, $G(x) \neq e^{-|x|^2/2\sigma^2}/(2\pi\sigma^2)$ for any $\sigma > 0$), g solves (1.1), there exist $A, B > 0$ such that $G(x) \le Ae^{-|x|^2/B}$ for all $x \in \mathbb{R}^2$, and

$$\int g(t,x)\,dx = \int G(x)\,dx = 1 \qquad \text{for all } t > 0.$$

The first step in proving Theorem 1.6 is to convert to the self-similar coordinates suggested by its statement. Letting $\tilde{g}$ be g in these new coordinates $(\tau, y) = (\log t, \frac{x}{\sqrt{t}})$, and $\tilde{g}(\tau, y) = e^{\frac{d}{2}\tau} g\left(e^\tau, e^{\tau/2} y\right)$, we see that

$$\partial_\tau \tilde{g} = \left[\frac{1}{2}\Delta\tilde{g} + \frac{y}{2}\cdot\nabla\tilde{g} + \frac{d}{2}\tilde{g}\right] + e^{-\frac{d-2}{2}\tau}\tilde{g}\left(\|\tilde{g}\|_2^2 - \tilde{g}\right). \tag{1.11}$$

The difference between $d = 2$ and $d \ge 3$ is clear from the above equation. When $d \ge 3$, the last term is exponentially decaying and we proceed by analyzing the spectrum of the operator in brackets, which is well-understood.

When $d = 2$, the last term is non-negligible. The construction then proceeds by finding a steady solution of (1.11). To begin, we pose the problem on a ball of radius r and examine the local (and slightly less nonlinear) problem where $\|\tilde{g}\|_2^2$ is replaced by a constant E. After finding a solution $\tilde{g}_E$ to this problem, we show that there is a critical value of E where $\|\tilde{g}_E\|_2^2$ is equal to E. A difficulty with this is that the dependence of $\tilde{g}$ on E is monotonic; that is, the larger E is, the larger $\tilde{g}_E$ is. Hence, it is difficult to simply look at small and large E and show that the ordering of E and $\|\tilde{g}_E\|_2^2$ switches. We overcome this by showing that the operator in brackets in (1.11) induces sufficient decay away from $x = 0$ to limit the growth of $\|\tilde{g}_E\|_2^2$ as E is increased. After finding this critical E value, the proof is concluded by taking $r \to \infty$.

We make two comments on the limitations of Theorem 1.6. First, we do not handle the case when R is continuous. In Theorem 1.6 (i), it is trivial to extend our proof to that case since the nonlinear terms are exponentially decaying in the self-similar variables. We believe that Theorem 1.6 (ii) can also be extended to the case when R is continuous albeit with more technical proofs using elliptic regularity theory. However, our construction of G is already quite involved and we opt instead for a clearer, more succinct construction.

Second, we do not address the stability of G; that is, we do not have a convergence result of $\tilde{g}$ to G when $d = 2$ as we do in Theorem 1.6 (i) with the Gaussian. The spectral theory based argument of part (i) does not apply to the stability of G when $d = 2$ because, as shown in (1.11), the nonlinear term plays a crucial role in the equation, thus, any stability result must use the nonlinearity in an essential way. The initial difficulty in establishing the stability of G is that $\|G\|_2$ is unknown and, it is not clear why $\|\tilde{g}(\tau)\|_2$ converges to a constant (and that that constant is $\|G\|_2$). Indeed, multiplying (1.11) by $\tilde{g}$ and integrating by parts yields

$$\frac{1}{2}\frac{d}{d\tau}\|\tilde{g}(\tau)\|_2^2 = -\frac{1}{2}\|\nabla\tilde{g}(\tau)\|_2^2 + \frac{1}{2}\|\tilde{g}(\tau)\|_2^2 + \|\tilde{g}(\tau)\|_2^4 - \|\tilde{g}(\tau)\|_3^3.$$

Unfortunately, it is not clear that this would yield convergence of the L^2-norm of $\tilde{g}$. For example, there is no obvious monotonicity of $\|\tilde{g}(\tau)\|_2$ imparted by the equation above. Until a better understanding of the fluctuations or convergence of $\|\tilde{g}(\tau)\|_2$ is gained, stability remains open, although we believe that G is stable.

The proofs of Theorem 1.5 and Theorem 1.6 are contained in Section 5.

Notation. Throughout the manuscript we use the notation $\lesssim$ for the following: $A \lesssim B$ if there exists $C > 0$ such that $A \le CB$, where C is any constant that does not depend on g (except for possibly on $\operatorname{supp}(g_0)$). We write $A \approx B$ to mean that $A \lesssim B$ and $B \lesssim A$.

All L^p norms are taken with respect to the spatial variable only unless explicitly indicated. Hence, $\|g\|_p$ and $\|g(t)\|_p$ both refer to

$$\left(\int_{\mathbb{R}^d} g(t,x)^p \, dx\right)^{1/p},$$

when $p \in [1, \infty)$. The analogous notation is used when $p = \infty$.

Various quantities play a special role in our analysis. We adopt the following notation: for any measurable function $f : [0,\infty) \times \mathbb{R}^d \to [0,\infty)$, we denote

$$\begin{aligned} E_f(t) &= \langle f(t), R * f(t)\rangle = \|\phi * f(t)\|_2^2, \\ D_f(t) &= \langle \nabla f(t), \nabla R * f(t)\rangle = \|\nabla\phi * f(t)\|_2^2, \quad \text{and} \\ M_f(t) &= \|f(t)\|_\infty, \end{aligned} \tag{1.12}$$

where we used (1.4) to get the relationship between the first and second characterizations of E_f and D_f (this also uses that convolutions can pass between functions in the L^2-inner product; see (2.9)). We point out that, when $R = \delta$, $E_f = \|f(t)\|_2^2$ and $D_f = \|\nabla f(t)\|_2^2$.

When writing lim, lim inf, and lim sup, we often omit the notation regarding the variable when no confusion will arise. For example, the conclusion of Proposition 1.4 (i) can be written

$$\liminf t^{2/3}\|g\|_2^2 = \liminf t^{2/3}\|g\|_\infty \le \theta_{\text{crit}}.$$

We use $B_r(x)$ to mean a ball of radius $r > 0$ centered at x in the spatial variables. When the ball is centered at the origin, we simply write B_r in place of $B_r(0)$.

2. The 2/3 power law when R is continuous.

Before beginning the proof, we notice that, due to Hölder's inequality

$$E_g \le M_g, \tag{2.1}$$

since $\int R * g = 1$, which comes from the assumption $\int R = 1$ and the fact that $\int g = 1$. In addition, by Young's inequality for convolutions,

$$E_g \le (\|g\|_2\|\phi\|_1)^2 = \|g\|_2^2. \tag{2.2}$$

A key aspect of the proof is understanding the precise relationship between g, $\phi * g$, and $R * g$. As such, it is useful to define more succinct notation for the latter two functions. Let

$$u = \phi * g \quad \text{and} \quad w = \phi * \phi * g = R * g. \tag{2.3}$$

We first show how to deduce Theorem 1.1 from Proposition 1.2 in the following subsection. Afterwards, in Section 2.2 and the following subsections, we prove Proposition 1.2. This is where the bulk of the work is undertaken.

2.1. **The proof of Theorem 1.1 from Proposition 1.2.** We establish the bounds on the moments via arguments very similar to [14, Theorem 1.3]; however, the slight alterations in the method here allows us to reduce the dependence of the estimates on g_0 to only on $\mathrm{supp}(g_0)$. When possible, we defer to the arguments in [14, Theorem 1.3] and omit them here.

Proof of Theorem 1.1. First, we obtain a pointwise upper bound on g. We have, from Proposition 1.2, that

$$g(t,\cdot) \lesssim t^{-2/3};$$

however, we require estimates on g when $|x| \gtrsim t^{2/3}$. To this end, let $L > 0$ be such that $\mathrm{supp}(g_0) \subset [-L, L]$, and define

$$\overline{g}(t,x) = e^{\int_0^t E_g(s)\,ds} h(t,x),$$

where h is the solution of

$$\begin{cases} h_t = \frac{1}{2}\Delta h & \text{in } (0,\infty)\times\mathbb{R}, \\ h = g_0 & \text{on } \{0\}\times\mathbb{R}. \end{cases}$$

It is straightforward to check that

$$\partial_t \overline{g} = \frac{1}{2}\Delta\overline{g} + E_g \overline{g}. \tag{2.4}$$

While (1.1) does not enjoy the comparison principle, (2.4) does. In addition, the non-negativity of g and (1.1) ensure that g is a subsolution of (2.4). Thus, the comparison principle implies that $g \le \overline{g}$. We deduce that, for all $t > 0$,

$$\begin{aligned} g(t,x) \le \overline{g}(t,x) &= e^{\int_0^t E_g(s)\,ds} \int_{\mathrm{supp}(g_0)} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}} g_0(y)\,dy \\ &\le e^{\int_0^t E_g(s)\,ds} \int_{\mathrm{supp}(g_0)} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{4t} + \frac{y^2}{t}} g_0(y)\,dy \\ &\le \frac{e^{\int_0^t E_g(s)\,ds - \frac{x^2}{4t} + \frac{L^2}{t}}}{\sqrt{2\pi t}} \int_{\mathrm{supp}(g_0)} g_0(y)\,dy = \frac{e^{\int_0^t E_g(s)\,ds - \frac{x^2}{4t} + \frac{L^2}{t}}}{\sqrt{2\pi t}}. \end{aligned} \tag{2.5}$$

In the first equality, we used the kernel representation of solutions to the heat equation, and in the second inequality, we used Young's inequality in the exponent.

Applying Proposition 1.2 and the fact that $E_g \le M_g$, we find $C > 0$ such that, if $t \ge 1$, then $\int_0^t E_g ds \le Ct^{1/3}$. Using this in (2.5) yields, for any $|x| \ge \sqrt{8C}t^{2/3}$,

$$g(t,x) \lesssim e^{Ct^{1/3} - \frac{x^2}{8t} - \frac{x^2}{8t} + \frac{L^2}{t}} \le e^{Ct^{1/3} - \frac{\left|\sqrt{8C}t^{2/3}\right|^2}{8t} - \frac{x^2}{8t} + \frac{L^2}{t}} = e^{-\frac{x^2}{8t} + \frac{L^2}{t}}.$$

Using this and Proposition 1.2, we obtain the crucial estimate:

$$g(t,x) \lesssim \begin{cases} t^{-2/3} & \text{if } |x| \le \sqrt{8C}t^{2/3}, \\ e^{\frac{L^2}{t} - \frac{|x|^2}{8t}} & \text{otherwise.} \end{cases}$$

A direct computation using this upper bound yields the upper bounds on $\int |x|^p g(t,x)\,dx$ for any $p > 0$ and $t \ge 1$. The proof of the lower bound is exactly as in [14, Theorem 1.3] (that is, it uses the upper bound and a variational argument, see [14, Lemma 4.2]). As such, we omit the details.

The last step is, thus, to obtain lower bounds on the L^2- and L^∞-norms of $g(t,\cdot)$. By Hölder's inequality, $\|g(t)\|_2^2 \le \|g(t)\|_\infty$ (recall that $\|g(t)\|_1 = 1$). Hence, it is sufficient to find a lower bound on $\|g(t)\|_2$ in order to finish the claim.

To this end, we utilize the previously-established moment bounds. Fix $L > 0$ to be determined. Then

$$\begin{aligned} 1 = \int g(t,x)\,dx &\leq \int_{-Lt^{2/3}}^{Lt^{2/3}} g(t,x)\,dx + \int_{[-Lt^{2/3},Lt^{2/3}]^c} \frac{|x|^2}{L^2 t^{4/3}} g(t,x)\,dx \\ &\leq \sqrt{2Lt^{2/3}} \|g(t)\|_2 + \frac{1}{L^2 t^{4/3}} \int |x|^2 g(t,x)\,dx. \end{aligned}$$

where, in the second inequality, we used Hölder's inequality. Choosing L sufficiently large, the last term is smaller than $1/2$ by the moment bounds above. Thus, $t^{1/3}\|g(t)\|_2$ is bounded below, as desired. This concludes the proof. □

2.2. The decay of M_g: proof of Proposition 1.2.

Although Proposition 1.2 and (2.1) yields that $E_g(t) \lesssim t^{-2/3}$, we, in fact, must establish the decay of E_g before proving Proposition 1.2 as the decay of E_g is a crucial element in its proof. This is in contrast to the proof in [14] for the case $R = \delta$, where we deduced the decay of M_g directly and then used this to establish the decay of E_g. We state this main ingredient in the following proposition.

Proposition 2.1. *We have, for all t, $E_g(t) \lesssim \min\left\{\frac{1}{\sqrt{t}}, \frac{1}{t^{2/3}}\right\}$.*

In addition, we require the following lemma, allowing us to show that, in some sense, g, u, and w cannot be "too different."

Lemma 2.2. *We have, for all $t \geq 1$,*

$$M_w(t) \leq M_u(t) \leq M_g(t) \lesssim M_w(t). \tag{2.6}$$

Furthermore, for any $(t,x) \in [1,\infty) \times \mathbb{R}$,

$$\frac{g(t,x)^2}{M_g(t)} \lesssim w(t,x), u(t,x). \tag{2.7}$$

Finally, we also have a small time bound on M_g.

Lemma 2.3. *We have, for all $t \in [0,1]$, $E_g(t) \leq M_g(t) \lesssim \frac{1}{\sqrt{t}}$. The implied constant does not depend on* $\operatorname{supp}(g_0)$.

We establish Proposition 2.1 in Section 2.3 up to a technical lemma. The technical lemma is proved in Section 2.4, and relies as well on Lemma 2.2. The latter is proved in Section 2.4.1. Finally, Lemma 2.3 has an elementary proof, that relies on the identity (2.12) established at the beginning of Section 2.3, which describes the evolution of E_g.

We now prove Proposition 1.2 assuming Proposition 2.1 and Lemmas 2.2 and 2.3.

Proof of Proposition 1.2. The bound for $t \leq 1$ follows directly from Lemma 2.3. Hence, we need only address the case $t \geq 1$. By evaluating (1.1) at the location of a spatial maximum (t, x_t), we find

$$\dot{M}_g \leq M_g(E_g - w(t,x_t)). \tag{2.8}$$

Using Proposition 2.1 and Lemma 2.2 in (2.8) yields, for some $C > 0$,

$$\dot{M}_g \leq M_g\left(\frac{C}{t^{2/3}} - \frac{M_g}{C}\right).$$

Let $\overline{M}(t) = At^{-2/3}$ for A to be determined. By Lemma 2.3, if A is sufficiently large, $\overline{M}(1) \geq M_g(1)$. In addition, we have

$$\dot{\overline{M}} - \overline{M}\left(\frac{C}{t^{2/3}} - \frac{\overline{M}}{C}\right) = -\frac{2A}{3t^{5/3}} + \frac{A^2}{t^{4/3}}\left(\frac{1}{C} - \frac{C}{A}\right) > 0,$$

where the last inequality holds as long as A is sufficiently large. The comparison principle implies that, for all $t \geq 1$,

$$M_g(t) \leq \overline{M}(t).$$

This concludes the proof. □

2.3. **A differential equation for E_g.** Before we embark on the proof, we note a useful identity for convolutions that is applied often in the sequel. By the symmetry of R (recall (1.4)),

$$\int (R * f_1) f_2 \, dx = \int f_1 (R * f_2) \, dx \qquad \text{for any } f_1, f_2. \tag{2.9}$$

We begin our proof by deriving a differential equation for E_g. Convolving (1.1) with R, we find

$$\partial_t w - \frac{1}{2}\Delta w = w E_g - R * (gw). \tag{2.10}$$

Multiplying by g and integrating yields

$$\int g \partial_t w \, dx + \frac{1}{2} D_g = E_g^2 - \int g R * (gw) \, dx.$$

Notice that

$$\dot{E}_g = \int (\partial_t g w + g \partial_t w) \, dx$$

and

$$\int g \partial_t w dx = \int g R * (\partial_t g) \, dx = \int (R * g) \partial_t g \, dx = \int w \partial_t g dx.$$

Above we used (2.9). Hence

$$\int g \partial_t w \, dx = \frac{1}{2}\dot{E}_g.$$

Thus, the above becomes

$$\frac{1}{2}\dot{E}_g + \frac{1}{2} D_g = E_g^2 - \int g R * (gw) \, dx. \tag{2.11}$$

We now derive a simplified form for the right hand side of (2.11). First, using (2.9) a second time, we find

$$\int g R * (gw) \, dx = \int (R * g) g w \, dx = \int g w^2 \, dx.$$

Recalling that g is a probability measure and $E_g = \langle g, w \rangle$, we obtain

$$\begin{aligned} E_g^2 - \int g w^2 \, dx &= -\int g(w^2 - E_g^2) \, dx = -\int g(w^2 - 2wE_g + E_g^2) \, dx \\ &= -\int g(w - E_g)^2 \, dx. \end{aligned}$$

Putting this together with (2.11), we find

$$\dot{E}_g + D_g = -2 \int g(w - E_g)^2 \, dx. \tag{2.12}$$

We note that (2.12) implies that E_g is decreasing. However, this is not sufficient for Proposition 2.1, and, instead, it is required to bound the integral term on the right hand side away from zero. Before proceeding with this proof, we show how to conclude Lemma 2.3.

We now establish the (much simpler) bound on E_g and M_g for small times.

Proof of Lemma 2.3. The bound on E_g follows from a general argument that works in all dimensions that is based on a generalization of the Nash inequality. As this decay is the focus of Theorem 1.5 and the one dimensional result is simply a side-effect of the analysis, we postpone it until Section 5. The bound is given in Proposition 5.1. Hence, to finish the proof we need only derive the bound for M_g from the bound for E_g.

Let C be such that $E_g(t) \le Ct^{-1/2}$. Define h to be the solution of $\partial_t h = (1/2)\Delta h$ with $h(0,\cdot) = g_0$. Let G be the heat kernel; that is,

$$G(t,x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}.$$

Then $h = G(t) * g_0$.

We now define our barrier function

$$\overline{g}(t,x) = e^{\int_0^\tau E_g(s)\,ds} h(t,x).$$

Notice that $\overline{g}$ solves

$$\partial_\tau \overline{g} = \frac{1}{2}\overline{g} + E_g \overline{g}. \tag{2.13}$$

By (1.1) and the non-negativity of g, we see that g is a subsolution of (2.13). Hence, the comparison principle implies that $g \le \overline{g}$. Thus, for all $t \in [0,1]$,

$$\begin{aligned} M_g(t) &\le M_{\overline{g}}(t) = e^{\int_0^t E_g(s)\,ds} M_h(t) \le e^{2Ct} M_h(t) \\ &\lesssim \|G(t) * g_0\|_\infty \le \|G(t)\|_\infty \|g_0\|_1 \lesssim \frac{1}{\sqrt{t}}, \end{aligned}$$

which concludes the proof. □

Having finished the proof of the bounds for small time, we focus on the case $t \ge 1$ for the remainder of the section. To that end, we seek to bound the integral term on the right hand side of (2.12) away from zero. We establish this in the following lemma.

Lemma 2.4. *We have, for every $t \ge 1$,*

$$\frac{E_g^5}{D_g} \lesssim \int g(w - E_g)^2\,dx.$$

Ideally, to prove Lemma 2.4, we would apply similar methods as were used to establish the lower bound on $\int g(M-g)\,dx$ in [14, Lemma 4.3] for the case $R = \delta$. This is, in spirit, possible; however, it is complicated by the fact that the right hand side of (2.12) involves both g and $w = R * g$, which take different values when $R \ne \delta$. As a result, the proof of Lemma 2.4 is significantly more technical than the proof of its counterpart in [14]. In fact, it is in the proof of this inequality that almost all of the difficulty lies. We delay its proof until Section 2.4. We now show how to conclude the upper bound on E_g assuming Lemma 2.4.

Proof of Proposition 2.1. We first note that Lemma 2.3 yields a bound on $E_g(1)$. This can be extended to a bound on $[1,2]$ since E_g is decreasing (recall (2.12)). Hence, we need only establish an upper bound on $[2,\infty)$.

We begin by applying (2.12) and Lemma 2.4 to find, for $t \geq 1$ and some $C > 0$,

$$\dot{E}_g + D_g + \frac{1}{C}\frac{E_g^5}{D_g} \leq 0.$$

Applying Young's inequality, we have

$$\frac{2}{\sqrt{C}} E_g^{5/2} = 2\sqrt{D_g}\sqrt{\frac{E_g^5}{C D_g}} \leq D_g + \frac{1}{C}\frac{E_g^5}{D_g}.$$

Combining the two above inequalities, we find

$$\dot{E}_g + \frac{2}{\sqrt{C}} E_g^{5/2} \leq 0.$$

Solving this differential inequality yields, for all $t \geq 1$,

$$E_g(t) \lesssim \frac{1}{(t-1)^{2/3}}.$$

This concludes the proof of the upper bound for all $t \geq 2$, which finishes the proof. □

2.4. **A lower bound on the integral term.** We begin by stating a key lemma related to the spatial regularity of g. This is the "local-in-time Harnack inequality." While this was introduced in [5], we use the more precise statement of [6] as we require the flexibility in the parameter t_0 in the sequel.

Proposition 2.5. [6, Proposition 1.2] *Fix any $t > t_0 > 0$. Let $\varepsilon \in (0,1)$. There exists C_ε, depending only on ε, such that*

$$g(t,x) \leq C_\varepsilon \exp\left\{ C_\varepsilon \|g\|_{L^\infty([t-t_0,t]\times\mathbb{R})} t_0 + \frac{C_\varepsilon |x-y|^2}{t_0} \right\} g(t,y)^{1-\varepsilon} \|g\|^\varepsilon_{L^\infty([t-t_0,t]\times\mathbb{R})}.$$

The main difference between Proposition 2.5 and the standard parabolic Harnack inequality is the fact that we do not require a "shift" in time; that is, the g terms on the right and left are both evaluated at the same t (up to the $\|g\|_\infty^\varepsilon$ error term). We use Proposition 2.5 often in the sequel.

2.4.1. *How the maxima evolve.* We now establish a preliminary upper bound on g and use it, along with the local-in-time Harnack inequality (Proposition 2.5) to establish the comparison between g, u, and w (Lemma 2.2).

We first show that the maxima of g, u, and w are bounded independent of $t \geq 1$. This allows us to obtain regularity of g that is uniform in t as $t \to \infty$.

Lemma 2.6. *For all $t \geq 1/4$, $M_g(t) \lesssim 1$.*

Proof. Fix any $t_0 \geq 1/4$ we establish a bound of $M_g(t_0)$. Let $\tilde{g}(t,x) = g(t + (t_0 - 1/4), x)$. Since (1.1) is an autonomous equation, $\tilde{g}$ satisfies (1.1) with initial data $\tilde{g}(0,\cdot) = g(t_0 - 1/4, \cdot)$ and $\int \tilde{g}(0,\cdot) = 1$. Applying Lemma 2.3, we find

$$M_g(t_0) = M_{\tilde{g}}(1/8) \lesssim \frac{1}{\sqrt{1/8}} \lesssim 1,$$

which concludes the proof. □

We next require that, over finite time intervals, the M_g does not change too much. This allows us to apply Proposition 2.5 in such a way that we can replace the $\|g\|_{L^\infty([t-t_0,t]\times\mathbb{R})}$ term by $M_g(t)$.

Lemma 2.7. *For all $t_1 \geq 1$ and $t_0 \in [0, 1/2]$, we have*

$$M_g(t_1) \lesssim M_g(t_1 - t_0) \lesssim M_g(t_1).$$

Proof. Fix $t_1 \geq 1$ and $t_0 \in [0, 1/2]$. Let h be the solution to

$$\begin{cases} h_t = \frac{1}{2}\Delta h & \text{in } (t_1 - 3/4, \infty) \times \mathbb{R}, \\ h = g(t_1 - 3/4, \cdot) & \text{on } \{t_1 - 3/4\} \times \mathbb{R}. \end{cases}$$

Recall that $M_g(t)$ is bounded above by Lemma 2.6 for $t \geq 1/4$. Let

$$A = \sup_{t \geq 1/4} M_g(t) < \infty \tag{2.14}$$

and define $\overline{g}(t, x) = e^{A(t - (t_1 - 3/4))} h(t, x)$ and $\underline{g}(t, x) = e^{-A(t-(t_1-3/4))} h(t, x)$. We claim that

$$\underline{g}(t, x) \leq g(t, x) \leq \overline{g}(t, x), \tag{2.15}$$

for all $t \in [t_1 - 3/4, t_1]$, and that

$$M_h(t_1) \leq M_h(t_1 - t_0) \lesssim M_h(t_1). \tag{2.16}$$

We prove this in the sequel. Let us momentarily assume both (2.15) and (2.16) are true and conclude the proof.

Indeed, then we have that

$$\begin{aligned} M_g(t_1) &\leq M_{\overline{g}}(t_1) = e^{3A/4} M_h(t_1) \leq e^{3A/4} M_h(t_1 - t_0) \\ &= e^{3A/4} \left(e^{A(3/4 - t_0)} M_{\underline{g}}(t_1 - t_0) \right) \leq e^{A(3/2 - t_0)} M_g(t_1 - t_0) \leq e^{3A/2} M_g(t_1 - t_0), \end{aligned}$$

and, similarly,

$$\begin{aligned} M_g(t_1 - t_0) &\leq M_{\overline{g}}(t_1 - t_0) = e^{A(3/4 - t_0)} M_h(t_1 - t_0) \lesssim e^{A(3/4 - t_0)} M_h(t_1) \\ &= e^{A(3/4 - t_0)} \left(e^{3A/4} M_{\underline{g}}(t_1) \right) \lesssim e^{3A/2} M_g(t_1). \end{aligned}$$

This establishes the claim up to proving (2.15) and (2.16).

We first establish (2.15). We show only the first inequality as the second is proved similarly. Indeed, by the fact that $\partial_t \underline{g} = \frac{1}{2}\Delta \underline{g} - A\underline{g}$, we have

$$\partial_t \underline{g} - \frac{1}{2}\Delta \underline{g} - \underline{g}\,(E_g - R * g) = -\underline{g}(A + E_g - R * g).$$

Since $A + E_g \geq M_g$ by (2.14) and $M_g \geq R * g$, $\underline{g}$ is a subsolution of (1.1). Hence, by the comparison principle, $\underline{g} \leq g$ on $[t_1 - 3/4, \infty) \times \mathbb{R}$, as claimed.

Now we establish (2.16). The first inequality follows from the maximum principle; that is, the maximum of a solution to the heat equation is decreasing in time. The second inequality follows indirectly from the parabolic Harnack inequality. Indeed, let x_m be the location of a maximum of $h(t_1 - 1/2, \cdot) = g(t_1 - 1/2, \cdot)$. Then the parabolic Harnack inequality implies that

$$M_h(t_1 - 1/2) = \sup_{x \in B_1(x_m)} h(t_1 - 1/2, x) \lesssim \inf_{x \in B_1(x_m)} h(t_1, x) \lesssim M_h(t_1).$$

Using the maximum principle again, we find $M_h(t_1 - t_0) \leq M_h(t_1 - 1/2)$ since $t_0 \in [0, 1/2]$. Combining this with the above inequality, establishes (2.16). This concludes the proof. □

We are now able to utilize Proposition 2.5 to show that g, u, and w are comparable; that is, we prove Lemma 2.2.

Proof of Lemma 2.2. The first and second inequalities in (2.6) are immediate by Young's inequality for convolutions: for all (t,x),

$$w(t,x) = \phi * u(t,x) \leq \|\phi\|_1 \|u(t,\cdot)\|_\infty = M_u(t).$$

The proof of the second inequality is the same and is thus omitted.

Finally, we point out that the third inequality in (2.6) follows directly from (2.7) (when applied at (t,x_t) that is the location of a spatial maximum of $g(t,\cdot)$). In addition, the proof of (2.7) is the same for either right hand side (w or u). Thus, we show it only for u.

Fix $t \geq 1$ and any $x \in \mathbb{R}$. Choose $r_0 > 0$ such that

$$\int_{B_{r_0}} \phi(y)dy = \frac{1}{2}.$$

Fix any $y \in B_{r_0}$. By Proposition 2.5 with $\varepsilon = 1/2$ and $t_0 = 1/2$, we have

$$g(t,x-y)^{1/2} \gtrsim \frac{g(t,x)}{\exp(C\sup_{s\in[t-1/2,t]} M_g(s) + \frac{Cr_0^2}{1/2})\sup_{s\in[t-1/2,t]} M_g(s)^{1/2}}.$$

Using Lemma 2.6, the exponential term in the denominator is bounded. In addition, Lemma 2.7 implies that

$$\sup_{s\in[t-1/2,t]} M_g(s) \lesssim M_g(t).$$

Hence, we find, for all $y \in B_{r_0}$,

$$g(t,x-y) \gtrsim \frac{g(t,x)^2}{M_g(t)}. \tag{2.17}$$

We now use this inequality to conclude. Recalling the definition of u and applying (2.17) yields

$$u(t,x) \geq \int_{B_{r_0}(0)} \phi(y)g(t,x-y)dy \gtrsim \int_{B_{r_0}(0)} \frac{g(t,x)^2}{M_g(t)}\phi(y)dy = \frac{g(t,x)^2}{2M_g(t)},$$

which concludes the proof. □

2.4.2. *The lower bound on the integral term: the proof of Lemma 2.4.* Having established the relationship between w, u, and g, we are now in a position to prove the main technical lemma in the proof of Proposition 1.2 apart from one final technical lemma. This lemma shows that if $g(t,x_1)$ is sufficiently small compared to $g(t,x_0)$, then $w(t,x_1)$ and $u(t,x_1)$ will be also be small compared to $g(t,x_0)$. We state this lemma now and prove it in the sequel.

Lemma 2.8. *For all $t \geq 1$, if $\frac{g(t,x_0)^2}{g(t,x_1)M_g(t)}$ is sufficiently large, depending only on R and $M_g(t)/g(t,x_0)$, then*

$$w(t,x_1), u(t,x_1) \leq \frac{1}{2} g(t,x_0).$$

We now prove Lemma 2.4.

Proof of Lemma 2.4. Since time plays no role here, except to allow us to apply the lemmas in Section 2.4.1, we omit $t \geq 1$ notationally for the remainder of the proof. We let $A > 1$ be a constant to be determined. There are two cases to consider, and A is determined in the second case.

Case one: $M_g \leq AE_g$. The constant A does not play a role in this case, and, as such, we absorb it into the constants in the $\lesssim$ notation.

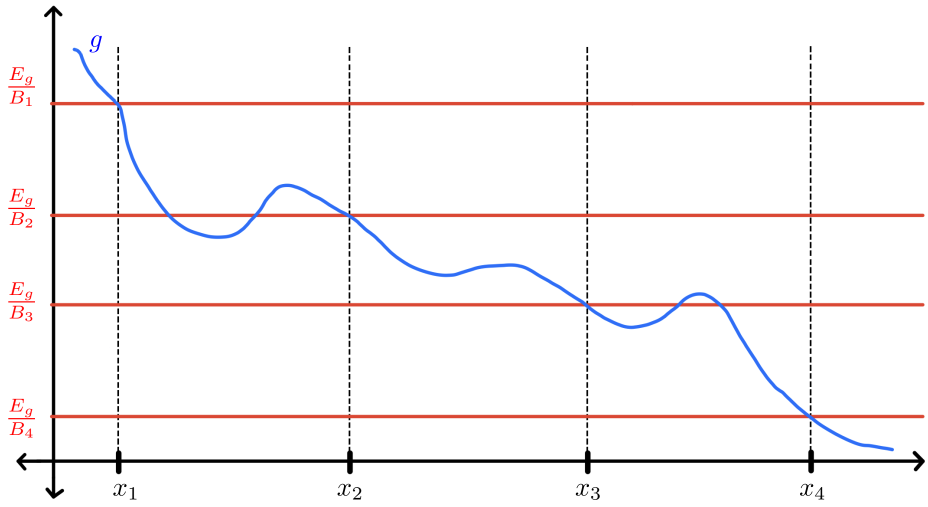

FIGURE 1. A cartoon depicting the relationship of x_i, B_i, and g.

Fix $1 = B_1 < B_2 < B_3 < B_4$ to be determined. We may find $x_1 < x_2 < x_3 < x_4$ such that

$$\begin{aligned} &g(x_i) = \frac{E_g}{B_i} && \text{for each } i \in \{1,2,3,4\}, \\ &g(x) \le \frac{E_g}{B_i} && \text{for all } x \in [x_i, x_4] \text{ and each } i \in \{1,2\} \\ &g(x) \ge \frac{E_g}{B_i} && \text{for all } x \in [x_1, x_i] \text{ and each } i \in \{3,4\} \end{aligned} \tag{2.18}$$

This can be achieved by making the choices

$$\begin{aligned} &x_1 = \sup\{x \in \mathbb{R} : g(x) = E_g\} && x_4 = \inf\{x \ge x_1 : g(x) = E_g/B_4\} \\ &x_2 = \sup\{x \le x_4 : g(x) = E_g/B_2\} && x_3 = \inf\{x \ge x_1 : g(x) = E_g/B_3\}. \end{aligned}$$

Roughly, x_1 and x_2 are the "last times" g takes the values E_g/B_1 and E_g/B_2, respectively, while x_3 and x_4 are the "first times" after x_2 that g takes the values E_g/B_3 and E_g/B_4, respectively. See Figure 1.

Fix any $x \in [x_2, x_4]$. Applying Lemma 2.8, we have that, choosing B_2 so that

$$\frac{g(x_1)^2}{g(x)M_g} = \frac{E_g^2}{g(x)M_g} \ge \frac{B_2}{A}$$

is sufficiently large, depending only on $M_g/g(x_0) = M_g/E_g \le A$, then

$$w(x) \le \frac{g(x_1)}{2} = \frac{E_g}{2}.$$

We conclude that $E_g - w \ge E_g/2$. Hence,

$$\int_{x_2}^{x_4} g(E_g - w)^2 dx \ge \int_{x_2}^{x_4} \frac{E_g}{B_4} \frac{E_g^2}{4} dx = \frac{E_g^3}{4B_4} |x_4 - x_2| \gtrsim E_g^3 |x_4 - x_2|. \tag{2.19}$$

Here we used that $g \ge E_g/B_4$ on $[x_2, x_4]$. On the other hand, we have

$$\begin{aligned} |u(x_2) - u(x_4)|^2 &= \left(\int_{x_2}^{x_4} \partial_x u dx \right)^2 \le |x_4 - x_2| \int_{x_2}^{x_4} |\partial_x u|^2 dx \\ &\le |x_4 - x_2| \int |\partial_x u|^2 dx = |x_4 - x_2| D_g. \end{aligned} \tag{2.20}$$

We now seek a lower bound on $|u(x_2) - u(x_3)|$. Using Lemma 2.2 to bound $u(x_2)$ from below, we find

$$u(x_2) \gtrsim \frac{g(x_2)^2}{M_g} = \frac{E_g^2}{M_g B_2^2} \gtrsim \frac{E_g}{A B_2^2}. \tag{2.21}$$

Choosing now B_3 to be AB_2^2 multiplied by a universal constant, we have

$$u(x_2) \geq \frac{E_g}{B_3}. \tag{2.22}$$

Next, we again apply Lemma 2.8 to conclude that, choosing B_4 so that

$$\frac{g(x_3)^2}{g(x_4) M_g} = \frac{E_g}{M_g} \frac{B_4}{B_3^2} \geq \frac{1}{A} \frac{B_4}{B_3^2}$$

is sufficiently large, depending only on $M_g / g(x_3) \leq AB_2$, then

$$u(x_4) \leq \frac{g(x_3)}{2} = \frac{E_g}{2B_3}. \tag{2.23}$$

Putting the bounds (2.22) and (2.23) together, we find

$$|u(x_2) - u(x_4)| \geq \frac{E_g}{2B_3}.$$

Including the above inequality in (2.20), we find

$$E_g^2 \lesssim |x_4 - x_2| D_g, \tag{2.24}$$

where we have absorbed the dependence of the B_i into the $\lesssim$ notation. Combining (2.19) and (2.24) finishes the proof in case one. Note that, while the estimate above depends on A, in the second case, we choose A to be a fixed large number depending only on R and independent of E_g.

Case two: $M_g \geq AE_g$**.** The argument in this case is similar; however, instead of bounding w *above* by $E_g/2$, we bound it *below* by $3E_g/2$. Indeed, let $1 = B_1 < B_2 < B_3$ and find $x_1 < x_2 < x_3$ such that

$$g(x_i) = \frac{M_g}{B_i} \qquad \text{for each } i \in \{1, 2, 3\},$$
$$g(x) \leq \frac{M_g}{B_2} \qquad \text{for all } x \in [x_2, x_3], \quad \text{and}$$
$$g(x) \geq \frac{M_g}{B_3} \qquad \text{for all } x \in [x_1, x_3].$$

Note that $B_1 < B_2 < B_3$.

Arguing exactly as in (2.20) in the previous case, we find

$$|u(x_1) - u(x_3)|^2 \leq |x_3 - x_1| D_g.$$

From Lemma 2.2, we have

$$u(x_1) \gtrsim \frac{g(x_1)^2}{M_g} = M_g. \tag{2.25}$$

The constant above does not depend on A or on any of the B_i's. Hence, we may select B_2 such that $g(x_2) \leq u(x_1)$.

Then, after increasing B_3 such that

$$\frac{g(x_2)^2}{g(x_3) M_g} = \frac{B_3}{B_2^2}$$

is sufficiently large, depending only on $M_g/g(x_2) = B_2$, Lemma 2.8 implies that

$$u(x_3) \le \frac{g(x_2)}{2} \le \frac{u(x_1)}{2}.$$

Using this and (2.25), we conclude that

$$M_g^2 \lesssim \left|\frac{u(x_1)}{2}\right|^2 \le |u(x_3) - u(x_1)|^2 \le |x_3 - x_1| D_g. \tag{2.26}$$

Applying Lemma 2.2, we have that, for $x \in [x_1, x_3]$,

$$w(x) \gtrsim \frac{g(x)^2}{M_g} \ge \frac{M_g}{B_3^2},$$

In the second inequality we used that $g \ge M_g/B_3$ on $[x_1, x_3]$. Since $E_g \le M_g/A$, then, choosing A such that A/B_3^2 is sufficiently large, we have $w(x) - E_g \gtrsim M_g$ for all $x \in [x_1, x_3]$. Thus, we find

$$\int_{x_1}^{x_3} g(E_g - w)^2 dx \gtrsim M_g^3 |x_1 - x_3|. \tag{2.27}$$

Combining (2.27) and (2.26) yields

$$\int g(E_g - h)^2 dx \gtrsim \frac{M_g^5}{D_g}.$$

The proof is finished in this case by using the fact that $M_g \ge A E_g$. Thus, we have concluded the proof in all cases. □

We now finish this section by proving the final lemma, Lemma 2.8. The idea behind this proof is that, by Proposition 2.5, if the ratio $g(x_0)^2/g(x_1)M_g$ is large enough, then g is closer to $g(x_1)$ than $g(x_0)$ on a large set. Plugging this into the convolutions defining u and w yields the result.

Proof of Lemma 2.8. We show the result for w. The proof is exactly the same for u and, hence, we omit it. We assume without loss of generality that $x_0 < x_1 = 0$ since the equation is invariant by reflection and translation. We suppress the time dependence in the proof as t plays no role.

Let

$$\bar\varepsilon = \frac{g(0) M_g}{g(x_0)^2}. \tag{2.28}$$

We may assume that $\bar\varepsilon < 1/4$. Define

$$r = \sup\{s > 0 : g(y) \le \frac{g(x_0)}{4} \text{ for all } y \in [-s, s]\}.$$

From (2.28) and the fact that $\bar\varepsilon < 1/4$, $g(0) < g(x_0)/4$, we conclude that $r > 0$ and $g \le g(x_0)/4$ on $[-r, r]$. In addition, by continuity, either $g(r) = g(x_0)/4$ or $g(-r) = g(x_0)/4$. We assume the former, although the proof is similar in the latter case.

Using this, we estimate the below directly:

$$\begin{aligned} w(0) = \int R(-y) g(y) dy &\le \int_{[-r,r]} R(-y) \frac{g(x_0)}{4} dy + \int_{[-r,r]^c} R(-y) M_g dy \\ &\le \frac{g(x_0)}{4} + 2M_g \int_r^\infty R(y) dy. \end{aligned} \tag{2.29}$$

The final inequality follows from the fact that R is symmetric[c].

[c]Symmetry is not necessary for this result, but it simplifies the notation in the proof.

We seek an estimate on the second term on the last line of (2.29). Indeed, we wish to show that the integral term is small, which corresponds to r being large. In order to conclude this, we appeal to Proposition 2.5 with $\varepsilon = 1/2$ and $t_0 = 1/2$, along with Lemma 2.7, to find

$$\frac{1}{4\sqrt{\bar\varepsilon}} = \frac{1}{4}\frac{g(x_0)}{\sqrt{g(0)M_g}} = \frac{g(r)}{\sqrt{g(0)M_g}} \lesssim \exp\left\{C(M_g + r^2)\right\} \lesssim \exp\left\{Cr^2\right\}.$$

The last line follows from Lemma 2.6. Choosing $\bar\varepsilon$ sufficiently small, makes r sufficiently large that

$$\int_r^\infty R(y)dy \le \frac{g(x_0)}{8M_g}.$$

Plugging this into (2.29), we conclude that $w(0) \le \frac{g(x_0)}{2}$, which finishes the proof. □

3. Long time dynamics in one dimension

For technical reasons that become clear in the sequel, we notice that, since (1.1) is autonomous, we may shift the initial data in time and assume that $g(1,\cdot) = g_0 \in C_c(\mathbb{R})$ without losing generality. In addition, recall that here we make the choice $R = \delta$ whence (1.1) becomes

$$\begin{cases} g_t = \frac{1}{2}\Delta g + g\left(E_g - g\right) & \text{in } (1,\infty)\times\mathbb{R},\\ g = g_0 & \text{on } \{1\}\times\mathbb{R}. \end{cases} \tag{3.1}$$

We begin with a heuristic argument that motivates the constants θ_{crit} and c_{crit} defined in (1.10). This argument also yields the main rescalings and objects of study for us and allows to outline our strategy and the main difficulties encountered in the proof.

3.1. The rescalings and the heuristic argument.

We appeal to two different scalings revealing different features of the dynamics. One suggests that g, properly rescaled, converges to a constant multiple of an indicator function while the second unveils the exact constant involved. We call the rescaled functions, respectively, h and u. Importantly, and somewhat surprisingly, u satisfies an equation reminiscent of the Fisher-KPP equation.

We begin with the new variables $\tau \approx t^{1/3}$ and $y \approx x/t^{2/3}$; that is, define

$$h(\tau,y) = (\tau+1)^2 g((\tau+1)^3, y(\tau+1)^2). \tag{3.2}$$

Notice that this respects the scaling in x and the decay of g shown in Theorem 1.1 and Proposition 1.2. Thus, we have that $C^{-1} < M_h, E_h < C$. We see that

$$\partial_\tau h = \frac{3}{2(\tau+1)^2}\Delta h + 3h(E_h - h) + \frac{2}{\tau+1}\left(h + y\partial_y h\right). \tag{3.3}$$

The meaning of the time shift of the initial conditions to be defined at $t = 1$, above, is now clear: (3.3) does not degenerate as $\tau \searrow 0$ and the initial data of h is defined at $\tau = 0$.

Notice that, heuristically, this indicates that if $h(\tau,y) \to H(y)$ as $\tau \to \infty$, then $H(E_H - H) \equiv 0$, implying that H has to be of the form $|A|^{-1}\mathbb{1}_A$ for some set A (recall that g is a probability measure and, hence, h is one too). Of course, we expect $A = [-c,c]$ for some $c > 0$.

Unfortunately, as the Laplacian term in (3.3) degenerates as $\tau \to \infty$, we can not lean on compactness in order to obtain convergence of h. Instead we need to obtain sharp estimates on h away from the boundary points $y = \pm c$, which leads us to the next scaling.

We pull apart the dynamics of g near $ct^{2/3}$ (i.e., h near c) to see the transition between $(2c)^{-1}t^{-2/3}$ and 0. Since the coefficient in front of the Laplacian is $O(\tau^{-2})$, we expect that this transition layer has width $O(\tau^{-1})$. Hence, for any $c \in \mathbb{R}$, we let

$$u_c(\tau, z) = h(\tau, c + z(\tau+1)^{-1}). \tag{3.4}$$

Using (3.3), we find

$$\partial_\tau u_c = \frac{3}{2}\Delta u_c + 3u_c(E_h - u_c) + 2c\partial_z u_c + \frac{2}{\tau+1}u_c + \frac{z}{\tau+1}\partial_z u_c. \tag{3.5}$$

We expect that, as $\tau \to \infty$, $E_h(\tau) \to \theta$ for some $\theta \geq 0$ and $h(\tau, y) \to H(y) = (2c)^{-1}\mathbb{1}_{[-c,c]}$. For these to be consistent, we have $\theta = (2c)^{-1}$. In addition, looking at (3.5) with $\tau = \infty$, we formally find

$$-c\partial_z u_c = \frac{3}{4}\Delta u_c + \frac{3\theta}{2}u_c\left(1 - \frac{u_c}{\theta}\right).$$

This is the equation for a traveling wave solution (of speed c) of the Fisher-KPP equation. Although there is a family of traveling wave solutions, whose speeds make up an infinite half-line, we expect the correct long-time dynamics to correspond to the minimal speed wave because our initial data is compactly supported (see, e.g. [11, 17]). The minimal speed is given by a formula in terms of the coefficients (see, e.g., [20, (1.25)]), which, in our setting, yields

$$c = 2\sqrt{\frac{3}{4}\cdot\frac{3\theta}{2}} = \frac{3}{\sqrt{2}}\sqrt{\theta}.$$

Recalling that $\theta = (2c)^{-1}$, the unique solution to the two equations is $(c, \theta) = (c_{\text{crit}}, \theta_{\text{crit}})$, where c_{crit} and θ_{crit} are defined in (1.10).

We now discuss the difficulties in establishing the above heuristics for u_c. The first issue is a subtle one. While the last two terms (3.5) appear to be error terms, this is only true for the last term with the correct choice of c, that is, *only in the correct moving frame*. Indeed, the coefficient of last term is approximately z/τ. If the transition from θ to 0 occurs at $\tilde{c}\tau$ with $\tilde{c} \neq c$, then non-trivial behavior for u_c occurs at $z \approx (\tilde{c} - c)\tau$. In this case, both $\partial_z u_c$ and $z/(\tau+1)$ are non-trivial at $(\tilde{c} - c)\tau$. Thus, if we have changed to the "incorrect" moving frame,

$$\liminf_{\tau\to\infty} \left\| \frac{z}{\tau+1}\partial_z u_c \right\|_\infty > 0,$$

and, hence, the heuristics above are no longer useful. It is, thus, crucial to work in the frame with $c = c_{\text{crit}}$.

The equation (3.5) is a non-local Fisher-KPP type equation similar to that considered in [1, 3, 7, 15], which takes the form

$$\partial_\tau u = D\Delta u + u(r - \phi * u), \tag{3.6}$$

for some non-negative function ϕ and some constants $D, r > 0$. Importantly, though, when finding the spreading speed in (3.6), the nonlocal term plays no role as it disappears in the linearization of the equation around zero $\partial_\tau u = D\Delta u + ru$. However, in contrast, the nonlocal term here E_h does not vanish in the linearization, which in our case is

$$\partial_\tau u_c = \frac{3}{2}\Delta u_c + 3E_h u_c + 2c\partial_z u_c + \frac{2}{\tau+1}u_c + \frac{z}{\tau+1}\partial_z u_c.$$

As such, the role of the nonlocality in the two equations is quite different, and we are not able to draw inspiration from the techniques introduced in the previous work to deal with the nonlocal term of (3.6).

The next complication is due to the coupling between E_h and u_c. Their codependence makes it impossible to first show convergence of E_h and then analyze u_c when $\tau \gg 1$ and E_h is almost constant. On the other hand, one might be tempted to consider E_h to be a time dependent prescribed coefficient; however, there is little robust theory on how the front of a reaction diffusion equation depends on the coefficients when their oscillations have no specific structure such as periodicity. This is in part because one can construct coefficients to have a diverse range of fronts if the coefficients oscillate in a complicated manner [2]. Indeed, there are even quite simple settings where there is no defined spreading speed due to oscillations of the coefficients [12].

To overcome this, we derive a differential inequality that shows that E_h can only increase slowly (although we do not rule out it decreasing arbitrarily quickly). Focusing our analysis on the resulting long upslopes, we can work with an almost constant E_h term. In this case, if E_h is too large, it will have been too large for a large time interval beforehand. Then the front, on that time interval, will move too quickly due to the large E_h term in (3.5). This corresponds to h having wide support and contradicts the fact that the integral of h is one. Similarly, if E_h is too small, it will be small for a long time afterwards. Then the front, on that time interval, will move too slowly, corresponding to h having too narrow of support and contradicting the fact that the integral of h is one.

Before proceeding with the proof, we note that the rescalings above, combined with Proposition 1.2 and the fact that $\int g\,dx = \int h\,dy = 1$, yield the following bounds, used often below.

Lemma 3.1. *For any $c \in \mathbb{R}$, We have, for all $\tau \geq 1$,*

$$1 \lesssim E_h(\tau) \leq M_h(\tau) = M_{u_c}(\tau) \lesssim 1.$$

3.2. **The weak bounds: proof of Proposition 1.4.** A key step in establishing the strong bounds in Theorem 1.3 is first establishing the weaker bounds in Proposition 1.4. We show this argument here.

3.2.1. *The lower bound on the limsup.* We begin with the lower bound on the lim sup. We require a simple relationship between M and E for large times, proved at the end of this section.

Lemma 3.2. *We have* $\limsup_{\tau\to\infty} M_h(\tau) = \limsup_{\tau\to\infty} E_h(\tau)$.

The need for this lemma is the following. We argue by contradiction, assuming that $\limsup E_h$ is small. Lemma 3.2 implies that M_h is eventually small as well. However, to be consistent with the requirement that $\int h\,dy = 1$, this forces h to be nontrivial near some $y_0 > c_{\text{crit}}$. Using the connection to (3.5), this corresponds to front propagation of speed y_0 starting from compactly supported initial data; however, y_0 is greater than the speed $2\sqrt{(3/4)\cdot(3E_h/2)} = 2\sqrt{9E_h/8}$ of the minimal speed traveling wave, which is known to be impossible. The last step is achieved by constructing a supersolution that is typical in studying Fisher-KPP.

We now show how to conclude Proposition 1.4.(ii) using Lemma 3.2.

Proof of Proposition 1.4.(ii). We prove this by contradiction, assuming that, for some $\varepsilon > 0$,

$$\limsup_{\tau\to\infty} E_h(\tau) \leq \theta_{\text{crit}} - \varepsilon.$$

Then, by Lemma 3.2, $\limsup M_h \leq \theta_{\text{crit}} - \varepsilon$, and, hence, there exists τ_0 such that $E_h(\tau), M_h(\tau) \leq \theta_{\text{crit}} - \varepsilon/2$ for all $\tau \geq \tau_0$.

First we show that, for any c and τ_0, there exists $D_c > 0$, depending on τ_0 and c, such that

$$u_c(\tau_0, z) \leq D_c \exp\left\{-\frac{z^2}{D_c}\right\}. \tag{3.7}$$

Notice that, by inverting the change of variables relating u_c and h, it is enough to show this for $c = 0$. The bound of $u_{c=0}$ of the form in (3.7) follows from changing variables from g to $u_{c=0}$ and using (2.5).

Fix $A > 0$ to be chosen. Let $c = c_{\text{crit}}$, $\lambda = 2c/3$, and let

$$\overline{u}_c(\tau, z) = Ae^{-\lambda z}.$$

We aim to show that $u_c \leq \overline{u}_c$ on $[\tau_0, \infty) \times \mathbb{R}$.

First, we consider the domain $[\tau_0, \infty) \times (-\infty, 0]$. By Lemma 3.1, u_c is uniformly bounded above, and $\overline{u}_c \geq A$ for any $z \leq 0$. Hence, if A is sufficiently large, then we have that $\overline{u}_c \geq u_c$ on $[\tau_0, \infty) \times (-\infty, 0]$.

Next we show that $\overline{u}_c \geq u_c$ on $[\tau_0, \infty) \times (0, \infty)$. We do so via the comparison principle. The first step is to show that $\overline{u}_c$ is a super-solution of (3.5) on $(\tau_0, \infty) \times (0, \infty)$. Indeed,

$$\begin{aligned}
&\partial_\tau \overline{u}_c - \frac{3}{2}\Delta \overline{u}_c - 3\overline{u}_c(E_h - \overline{u}_c) - 2c\partial_z \overline{u}_c - \frac{2}{\tau+1}\overline{u}_c - \frac{z}{\tau+1}\partial_z \overline{u}_c \\
&\quad = \overline{u}_c \left(-\frac{3}{2}\lambda^2 - 3E_h + 3\overline{u}_c + 2c\lambda - \frac{2}{\tau+1} + \frac{\lambda}{\tau+1} z\right) \\
&\quad \geq \overline{u}_c \left(-\frac{3}{2}\lambda^2 - 3(\theta_{\text{crit}} - \varepsilon/2) + 2c\lambda - \frac{2}{\tau+1}\right) = \overline{u}_c \left(\frac{2c^2}{3} - 3(\theta_{\text{crit}} - \varepsilon/2) - \frac{2}{\tau+1}\right).
\end{aligned}$$

By our choice of c, it is clear that, up to increasing τ_0 if necessary, the last line is non-negative and, thus, $\overline{u}_c$ is a supersolution of (3.5).

In order to apply the comparison principle, we address the parabolic boundary. First, we have $u_c \leq \overline{u}_c$ on $[\tau_0, \infty) \times \{0\}$, as established above. Second, up to increasing A, we have, via (3.7),

$$u_c(\tau_0, z) \leq D_c \exp\left\{-\frac{z^2}{D_c}\right\} \leq D_c \exp\left\{D_c\lambda^2 - \lambda z\right\} \leq Ae^{-\lambda z} \qquad \text{for all } z \geq 0.$$

Hence, $u_c \leq \overline{u}_c$ on $\{\tau_0\} \times (0, \infty)$.

We conclude that $u_c \leq \overline{u}_c$ on the parabolic boundary of $(\tau_0, \infty) \times (0, \infty)$ and that $\overline{u}_c$ is a supersolution of (3.5). We can, thus, apply the comparison principle, which implies that $\overline{u}_c \geq u_c$ on $(\tau_0, \infty) \times \mathbb{R}$. This concludes the proof that $u_c \leq \overline{u}_c$ on $[\tau_0, \infty) \times \mathbb{R}$.

From the definition of $\overline{u}_c$, it is clear that, for any $c' > c$,

$$\begin{aligned}
\limsup_{\tau\to\infty} \int_{y>c'} h(\tau, y)\, dy &= \limsup_{\tau\to\infty} \frac{1}{\tau+1} \int_{z>(c'-c)(\tau+1)} u_c(\tau, z)\, dz \\
&\leq \limsup_{\tau\to\infty} \frac{1}{\tau+1} \int_{z>(c'-c)(\tau+1)} \overline{u}_c(\tau, z)\, dz = 0.
\end{aligned}$$

A similar argument as above can be used to obtain a supersolution of u_{-c} and bound the integral for $y < -c'$. Thus, we find

$$\limsup_{\tau\to\infty} \int_{|y|>c'} h(\tau, y) dy = 0.$$

In addition, we notice that

$$\limsup_{\tau\to\infty}\int_{|y|\le c'} h(\tau,y)\,dy \le \limsup_{\tau\to\infty} 2c' M_h(\tau) = 2c' \limsup_{\tau\to\infty} E_h(\tau) \le 2c'(\theta_{\text{crit}}-\varepsilon).$$

Here we used Lemma 3.2.

On the other hand, $\int h(\tau,y)\,dy = 1$. Combining all above estimates, we have

$$1 \le \limsup_{\tau\to\infty}\int_{|y|>c'} h(\tau,y)\,dy + \limsup_{\tau\to\infty}\int_{|y|\le c'} h(\tau,y)\,dy \le 2c'(\theta_{\text{crit}}-\varepsilon).$$

Since this is true of all $c' > c = c_{\text{crit}}$, it follows for $c' = c_{\text{crit}}$ in the limit. Hence,

$$1 \le 2c_{\text{crit}}(\theta_{\text{crit}}-\varepsilon).$$

Recall that $2c_{\text{crit}}\theta_{\text{crit}} = 1$. It follows that the right hand side is strictly less than one. This is a contradiction, which concludes the proof. □

We now prove Lemma 3.2.

Proof of Lemma 3.2. Since $M_h(\tau) \ge E_h(\tau)$ for all τ, it is enough to establish only the "≤" in the claim above. For notational reasons, let $\theta_M = \limsup M_h$ and $\theta_E = \limsup E_h$.

There are two cases. Either M_h is eventually monotonic in τ or not. Consider first the latter case.

If M_h is not eventually monotonic, then we may select τ_n that are local maxima of M_h such that $M_h(\tau_n) \to \theta_M$. It follows that $u_{c=0}$ has local maxima at (τ_n, z_n) for some z_n. Using the equation (3.5), we find, at (τ_n, z_n),

$$0 \le \partial_\tau u_{c=0} - \frac{3}{2}\Delta u_{c=0} = 3u_{c=0}\left(E_h(\tau_n) - u_{c=0} + \frac{2}{3(\tau_n+1)}u_{c=0}\right).$$

Taking a limit as $n\to\infty$ and using the fact that $u_{c=0}(\tau_n,z_n) = M(\tau_n) \to \theta_M$ and $\limsup E_h(\tau_n) \le \theta_E$, we find

$$0 \le 3\theta_M\left(\theta_E - \theta_M\right). \tag{3.8}$$

Since $\theta_M > 0$ by Theorem 1.1, it follows that $\theta_M \le \theta_E$, which concludes the proof in this case.

We now consider the case where M_h is eventually monotonic. The case where M_h is eventually nonincreasing and the case where it is eventually nondecreasing are handled similarly, so we show only the argument for when it is eventually nonincreasing. In this case, we can select τ_n tending to infinity such that $\partial_\tau M_h(\tau_n) \to 0$ as $n\to\infty$, since, otherwise, $\partial_\tau M_h$ is uniformly negative, which implies that $M_h(\tau) \to -\infty$ as $\tau\to\infty$, a contradiction. Since M_h is eventually monotonic, $M_h(\tau_n) \to \theta_M$ as $n\to\infty$. Note that $\Delta u_{c=0}(\tau_n,z_n) \le 0$ when z_n is the location of the spatial maximum of $u_{c=0}(\tau_n,\cdot)$. Using the equation (3.5), we find, at (τ_n,z_n),

$$0 \le -\frac{3}{2}\Delta u_{c=0} = 3u_{c=0}\left(E_h(\tau_n) - u_{c=0} + \frac{2}{3(\tau_n+1)}u_{c=0}\right) - \partial_\tau u_{c=0}.$$

Taking a limit as $n\to\infty$ and using the fact that $u_{c=0}(\tau_n,z_n) = M(\tau_n) \to \theta_M$, $\limsup E_h(\tau_n) \le \theta_E$, and $\lim \partial_\tau u_{c=0} = 0$, we find, again, (3.8) in this case. The proof is then concluded in the same way as in the previous case. □

3.2.2. *The upper bound on the liminf.* We begin by stating the proposition that is the crucial step. It roughly states that if E_h remains large enough (depending on c) for long enough, then u_c grows up to E_h. This is shown by approximating from below by solutions of the Fisher-KPP equation.

Proposition 3.3. *Fix any τ_0, τ_1, R, μ, ε, δ, and c. Suppose that*

$$u_c(\tau_0,\cdot) \geq \mu \mathbb{1}_{B_R} \quad \text{and} \quad 18E_h(\tau) - 4c^2 > \delta \quad \text{for all } \tau \in [\tau_0,\tau_1].$$

If τ_0 is sufficiently large, depending only on δ and ε, then there exists $\underline{\tau}$, depending only on μ, R, ε, and δ, such that, if $\tau_1 - \tau_0 \geq \underline{\tau}$, then

$$u_c(\tau_1, 0) \geq E_h(\tau_1) - \varepsilon.$$

A crucial point is that $\underline{\tau}$ does not depend on c; that is the lower bound is uniform over all admissible c. We prove Proposition 3.3 in Section 4.3. We show how to conclude Proposition 1.4.(i) assuming Proposition 3.3.

Proof of Proposition 1.4.(i). We prove this by contradiction. Assume that $\liminf E_h \geq \theta_{\mathrm{crit}} + \varepsilon$ for some $\varepsilon > 0$. We may find $\tau_0 > 0$ such that $E_h(\tau) \geq \theta_{\mathrm{crit}} + \varepsilon/2$ for all $\tau \geq \tau_0$.

Notice that, for $\tau \geq \tau_0$,

$$18E_h(\tau) - 4c_{\mathrm{crit}}^2 \geq 18(\theta_{\mathrm{crit}} + \varepsilon/2) - 4c_{\mathrm{crit}}^2 = 18\theta_{\mathrm{crit}} + 9\varepsilon - 4(2\theta_{\mathrm{crit}})^{-2} = 9\varepsilon.$$

In the last line we used that $18\theta_{\mathrm{crit}}^3 = 1$. Thus, a straightforward application of Proposition 3.3 yields

$$\liminf_{\tau\to\infty} \inf_{|y|\leq c_{\mathrm{crit}}} h(\tau,y) = \liminf_{\tau\to\infty} \inf_{|c|\leq c_{\mathrm{crit}}} u_c(\tau,0) \geq \theta_{\mathrm{crit}} + \varepsilon/4. \tag{3.9}$$

Using (3.9) and that $2c_{\mathrm{crit}}\theta_{\mathrm{crit}} = 1$, we find

$$1 = \liminf_{\tau\to\infty} \int h(\tau,y)\,dy \geq \liminf_{\tau\to\infty} \int_{-c_{\mathrm{crit}}}^{c_{\mathrm{crit}}} h(\tau,y)\,dy \geq 2c_{\mathrm{crit}}(\theta_{\mathrm{crit}} + \varepsilon/4) = 1 + \frac{c_{\mathrm{crit}}\varepsilon}{2}.$$

This is clearly a contradiction, which concludes the proof. □

4. The strong bounds with radial symmetry

We now show the significantly stronger bounds under the assumption that g_0 is even and radially decreasing. There are two major steps here. First, we show that E_h converges as $\tau \to \infty$. Then we use that to obtain convergence of h. The first step is restated in the following proposition:

Proposition 4.1. *If g solves (3.1) with initial data $0 \leq g_0 \in C_c(\mathbb{R})$ satisfying $\int g_0\,dx = 1$ that is even and radially decreasing, then*

$$\lim_{\tau\to\infty} E_h(\tau) = \lim_{\tau\to\infty} M_h(\tau) = \theta_{\mathrm{crit}}.$$

We perform these steps in reverse order. First, in Section 4.1, we show that h converges as claimed assuming that E_h and M_h do; that is, Proposition 4.1 holds. Then, in Section 4.2 we prove Proposition 4.1.

4.1. **Convergence of h given convergence of E_h and M_h.** We now show how to conclude Theorem 1.3 via Proposition 4.1. While we use radial symmetry in this step, it is not required and is only used here to simplify the proof. Indeed, a close inspection of the arguments reveals how to construct explicit sub- and supersolutions of u_c (and, therefore, h) in order to obtain such pointwise bounds without using radial symmetry.

A main idea in the proof is that, by Hölder's inequality, M_h and E_h can only be equal if h is an indicator function. Since h is even and radially decreasing, this implies that $h = \theta \mathbb{1}_{[-c,c]}$ for some θ and c, which must be θ_{crit} and c_{crit} by the rescalings described in Section 3.1. Of course, for all finite t, M_h and E_h are not exactly equal, so we must understand the stability of Hölder's inequality in our setting. This is encoded in the following general lemma, which essentially yields that if $M_h \approx E_h$, then $h \approx \theta \mathbb{1}_{[-c,c]}$.

Lemma 4.2. *Fix $d \in \mathbb{N}$, and let ω_d be the volume of the unit ball in $\mathbb{R}^d$. Suppose that $f : \mathbb{R}^d \to [0,\infty)$ is a rotationally symmetric and radially decreasing function with $\int f\,dx = 1$. Then, for every $r > 0$:*

(i) if $x_0 \in B_r$ and $\int_{B_r} f\,dx < 1$,

$$f(x_0) \geq M_f \frac{\frac{E_f}{M_f} - \int_{B_r} f\,dx}{1 - \int_{B_r} f\,dx}.$$

(ii) if $x_0 \notin B_r$,

$$f(x_0) \leq \frac{1 - \int_{B_r} f\,dx}{\omega_d(|x_0|^d - r^d)}.$$

Notice that if $\|f\|_2^2 = E_f = M_f = \|f\|_\infty$, we find $f(x) \geq M_f$ in (i) for all $|x| < (\omega_d M_f)^{-1/d}$. Then, using this with the constraint $\int f\,dx = 1$, we see that $f(x) = M_f$ for all $|x| < (\omega_d M_f)^{-1/d}$ and $f(x) \equiv 0$ for all $|x| > (\omega_d M_f)^{-1/d}$. Hence, f is an indicator function, which is the only function for which Hölder's inequality is an equality:

$$E_f = \int f^2\,dx \leq \|f\|_\infty \|f\|_1 = M_f \int f\,dx = M_f.$$

This is why we describe Lemma 4.2 as a stability estimate for Hölder's inequality. We note that when $E_f < M_f$, both (i) and (ii) are necessary to obtain precise bounds on f. This is in contrast to the special case described above of $E_f = M_f$ where only (i) is used.

Lemma 4.2 is proved in Section 4.3. We now show how to deduce Theorem 1.3 using the above results.

Proof of Theorem 1.3. First, we notice that Proposition 4.1 yields the claim about the long time limits of E_g and M_g after suitable rescaling. Next, we notice that the claim about the profile of g in long times is equivalent to showing that

$$h(\tau,y) \to \theta_{\text{crit}} \mathbb{1}_{[-c_{\text{crit}},c_{\text{crit}}]} \qquad \text{as } \tau \to \infty, \tag{4.1}$$

which we now show.

Fix any $\varepsilon > 0$. Applying Lemma 4.2.(i) and using Proposition 4.1, we see that

$$\liminf_{\tau\to\infty} \min_{|y|\leq c_{\text{crit}}-\varepsilon} h(\tau,y) \geq \theta_{\text{crit}},$$

and, recalling that h is even and radially decreasing,

$$\limsup_{\tau\to\infty} \max_{|y|\leq c_{\text{crit}}-\varepsilon} h(\tau,y) \leq \limsup_{\tau\to\infty} h(\tau,0) = \limsup_{\tau\to\infty} M_h(\tau) = \theta_{\text{crit}}.$$

The last equality follows by Proposition 4.1. Thus we have established (4.1) in the case where $|y| < c_{\text{crit}}$.

We now investigate the upper bound. Applying Lemma 4.2.(ii), we have

$$h(\tau, c_{\text{crit}} + \varepsilon) \le \frac{1 - \int_{-c_{\text{crit}}}^{c_{\text{crit}}} h(\tau, y')dy'}{2\varepsilon}.$$

From the lower bound established above, we have that $\lim \int_{-c_{\text{crit}}}^{c_{\text{crit}}} h(\tau, y)dy \to 1$ as $\tau \to \infty$. Thus, we conclude that

$$h(\tau, c_{\text{crit}} + \varepsilon) \to 0 \qquad \text{as } \tau \to \infty.$$

Since h is even and radially decreasing, we find

$$\limsup_{\tau\to\infty} \sup_{|y| \ge c_{\text{crit}}+\varepsilon} h(\tau, y) = 0,$$

which concludes the proof by the arbitrariness of ε. □

4.2. **The convergence of E_h and M_h.** In order to provide clearer references and to more closely mirror the structure of the proof, we split this into two separate propositions.

Proposition 4.3. *Under the assumptions of Proposition 4.1,*

$$\limsup_{\tau\to\infty} E_h(\tau) = \limsup_{\tau\to\infty} M_h(\tau) = \theta_{\text{crit}}.$$

Proposition 4.4. *Under the assumptions of Proposition 4.1,*

$$\liminf_{\tau\to\infty} E_h(\tau) = \liminf_{\tau\to\infty} M_h(\tau) = \theta_{\text{crit}}.$$

Clearly Propositions 4.3 and 4.4 imply Proposition 4.1. Hence, we focus on proving each of the above propositions in turn in Sections 4.2.2 and 4.2.3, respectively. Interestingly, the proof of Proposition 4.4 uses Proposition 4.3, and, hence, the order in which these propositions are proved is important.

4.2.1. *Technical lemmas.* We begin by stating and proving two crucial lemmas.

Lemma 4.5. *For any $\tau' > \tau > 0$,*

$$E_h(\tau') \le \left(\frac{\tau'+1}{\tau+1}\right)^2 E_h(\tau).$$

Proof. Multiplying (3.3) by h and integrating by parts implies that $\dot{E}_h \le \frac{2}{\tau+1} E_h$. Solving this differential inequality yields the claim: $E_h(\tau') \le \left(\frac{\tau'+1}{\tau+1}\right)^2 E_h(\tau)$. □

Lemma 4.6. *For any $\tau' > \tau > 0$,*

$$M_h(\tau') \le \left(\frac{\tau'+1}{\tau+1}\right)^2 M_h(\tau).$$

The proof of this fact is exactly as in Lemma 4.5, though using the comparison principle in place of energy estimates, so we omit its proof.

We require another technical lemma; however, its proof is quite involved and so we postpone it until Section 4.3.

Lemma 4.7. *Fix any* $\varepsilon > 0$. *There exists* $\tau_\varepsilon > 0$ *and* τ_{shift}, *depending only on* ε, *such that if* $\tau \geq \tau_\varepsilon$, *then*

$$M_h(\tau + \tau_{\text{shift}}) \leq (1+\varepsilon)E_h(\tau).$$

Moreover, there exists a sequence τ_n *tending to infinity such that*

$$\lim_{n\to\infty} M_h(\tau_n) = \lim_{n\to\infty} E_h(\tau_n) = \liminf_{\tau\to\infty} E_h = \liminf_{\tau\to\infty} M_h.$$

Roughly, we require this, in conjunction with the two previous lemmas, to find large time intervals on which M_h and E_h take approximately the same value.

4.2.2. *The upper bound on* $\limsup E_h$ *and* $\limsup M_h$*: Proposition 4.3.*

Proof of Proposition 4.3. By Lemma 3.2, $\limsup M_h = \limsup E_h$; hence, for the remainder of this proof, we focus our attention only on $\limsup E_h$. We prove the bound on E_h by contradiction. Suppose that

$$\limsup_{\tau\to\infty} E_h(\tau) = \theta > \theta_{\text{crit}}.$$

The main idea is the following. First, find a large time interval where E_h transitions between θ_{crit} to θ. When $E_h \approx \theta_{\text{crit}}$, the closeness of M_h and E_h forces h to be $O(1)$ near c_{crit} (recall Lemma 4.2). Then, as E_h grows to be approximately θ, so does h, at least on the set $[-c_{\text{crit}}, c_{\text{crit}}]$. It follows that the integral of h is at least approximately $2\theta c_{\text{crit}} > 2\theta_{\text{crit}} c_{\text{crit}} = 1$, which is a contradiction.

Before beginning we set two parameters. Fix

$$\varepsilon \in \left(0, \frac{\theta - \theta_{\text{crit}}}{100(1+\theta_{\text{crit}})}\right) \quad\text{and}\quad \underline{c} = \frac{c_{\text{crit}}}{(1+2\varepsilon)^2}. \tag{4.2}$$

such that

$$2(\underline{c} - \varepsilon)(\theta - 2\varepsilon) > 1. \tag{4.3}$$

The above is possible since $\theta > \theta_{\text{crit}}$ and $2\theta_{\text{crit}} c_{\text{crit}} = 1$.

Step 1: a large time interval in which E_h transitions from θ_{crit} to θ.

Let $\overline{\tau} > 0$ be a large time to be determined. By Proposition 1.4, we have that $\liminf E_h \leq \theta_{\text{crit}}$. Hence, we can find $\overline{\tau} < \tilde{\tau}_1 < \tau_2$ such that

$$E_h(\tilde{\tau}_1) = \theta_{\text{crit}}(1+\varepsilon), \quad E_h(\tau_2) = \theta - \varepsilon, \quad\text{and}\quad E_h(\tau) \in [\theta_{\text{crit}}(1+\varepsilon), \theta - \varepsilon] \tag{4.4}$$

for all $\tau \in [\tilde{\tau}_1, \tau_2]$. By Lemma 4.5,

$$\tau_2 \geq \tilde{\tau}_1 \sqrt{\tfrac{\theta - \varepsilon}{\theta_{\text{crit}}(1+\varepsilon)}}.$$

We note that, by the choice of ε (4.2), the coefficient of $\tilde{\tau}_1$ is greater than 1 so that $\tau_2 \geq (1+2\rho)\tilde{\tau}_1$ for some $\rho > 0$. We note that ρ can be chosen independent of ε over all ε satisfying (4.2).

Using Lemma 4.7 and increasing $\overline{\tau}$ if necessary, we find a universal constant τ_{shift} such that

$$M_h(\tilde{\tau}_1 + \tau_{\text{shift}}) \leq \frac{1+2\varepsilon}{1+\varepsilon} E_h(\tilde{\tau}_1) = (1+2\varepsilon)\theta_{\text{crit}}. \tag{4.5}$$

Let $\tau_1 = \tilde{\tau}_1 + \tau_{\text{shift}}$. Up to increasing $\overline{\tau}$ in a way depending only on ρ, we have that $(1+\rho)\tau_{\text{shift}} \leq \rho\tilde{\tau}_1$ and, hence

$$(1+\rho)\tau_1 = (1+\rho)(\tilde{\tau}_1 + \tau_{\text{shift}}) \leq (1+\rho)\tilde{\tau}_1 + \rho\tilde{\tau}_1 = (1+2\rho)\tau_1 \leq \tau_2. \tag{4.6}$$

Combining (4.5) with the fact that $E_h(\tau_1) \geq \theta_{\rm crit}(1+\varepsilon)$ by (4.4), we deduce the two following inequalities:

$$M_h(\tau_1) \leq \theta_{\rm crit}(1+2\varepsilon) \qquad \text{and} \qquad \frac{E_h(\tau_1)}{M_h(\tau_1)} \geq \frac{1+\varepsilon}{1+2\varepsilon}. \tag{4.7}$$

Step 2: $h = O(1)$ near $c_{\rm crit}$. In this step, we obtain a preliminary bound on h so that we may apply Proposition 3.3.

Applying Lemma 4.2.(i), we find

$$h(\tau_1, \underline{c}) \geq M_h(\tau_1) \frac{\frac{E_h(\tau_1)}{M_h(\tau_1)} - \int_{-\underline{c}}^{\underline{c}} h\, dy}{1 - \int_{-\underline{c}}^{\underline{c}} h\, dy}. \tag{4.8}$$

In order to obtain a lower bound on the right hand side above, we define the following auxiliary function. For any $s < E_h/M_h \leq 1$, let

$$\phi(s) = \frac{\frac{E_h(\tau_1)}{M_h(\tau_1)} - s}{1-s}.$$

Notice that ϕ is decreasing on its domain. In addition, recalling the definition of M_h, the choice of $\underline{c}$ (4.2), the fact that $2c_{\rm crit}\theta_{\rm crit} = 1$, and the bounds on M_h and E_h/M_h (4.7), yields

$$\begin{aligned}\int_{-\underline{c}}^{\underline{c}} h\, dy &\leq 2\underline{c} M_h(\tau_1) = 2\frac{c_{\rm crit}}{(1+2\varepsilon)^2} M_h(\tau_1)\\ &\leq 2\frac{c_{\rm crit}}{1+2\varepsilon}\theta_{\rm crit} = \frac{1}{1+2\varepsilon} < \frac{1+\varepsilon}{1+2\varepsilon} \leq \frac{E_h(\tau_1)}{M_h(\tau_1)}.\end{aligned}$$

Thus, $\int_{-\underline{c}}^{\underline{c}} h\, dy \leq 1/(1+2\varepsilon)$ and both quantities are in the domain of ϕ. Hence,

$$\frac{\frac{E_h(\tau_1)}{M_h(\tau_1)} - \int_{-\underline{c}}^{\underline{c}} h\, dy}{1 - \int_{-\underline{c}}^{\underline{c}} h\, dy} = \phi\left(\int_{-\underline{c}}^{\underline{c}} h\, dy\right) \geq \phi\left(\frac{1}{1+2\varepsilon}\right) = \frac{\frac{E_h(\tau_1)}{M_h(\tau_1)} - \frac{1}{1+2\varepsilon}}{1 - \frac{1}{1+2\varepsilon}}.$$

Using this in (4.8) and applying (4.7) yields

$$h(\tau_1, \underline{c}) \geq M_h(\tau_1) \frac{\frac{E_h(\tau_1)}{M_h(\tau_1)} - \frac{1}{1+2\varepsilon}}{1 - \frac{1}{1+2\varepsilon}} \geq M_h(\tau_1) \frac{\frac{1+\varepsilon}{1+2\varepsilon} - \frac{1}{1+2\varepsilon}}{1 - \frac{1}{1+2\varepsilon}} = \frac{M_h(\tau_1)}{2}.$$

By Lemma 3.1 and the fact that h is radially decreasing, we conclude that

$$\min_{|x| \leq \underline{c}} h(\tau_1, x) = h(\tau_1, \underline{c}) \gtrsim 1. \tag{4.9}$$

Step 3: h grows up to θ. Up to increasing $\underline{\tau}$, which also increases $\tau_2 - \tau_1$ (see (4.6)), we now apply Proposition 3.3 to conclude that

$$u_c(\tau_2, 0) \geq \theta - 2\varepsilon$$

for all $c \in [0, \underline{c} - \varepsilon]$. Here it was crucial that the lower bound (4.9) on h was $\mathcal{O}(1)$ and uniform over all $x \in [-\underline{c}, \underline{c}]$. Thus, we find

$$1 = \int_{-\infty}^{\infty} h(\tau_2, c) dc \geq \int_{-\underline{c}+\varepsilon}^{\underline{c}-\varepsilon} u_c(\tau_2, 0) dc \geq 2(\underline{c} - \varepsilon)(\theta - 2\varepsilon).$$

However, $2(\underline{c} - \varepsilon)(\theta - 2\varepsilon) > 1$ by (4.3). This is a contradiction, concluding the proof. □

4.2.3. *Lower bound on liminf.* We now prove Proposition 4.4, which completes the proof of Proposition 4.1. As in the previous section, we note that we do not directly use the symmetries of g_0 in a strong way here. It only arises in this section through Proposition 4.3, which was established using the symmetry.

A useful quantity in the sequel is the time average of E_h. In general, given a function $f : [0,\infty) \to \mathbb{R}$, we define its time average to be, for any $\tau > 0$,

$$\overline{f}(\tau) = \frac{1}{\tau}\int_0^\tau f(s)ds.$$

The main heuristic of the proof is the following. If E_h oscillates below $\theta_{\rm crit}$ by a fixed amount at a large time, then so does M_h, by Lemma 4.7. Hence, on $[-c_{\rm crit}, c_{\rm crit}]$, h has to be smaller than $\theta_{\rm crit}$, which implies that

$$u_{c_{\rm crit}} < \theta_{\rm crit} \qquad \text{for } z < 0.$$

On the other hand, using Proposition 4.3, the "accumulated reaction" $3\int_0^\tau E_h(s)ds = 3\tau\overline{E}_h(\tau)$ can be no larger than $3\tau\theta_{\rm crit}$ plus a small correction due to the difference in time domains between the integral and the interval on which E_h is small. Heuristically, this means the front $u_{c_{\rm crit}}$ in (3.5) cannot exceed the speed 0 in the moving frame (speed $c_{\rm crit}$ in the physical variables). This yields

$$u_{c_{\rm crit}} = o(1) \qquad \text{for } z > 0.$$

Thus, after suitably quantifying everything, we obtain the contradiction:

$$\begin{aligned}\frac{1}{2} &= \int_0^\infty h(\tau,y)\,dy = \frac{1}{\tau}\int_{-c_{\rm crit}(\tau+1)}^\infty u_{c_{\rm crit}}(\tau,z)\,dz\\ &< \underbrace{\frac{1}{\tau}\int_{-c_{\rm crit}(\tau+1)}^0 \theta_{\rm crit}\,dz}_{h<\theta_{\rm crit}\text{ on }[-c_{\rm crit},c_{\rm crit}]} + \underbrace{\frac{1}{\tau}\int_0^\infty o(1)\,dz}_{\substack{u_{c_{\rm crit}}\text{ is small}\\ \text{beyond the front}}} \approx \theta_{\rm crit}c_{\rm crit} = \frac{1}{2}.\end{aligned}$$

Proof of Proposition 4.4. Recall that $E_h \le M_h$, and hence that $\liminf E_h \le \liminf M_h$. By Proposition 4.3, we have that $\limsup M_h = \theta_{\rm crit}$. Hence, if we prove that $\liminf E_h = \theta_{\rm crit}$, we may conclude that $\liminf M_h = \theta_{\rm crit}$. As such, we consider only $\liminf E_h$ for the remainder of the proof.

We proceed by contradiction. Assume that $\liminf E_h < \theta_{\rm crit}$. By Lemma 4.7, there exists τ_n tending to infinity such that

$$\lim_{n\to\infty} M_h(\tau_n), \lim_{n\to\infty} E_h(\tau_n) < \theta_{\rm crit}.$$

Define $E_+(\tau) = \max\{E_h(\tau), \theta_{\rm crit}\}$. We note that $\overline{E}_+(\tau) \ge \theta_{\rm crit}$ for all τ and that $\lim \overline{E}_+ = \theta_{\rm crit}$. The latter follows from the fact that, by Proposition 4.3, $\limsup E_h = \theta_{\rm crit}$. Recall that $\overline{E}_+$ was defined as the time average of E_+.

We now construct a super-solution of $u_{c_{\rm crit}}$ on the parabolic domain $\mathcal{P} = (0,\infty)\times(0,\infty)$. Indeed, for $A > 0$ to be determined and any $(\tau,z) \in \overline{\mathcal{P}}$, let

$$\overline{u}(\tau,z) = A(\tau+1)\exp\left\{3\int_0^\tau E_+(s)ds - \frac{2c_{\rm crit}^2}{3}\tau - \frac{2c_{\rm crit}}{3}z - \frac{z^2}{2(\tau+1)}\right\}.$$

The first three factors in the exponential are typical of supersolutions for Fisher-KPP. The last factor is to cancel the contributions due to the $\frac{z}{\tau+1}\partial_z$ term in (3.5).

Define the parabolic domain $\mathcal{P} = (0,\infty)\times(0,\infty)$. We aim to use the comparison principle to bound $u_{c_{\rm crit}}$ from above by $\overline{u}$ on $\mathcal{P}$. To that end, we first check that $\overline{u} \ge u_{c_{\rm crit}}$ on the parabolic boundary of $\mathcal{P}$. There are two components to check: when $\tau = 0$ and when $z = 0$. In the former case, this is clearly satisfied by choosing

A sufficiently large as $u_{c_{\text{crit}}}(\tau = 0, \cdot) = g_0(c_{\text{crit}} + \cdot) \in C_c(\mathbb{R})$. In the latter case, we note that

$$\overline{u}(\tau, 0) = A(\tau+1)\exp\left\{\tau\left(3\overline{E}_+(\tau) - \frac{2c_{\text{crit}}^2}{3}\right)\right\} = A(\tau+1)\exp\left\{3\tau\left(\overline{E}_+(\tau) - \theta_{\text{crit}}\right)\right\}.$$

where the second equality follows by the definition of θ_{crit} and c_{crit} (see (1.10)). Since, by construction, $\overline{E}_+ \geq \theta_{\text{crit}}$, it follows that $\overline{u}(\tau, 0) \geq A$ for all $\tau \geq 0$. By Theorem 1.1 and its definition, $u_{c_{\text{crit}}}$ is bounded above uniformly for all τ. Hence, there is some A such that $\overline{u}(\tau, 0) \geq u_{c_{\text{crit}}}(\tau, 0)$ for all $\tau \geq 0$. We conclude that $\overline{u} \geq u_{c_{\text{crit}}}$ on the parabolic boundary of $\mathcal{P}$.

The last step is to check that $\overline{u}$ is, in fact, a supersolution of (3.5) in $\mathcal{P}$. We compute:

$$\begin{aligned}
&\overline{u}_\tau - \frac{3}{2}\Delta\overline{u} - 3\overline{u}(E_h - \overline{u}) - 2c_{\text{crit}}\partial_z\overline{u} - \frac{2}{\tau+1}\overline{u} - \frac{z}{\tau+1}\partial_z\overline{u} \\
&\quad = \overline{u}\left(\frac{1}{\tau+1} + 3E_+ - \frac{2c_{\text{crit}}^2}{3} + \frac{z^2}{2(\tau+1)^2} - \frac{3}{2}\left(\left(\frac{2c_{\text{crit}}}{3} + \frac{z}{(\tau+1)}\right)^2 - \frac{1}{\tau+1}\right)\right. \\
&\qquad\quad \left. - 3E_h + 3\overline{u} + 2c_{\text{crit}}\left(\frac{2c_{\text{crit}}}{3} + \frac{z}{\tau+1}\right) - \frac{2}{\tau+1} + \frac{2c_{\text{crit}}z}{3(\tau+1)} + \frac{z^2}{(\tau+1)^2}\right) \\
&\quad = \overline{u}\left(\frac{1}{2(\tau+1)} + 3(E_+ - E_h) + \frac{2c_{\text{crit}}z}{\tau+1} + 3\overline{u}\right) \geq 0.
\end{aligned}$$

In the inequality, we used that $E_+ \geq E_h$ and $\overline{u} \geq 0$. Hence $\overline{u}$ is a supersolution of (3.5).

Combining all the work above, we apply the comparison principle and conclude that $\overline{u} \geq u_{c_{\text{crit}}}$ on $\mathcal{P}$.

We now conclude the proof. Let

$$\overline{c} = \frac{1}{\theta_{\text{crit}} + \lim M_h(\tau_n)} > \frac{1}{\theta_{\text{crit}} + \theta_{\text{crit}}} = c_{\text{crit}}$$

Using that $\int h\, dy = 1$ and that h is even, we have that

$$\begin{aligned}
\frac{1}{2} &= \int_0^\infty h(y)\, dy = \int_0^{\overline{c}} h(y)\, dy + \frac{1}{\tau+1}\int_{(\overline{c}-c_{\text{crit}})(\tau+1)}^\infty u_{c_{\text{crit}}}(\tau, z)\, dz \\
&\leq \overline{c}M_h(\tau) + \frac{1}{\tau+1}\int_{(\overline{c}-c_{\text{crit}})(\tau+1)}^\infty \overline{u}(\tau, z)\, dz \\
&= \overline{c}M_h(\tau) + Ae^{3\tau\left(\overline{E}_+ - \frac{2c_{\text{crit}}^2}{9}\right)}\int_{(\overline{c}-c_{\text{crit}})(\tau+1)}^\infty e^{-\frac{2c_{\text{crit}}}{3}z - \frac{z^2}{2(\tau+1)}}\, dz \\
&\leq \overline{c}M_h(\tau) + \frac{3A}{2c_{\text{crit}}}\exp\left\{3\tau\left(\overline{E}_+ - \frac{2c_{\text{crit}}^2}{9} - \frac{2c_{\text{crit}}}{9}(\overline{c} - c_{\text{crit}})\frac{\tau+1}{\tau}\right)\right\}.
\end{aligned} \tag{4.10}$$

Recall that $\lim \overline{E}_+ = \theta_{\text{crit}} = 2c_{\text{crit}}^2/9$ and $\overline{c} > c_{\text{crit}}$. Evaluating the above at τ_n and taking n to infinity, we find

$$\frac{1}{2} \leq \overline{c}\lim M_h(\tau_n) + 0 = \frac{\lim M_h(\tau_n)}{\theta_{\text{crit}} + \lim M_h(\tau_n)} < \frac{1}{2},$$

where the last inequality follows because $\lim M_h(\tau_n) < \theta_{\text{crit}}$. This is a contradiction. Hence, the proof is complete. □

4.3. **Proof of technical lemmas.**

4.3.1. *The proof of Lemma 4.2.* There are two parts to prove. We show them in order.

Proof of Lemma 4.2.(i). Fix r as in the statement of the lemma. We abuse notation by denoting $f(r) = f(x)$ for any $|x| = r$. We compute:

$$E_f \le \int_{B_r} f^2\,dx + \int_{B_r^c} f^2\,dx \le M_f \int_{B_r} f\,dx + f(r)\int_{B_r^c} f\,dx$$
$$= M_f \int_{B_r} f\,dx + f(r)\left(1 - \int_{B_r} f\,dx\right).$$

Recall that $\int_{B_r} f dx < 1$. Thus, rearranging the above yields

$$M_f \frac{\frac{E_f}{M_f} - \int_{B_r} f\,dx}{1 - \int_{B_r} f\,dx} = \frac{E_g - M_f \int_{B_r} f\,dx}{1 - \int_{B_r} f\,dx} \le f(r).$$

This completes the proof. □

Proof of Lemma 4.2.(ii). First we compute:

$$1 = \int f\,dx \ge \int_{B_r} f\,dx + \int_{B_{|x_0|}\setminus B_r} f\,dx \ge \int_{B_r} f\,dx + \omega_d(|x_0|^d - r^d)f(x_0).$$

The last inequality uses the fact that f is rotationally symmetric and radially decreasing. The proof is concluded by rearranging the above inequality. □

4.3.2. *The proof of Lemma 4.7.*

Proof. We begin with the first claim. Fix τ_ε and τ_{shift} to be determined. In order to prove this, we first note that E_h cannot stray too far above $E_h(\tau)$. Indeed, by Lemma 4.5, we have

$$E_h(\tau') \le \left(\frac{\tau'}{\tau}\right)^2 E_h(\tau).$$

Hence, there is $\mu_\varepsilon > 0$ such that if $\tau_{\text{shift}} < \mu_\varepsilon \tau$ then $E_h(\tau') \le (1 + \varepsilon/4)E_h(\tau)$ for all $\tau' \in [\tau, \tau + \tau_{\text{shift}}]$. Notice that μ_ε depends only on ε.

We now construct a simple super-solution that "pushes" M_h down to E_h. Indeed, for $\tau' \ge \tau$, let

$$\overline{h}(\tau', y) = \left(1 + \frac{\varepsilon}{2}\right)\left[M_h(\tau)e^{-\gamma(\tau'-\tau)} + \left(1 - e^{-\gamma(\tau'-\tau)}\right)E_h(\tau)\right],$$

where $\gamma > 0$ is a constant to be determined. Notice that $\overline{h}(\tau, y) \ge M_h(\tau) \ge h(\tau, y)$ for every y.

We next show that $\overline{h}$ is a supersolution of (3.3). Indeed, for any $\tau' \in [\tau, \tau + \tau_{\text{shift}}]$,

$$\partial_{\tau'}\overline{h} - \frac{3}{2(\tau'+1)^2}\Delta\overline{h} - 3\overline{h}(E_h(\tau') - \overline{h}) - \frac{2}{\tau'+1}\left(\overline{h} + y\partial_y\overline{h}\right)$$
$$= \gamma(1+\varepsilon/2)E_h(\tau) + 3\overline{h}\left(\overline{h} - \frac{\gamma}{3} - E_h(\tau') - \frac{2}{3(\tau'+1)}\right)$$
$$\ge \gamma(1+\varepsilon/2)E_h(\tau) + 3\overline{h}\left(\overline{h} - \frac{\gamma}{3} - \left(1+\frac{\varepsilon}{4}\right)E_h(\tau) - \frac{2}{3(\tau_\varepsilon+1)}\right).$$

In the last step, we used that $E_h(\tau') \leq (1+\varepsilon/4)E_h(\tau)$ for all $\tau' \in [\tau, \tau+\tau_{\text{shift}}]$. Recall that $M_h(\tau) \geq E_h(\tau)$. Thus, $\overline{h} \geq (1+\varepsilon/2)E_h(\tau)$, and we find

$$\partial_\tau \overline{h} - \frac{3}{2(\tau'+1)^2}\Delta \overline{h} - 3\overline{h}(E_h(\tau') - \overline{h}) - \frac{2}{\tau'+1}\left(\overline{h} + y\partial_y \overline{h}\right)$$
$$\geq \gamma(1+\varepsilon/2)E_h(\tau) + 3\overline{h}\left(\frac{\varepsilon}{4}E_h(\tau) - \frac{\gamma}{3} - \frac{2}{3(\tau_\varepsilon+1)}\right).$$

From Theorem 1.1, there exists $C > 0$ such that $C^{-1} < E_h(\tau) \leq M_h(\tau) < C$ for all $\tau \geq 1$. Thus, choosing $\gamma = \varepsilon/(24C)$ and increasing τ_ε if necessary, we find

$$\partial_\tau \overline{h} - \frac{3}{2(\tau'+1)^2}\Delta \overline{h} - 3\overline{h}(E_h(\tau') - \overline{h}) - \frac{2}{\tau'+1}\left(\overline{h} + y\partial_y \overline{h}\right) \geq 0.$$

It follows from the comparison principle that $\overline{h}(\tau', \cdot) \geq h(\tau', \cdot)$ for all $\tau' \in [\tau, \tau + \tau_{\text{shift}}]$. Thus, we conclude that

$$M_h(\tau+\tau_{\text{shift}}) \leq \overline{h}(\tau+\tau_{\text{shift}}, 0) \leq (1+\varepsilon/2)Ce^{-\frac{\varepsilon}{24C}\tau_{\text{shift}}} + (1+\varepsilon/2)E_h(\tau).$$

It is clear that if τ_{shift} is chosen large enough, depending only on ε, then the right hand side above is bounded by $(1+\varepsilon)E_h(\tau)$, as desired. This finishes the proof of the first claim.

We now prove the second claim. First note that $\liminf E_h \leq \liminf M_h$ since $E_h \leq M_h$. Hence, if we find the sequence τ_n in the statement of the lemma, then it follows that $\liminf E_h = \liminf M_h$ and the proof is concluded. In order to find such a sequence, it is enough to establish the following claim:

for every $\varepsilon, \tau_0 > 0$, there is $\tau_1 \geq \tau_0$ such that $M_h(\tau_1), E_h(\tau_1) \leq (1+\varepsilon)^2 \liminf E_h$.

It is clear that the procedure used to establish the first claim yields the desired τ_1. Indeed, after choosing, in the notation above, $\tau_\varepsilon > \tau_0$, $\tau \geq \tau_\varepsilon$ such that $E_h(\tau) \leq (1+\varepsilon)\liminf E_h$, and letting $\tau_1 = \tau + \tau_{\text{shift}}$, we find

$$M_h(\tau_1) \leq (1+\varepsilon)E_h(\tau) \leq (1+\varepsilon)^2 \liminf E_h \quad \text{and}$$
$$E_h(\tau_1) \leq (1+\varepsilon/4)E_h(\tau) \leq (1+\varepsilon/4)(1+\varepsilon)\liminf E_h.$$

This concludes the proof. □

4.3.3. *The proof of Proposition 3.3.* We break this proposition into two smaller lemmas. The first (Lemma 4.8) shows that u_c remains locally uniformly bounded below over the entire time interval $[\tau_0, \tau_1]$, while the second (Lemma 4.9) shows that u_c grows from this initial weak lower bound to the claimed value $E(\tau_1) - \varepsilon$ over a terminal boundary layer $[\tau_1(1-\beta_\varepsilon), \tau_1]$ for $\beta_\varepsilon \approx \varepsilon$. In this final step, it is crucial that E_h grows "slowly" as in Lemma 4.5. We state these lemmas here and then show how to use them to conclude Proposition 3.3. Afterwards, we prove them.

Our first lemma is below. Similar results exist in the literature (see, e.g., a very general work of Berestycki, Hamel, and Nadin [2]); however, we are unable to find one that, applied out-of-the-box, yields the result below with the uniformity in all parameters and allows for the particular assumptions that we require. As such, we provide a proof below, although the ideas are standard.

Lemma 4.8. *Fix any $T_0 < T_1$, μ, δ, and c. Suppose that $R > 1 + 10\pi/\sqrt{\delta}$ and*

$$\begin{cases} \partial_\tau u = \frac{3}{2}\Delta u + 3u(f(\tau) - u) + 2c\partial_z u + a(\tau,z)\partial_z u & \text{in } (T_0, T_1) \times B_{2R}, \\ u \geq \mu \mathbb{1}_{B_R} & \text{on } \{T_0\} \times B_{2R}(0), \end{cases} \tag{4.11}$$

where f and a are continuous and f satisfies

$$18 f(\tau) - 4c^2 > \delta \quad \text{for all } \tau \in [T_0, T_1]. \tag{4.12}$$

If $\|a\|_{L^\infty([T_0,T_1]\times B_{2R})}$ is sufficiently small depending only on δ and c, then there exists $C_{\delta,f}$, depending only on δ and $\|f\|_\infty$, and T, depending only on $\|f\|_\infty$, $\|a\|_\infty$, μ, δ, and R, such that, if $\tau \in [T_0+T, T_1]$, then

$$\min_{z\in B_R} u(\tau,z) \geq \frac{1}{C_{\delta,f}}.$$

The heuristic of Lemma 4.8 is that if the reaction f, advection a, and speed c satisfy a sub-minimal speed condition (4.12) in a uniform way, then u propagates at speed c in the sense that it remains $O(1)$ regardless of the particular fluctuations of f and g. We note that it is crucial for our estimates that this is uniform in c and f.

Our next lemma is the following.

Lemma 4.9. *Fix any τ_0, τ_1, μ, R, ρ, ε, δ, and c. Suppose that u solves*

$$\begin{cases} \partial_\tau u = \frac{3}{2}\Delta u + 3u(\rho-u) + 2c\partial_z u + \frac{2}{\tau+1}u + \frac{z}{\tau+1}\partial_z u & \text{in } (\tau_0,\tau_1)\times B_R,\\ u=\mu & \text{on } \{\tau_0\}\times B_R,\\ u=0 & \text{on } (\tau_0,\tau_1)\times \partial B_R \end{cases}$$

and $18\rho - 4c^2 > \delta$. If τ_0 and R are sufficiently large, depending only on δ and ϵ, then there exists $\underline{\tau}$, depending only on μ, ε, and δ, such that, if $\tau_1-\tau_0 \geq \underline{\tau}$, then

$$u(\tau_1,0) \geq \rho - \varepsilon.$$

The main difference between the following and Lemma 4.8 is that the lower bound at the final time is much stronger at the cost of having a constant reaction term 3ρ. In particular, as long as the sub-minimal speed condition ($18\rho - 4c^2 > \delta$) is satisfied and τ is sufficiently large that the last two terms in (3.5) are negligible, then u must grow to the carrying capacity ρ *uniformly* among all speeds.

We first show how to conclude Proposition 3.3 from Lemmas 4.8 and 4.9.

Proof of Proposition 3.3. First we notice that, by Lemma 4.5, there is $\beta_\varepsilon \in (0,1)$, depending only on ε, such that

$$E_h(\tau_1) \leq \left(\frac{\tau_1+1}{\tau+1}\right)^2 E_h(\tau) \leq \frac{1}{(1-\beta_\varepsilon)^2}E_h(\tau) \leq \frac{1}{1-\frac{\varepsilon}{2E_h(\tau_1)}}E_h(\tau),$$

for all $\tau \in [\tau_1(1-\beta_\varepsilon),\tau_1]$. Recall that $E_h(\tau_1)$ is bounded uniformly above and below by Theorem 1.1, and, hence, β_ε can be chosen independent of $E_h(\tau_1)$. Thus,

$$E_h(\tau) \geq E_h(\tau_1) - \frac{\varepsilon}{2} \qquad \text{for all } \tau \in [\tau_1(1-\beta_\varepsilon),\tau_1]. \tag{4.13}$$

Fix $R > 1 + 10\pi/\sqrt{\delta}$ large enough such that Lemma 4.9 can be applied with the choice

$$\rho = \min_{\tau'\in[\tau_1(1-\beta_\varepsilon),\tau_1]} E_h(\tau') \quad \left(\geq \max\left\{E(\tau_1)-\frac{\varepsilon}{2}, \frac{1}{C}\right\}\right) \tag{4.14}$$

and the ε in Lemma 4.9 taking the value of $\varepsilon/2$ in the current proof. As $18E_h - 4c^2 > \delta$, by assumption, it follows that $18\rho - 4c^2 > \delta$.

Fix an intermediate time $\tau_{\text{int}} \in [\tau_1(1-\beta_\varepsilon),\tau_1]$ to be determined. Applying Lemma 4.8 on the time interval $[\tau_0,\tau_{\text{int}}]$, we find that, up to increasing τ_0 if necessary (to make the coefficients in (3.5) sufficiently small) and choosing $\tau_{\text{int}}-\tau_0$ sufficiently large, there is $C_\delta > 0$ such that

$$u_c(\tau_{\text{int}}(1-\beta_\varepsilon),z) \geq \frac{1}{C_\delta} \qquad \text{for all } z \in B_R. \tag{4.15}$$

We now let $\underline{u}$ be the solution of

$$\begin{cases} \partial_\tau \underline{u} = \frac{3}{2}\Delta \underline{u} + 3\underline{u}(\rho - \underline{u}) + 2c\partial_z \underline{u} + \frac{2}{\tau+1}\underline{u} + \frac{z}{\tau+1}\partial_z \underline{u}, & \text{in } (\tau_{\text{int}}, \infty) \times B_R, \\ \underline{u} = \frac{1}{C_\delta}, & \text{on } \{\tau_{\text{int}}\} \times B_R, \\ \underline{u} = 0 & \text{on } (\tau_{\text{int}}, \infty) \times \partial B_R. \end{cases}$$

By (4.15), $\underline{u} \leq u_c$ on $\{\tau_{\text{int}}\} \times B_R$. By the choice of boundary conditions $\underline{u} \leq u_c$ on $[\tau_{\text{int}}, \tau_1] \times \partial B_R$. By our choice of ρ and by (4.13), $\underline{u}$ is a subsolution of (3.5). Thus, the comparison principle implies that $\underline{u} \leq u_c$ on $[\tau_{\text{int}}, \tau_1] \times B_R$.

Up to increasing $\tau_1 - \tau_0$ (recall that τ_{int} has already been fixed relative to τ_0), we have that $\tau_1 - \tau_{\text{int}}$, the length of the time interval $[\tau_{\text{int}}, \tau_1]$, is sufficiently large to apply Lemma 4.9 and conclude that

$$E(\tau_1) - \varepsilon \leq \rho - \frac{\varepsilon}{2} \leq \underline{u}(\tau_1, 0) \leq u_c(\tau_1, 0).$$

The first inequality is by (4.14), the second by Lemma 4.9, and the third by the ordering $\underline{u} \leq u_c$ outlined in the previous paragraph. This concludes the proof. □

We now prove the two lemmas.

Proof of Lemma 4.8. Let $\rho = 10\pi/\sqrt{\delta}$. By assumption $R + \rho + 1 < 2R$. By the Harnack inequality, there exists $\underline{\mu}$, depending only on $\|f\|_{L^\infty[T_0,T_1]}$, $\|g\|_{L^\infty([T_0,T_1]\times B_{2R})}$, μ, and ρ, such that

$$u(T_0 + 1, \cdot) \geq \underline{\mu} \mathbb{1}_{B_{R+\rho}}.$$

In particular, we have the bound $u(T_0 + 1, \cdot) \geq \underline{\mu}\mathbb{1}_{B_\rho(z_0)}$ for any $z_0 \in B_R$. Our goal is to obtain a lower bound on $u(T_1, z_0)$ via the construction of a subsolution $\underline{u}$ on $[T_0 + 1, T_1] \times B_\rho(z_0)$. For notational ease, we assume $z_0 = 0$ and $T_0 + 1 = 0$ for the remainder of the proof since all arguments are translation invariant.

We now construct $\underline{u}$. Let $\lambda > 0$ and $\phi : [0, T_1] \to \mathbb{R}$ be a constant and function determined, respectively. Then let

$$\underline{u}(\tau, z) = \underline{\mu} e^{-\lambda(\rho+z)} e^{\int_0^\tau \phi(s)ds} \cos\left(\frac{z\pi}{2\rho}\right)^2.$$

Up to decreasing $\underline{\mu}$, we can choose ϕ in the sequel such that

$$\underline{u} \leq \frac{\delta}{100} \qquad \text{on } [0, T_1] \times B_\rho. \tag{4.16}$$

Notice that $\underline{u} \leq \underline{\mu} \leq u$ on $\{0\} \times B_\rho$. Moreover, $\underline{u} \leq u$ on ∂B_ρ since u is positive. Hence, we need only check that $\underline{u}$ is a subsolution of (4.11) in order to conclude, via the comparison principle, that $\underline{u} \leq u$ on $[0, T_1] \times B_\rho$. To this end, we compute that, using (4.16),

$$\begin{aligned}
&\partial_\tau \underline{u} - \frac{3}{2}\Delta \underline{u} - 3\underline{u}(f - \underline{u}) - (2c + a)\partial_z \underline{u} \\
&\leq \phi \underline{u} - \frac{3}{2}\Big(\lambda^2 \underline{u} + \frac{2\pi\lambda \underline{\mu} e^{-\lambda(\rho+z)+\int_0^\tau \phi(s)ds}}{\rho} \cos\sin + \frac{\pi^2 \underline{\mu} e^{-\lambda(\rho+z)+\int_0^\tau \phi(s)ds}}{2\rho^2}\big(\sin^2 - \cos^2\big)\Big) \\
&\qquad - 3\underline{u}\left(f - \frac{\delta}{100}\right) + (2c + a)\left(\lambda \underline{u} + \frac{\pi \underline{\mu} e^{-\lambda(\rho+z)+\int_0^\tau \phi(s)ds}}{\rho} \cos\sin\right) \\
&= -\underline{u}\left(\frac{3\lambda^2}{2} + 3f - 3\frac{\delta}{100} - \frac{\pi^2}{2\rho^2} - (2c + a)\lambda - \phi\right) + \underline{u}\frac{\sin}{\cos}\frac{(2c + a - 3\lambda)\pi}{\rho} - \underline{u}\frac{\sin^2}{\cos^2}\frac{\pi^2}{2\rho^2}.
\end{aligned}$$

We now select $\lambda = 2c/3$. Using this choice of λ and condition (4.12) to find

$$\begin{aligned}&\partial_\tau \underline{u} - \frac{3}{2}\Delta \underline{u} - 3\underline{u}(f-\underline{u}) - (2c+a)\partial_z \underline{u}\\ &\le -\underline{u}\left(\frac{\delta}{10} - \frac{\pi^2}{2\rho^2} - \frac{2ca}{3} - \phi\right) + \underline{u}\frac{\sin}{\cos}\frac{a\pi 2c}{3\rho} - \underline{u}\frac{\sin^2}{\cos^2}\frac{\pi^2}{2\rho^2}.\end{aligned}$$

Next, see that, by Young's inequality

$$\frac{\sin}{\cos}\frac{a\pi 2c}{3\rho} \le \frac{\sin^2}{\cos^2}\frac{\pi^2}{2\rho^2} + \frac{2a^2c^2}{9}.$$

Thus,

$$\partial_\tau \underline{u} - \frac{3}{2}\Delta \underline{u} - 3\underline{u}(f-\underline{u}) - (2c+a)\partial_z \underline{u} \le -\underline{u}\left(\frac{\delta}{10} - \frac{\pi^2}{2\rho^2} - \frac{2ca}{3} - \phi - \frac{2a^2c^2}{9}\right). \tag{4.17}$$

Using the choice $\rho = 10\pi/\sqrt{\delta}$ and letting

$$\phi(\tau) = \frac{\delta}{100}\mathbb{1}_{[0,T]}(\tau) \qquad \text{with } T = \frac{100}{\delta}\log\left(\max\left\{1, \frac{\delta}{100\underline{\mu}}\right\}\right),$$

(clearly $T \le T_1$ if T_1 is sufficiently large), we find

$$\partial_\tau \underline{u} - \frac{3}{2}\Delta \underline{u} - 3\underline{u}(f-\underline{u}) - (2c+g)\partial_z \underline{u} \le -\underline{u}\left(\frac{\delta}{25} - \frac{2ca}{3} - \frac{2a^2c^2}{9}\right)$$

Thus $\partial_\tau \underline{u} - \frac{3}{2}\Delta \underline{u} - 3\underline{u}(f-\underline{u}) - (2c+a)\partial_z \underline{u} \le 0$ if $\|a\|_\infty$ is sufficiently small. We conclude, via the comparison principle, that $\underline{u} \le u$. Using the form of ϕ yields

$$\underline{u}(T_1, 0) \ge \frac{\delta}{100}e^{-\lambda\rho}.$$

Recall that ρ is given explicitly in terms of δ and λ is bounded by $3\sqrt{\|f\|_\infty/2}$ due to (4.12). Hence $u(T_1, 0)$ is bounded below in a way only depending on δ and $\|f\|_\infty$, as claimed. This concludes the proof. □

Proof of Lemma 4.9. We prove this by contradiction. Suppose there exists $\underline{\tau}_n$, $\overline{\tau}_n$, and $R_n \le \sqrt{\underline{\tau}_n}$ all tending to infinity such that $\underline{u}(\underline{\tau}_n, \cdot) \ge \mu$ on $B_{R_n}(0)$ and $\underline{u}(\underline{\tau}_n + \overline{\tau}_n, 0) < \rho - \varepsilon$.

Let $u_n(\tau, z) = \underline{u}(\underline{\tau}_n + \overline{\tau}_n + \tau, z)$. By the maximum principle, $\underline{u} \le \max\{\mu, \rho + 2/3\}$. Thus u_n is uniformly bounded in L^∞, which, using parabolic regularity theory[d], yields a uniform bound in $C^{2+\alpha}_{\text{parabolic}}$ for any $\alpha \in (0,1)$. Thus, there exists u_∞ such that $u_n \to u_\infty$ locally uniformly in $C^2_{\text{parabolic}}$ and u_∞ solves

$$\partial_\tau u_\infty = \frac{3}{2}\Delta u_\infty + 3u_\infty(\rho - u_\infty) + 2c\partial_z u_\infty \qquad \text{in } \mathbb{R}\times\mathbb{R}.$$

Applying Lemma 4.8, we have that $u_\infty \ge \underline{\mu}$ on $\mathbb{R}\times\mathbb{R}$ for some $\underline{\mu} > 0$.

Notice that

$$u_\infty(0,0) = \lim_{n\to\infty} \underline{u}(\underline{\tau}_n + \overline{\tau}_n, 0) \le \rho - \varepsilon,$$

where the inequality follows by assumption, it follows that $\inf_{\mathbb{R}\times\mathbb{R}} u_\infty \in [\underline{\mu}, \rho - \varepsilon]$. There are two cases to consider.

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[d]See, e.g., [18, Theorem 4.9, Theorem 6.9] which correspond to Schauder and De Giorgi estimates, respectively. Also, $C^{2+\alpha}_{\text{parabolic}}$ mentioned here refers to the standard parabolic Hölder spaces. Roughly, this corresponds to $C^{1+\alpha/2}$ regularity in t and $C^{2+\alpha}$ regularity in z.

Case one: u_∞ achieves its minimum at a point $(\tau_{\min}, z_{\min})$. Then, $u_\infty(\tau_{\min}, z_{\min}) \in [\underline{\mu}, \rho - \varepsilon]$ and, at $(\tau_{\min}, z_{\min})$, we have

$$0 \geq \partial_\tau u_\infty - \frac{3}{2}\Delta u_\infty - 2c\partial_z u_\infty = 3u_\infty(\rho - u_\infty).$$

This is a contradiction since $u_\infty(\tau_{\min}, z_{\min}) \in [\underline{\mu}, \rho - \varepsilon]$ and, hence, at $(\tau_{\min}, z_{\min})$,

$$u_\infty(\rho - u_\infty) \geq \min\{\underline{\mu}(\rho - \underline{\mu}), \varepsilon(\rho - \varepsilon)\} > 0.$$

Thus, this case cannot occur.

Case two: u_∞ does not achieve a minimum. Here we use a fairly standard re-centering trick. Indeed, let (τ_n, z_n) be a sequence such that $u_\infty(\tau_n, z_n) \to \inf u_\infty$ as $n \to \infty$. Let

$$v_n(\tau, z) = u_\infty(\tau_n + \tau, z_n + z) \qquad \text{for all } (\tau, z) \in \mathbb{R} \times \mathbb{R}.$$

We conclude as above that $v_n \to v_\infty$ for some smooth function v_∞ solving the same equation as u_∞. In addition, $v_\infty(0,0) = \inf v_\infty \in [\underline{\mu}, \rho - \varepsilon]$. At this point, the proof proceeds exactly as in case one, leading to a contradiction.

Since we obtain a contradiction in all cases, the proof is finished. □

5. Long time dynamics of solutions in higher dimensions

We begin with the (somewhat simpler) proof of the moments estimate Theorem 1.5. Afterwards we proceed with the construction of non-Gaussian self-similar solutions for the equation in two dimensions.

5.1. Higher dimensional moments estimates.

As in the one-dimensional case, the key estimate to establish is an upper bound on E_g. We state this here and prove it in Section 5.1.2.

Proposition 5.1. *Suppose that $d \geq 2$. Then, for all $t > 0$,*

$$E_g(t) \lesssim \left(t + E_g(0)^{-\frac{2}{d}}\right)^{-\frac{d}{2}}.$$

We note that, in the course of establishing Theorem 1.5 from Proposition 5.1, we obtain a similar bound on M_g.

5.1.1. *Moment bounds.* We show how to conclude bounds on the moments of g using Proposition 5.1. The main difficulty is in establishing the upper bounds on the moments as all other conclusions in Theorem 1.5 are either obtained along the way or a simple consequence of Proposition 5.1 and the moment upper bound.

We establish these upper bounds through the construction of a supersolution. In the one dimensional case, this was made up of a solution to the heat equation with an exponential integrating factor depending on E_g. Trying to apply this directly here yields an issue in the 2d case: $\int_1^t E_g(s)ds$ grows logarithmically in t (it is bounded if $d > 2$). As such, a simple proof mirroring the 1d proof closely can be established in dimensions $d \geq 3$ but will not be sharp when $d = 2$. Thus, we mainly focus below on the case $d = 2$. The key step here is to obtain and use a lower bound on the $R * g$ term when $|x|^2/t = \mathcal{O}(1)$.

Proof of Theorem 1.5. We begin with the proof of the upper bound on the moments in Theorem 1.5. We claim that there exists $A > 0$ such that for all $t \geq 0$,

$$g(t,x) \leq \frac{A}{(t+1)^{\frac{d}{2}}} e^{-\frac{x^2}{A(t+1)}}. \tag{5.1}$$

Before establishing this, we note that the proof of the upper bound follows immediately via a direct computation using (5.1) (indeed, this is, up to a time change, equivalent to the fact that the pth moment of a Brownian motion is $\mathcal{O}(t^{p/2})$).

We now establish (5.1) via the construction of a supersolution. Let

$$\overline{g}(t,x) = \frac{A}{t+1} e^{-\frac{x^2}{2A(t+1)}} \qquad \text{if } d = 2,$$
$$\overline{g}(t,x) = \frac{A}{(t+1)^{d/2}} e^{\int_0^t E_g(s)\,ds - \frac{x^2}{2(t+1)}} \qquad \text{if } d \geq 3.$$

Again, note that the choice of $\overline{g}$ for $d \geq 3$ would yield an extra logarithmic factor were we to use it in the case $d = 2$ as $E_g(s) \approx (s+1)^{-1}$ and, hence, would not yield the sharp asymptotics. In fact, that $\overline{g}$ is a supersolution when $d \geq 3$ is clear as it is simply a solution to the heat equation along with an integrating factor. Thus, we focus our efforts on the case $d = 2$.

Up to increasing A, we have that $\overline{g} > g$ at $t = 0$ since g_0 is compactly supported and bounded. We show that $\overline{g} \geq g$ on $(0,\infty) \times \mathbb{R}^2$ by contradiction, taking $t_0 > 0$ to be the first time that $\overline{g}$ and g "touch." Let x_0 be the point at which they touch. It follows that, at (t_0, x_0),

$$\partial_t \overline{g} - \frac{1}{2}\Delta \overline{g} \leq \partial_t g - \frac{1}{2}\Delta g. \tag{5.2}$$

Our goal is to use (5.2) to obtain a contradiction.

We claim that, at (t_0, x_0),

$$\partial_t \overline{g} - \frac{1}{2}\Delta \overline{g} - \overline{g}\left(E_g - R * g\right) > 0. \tag{5.3}$$

Postponing the proof of (5.3) momentarily, we show how to conclude using it. Indeed, at (t_0, x_0), we have, from (5.2), (1.1), and then (5.3),

$$\partial_t \overline{g} - \frac{1}{2}\Delta \overline{g} \leq \partial_t g - \frac{1}{2}\Delta g = g\left(E_g - R * g\right) = \overline{g}\left(E_g - R * g\right) < \partial_t \overline{g} - \frac{1}{2}\Delta \overline{g}.$$

This is clearly a contradicton. Hence, (5.1) follows from (5.3), which we prove now.

With arguments reminiscent of those in Section 2 (see, e.g., the proof of Lemma 2.3), it is easy to check that, up to further increasing A, we may assume that $t_0 > 1$. Next, using arguments exactly as in Lemma 2.2 (cf. (2.7)), there exists $C_R > 0$, depending only on R, such that

$$R * g(t_0, x_0) \geq \frac{A}{C_R(t_0+1)} e^{-\frac{2x_0^2}{A(t_0+1)}}. \tag{5.4}$$

The above relies on the fact that $g(t_0,\cdot) \leq \overline{g}(t_0,\cdot) \leq (1+t_0)^{-1}$ and that

$$g(t_0,x_0) = \overline{g}(t_0,x_0) = \frac{A}{t_0+1} e^{-\frac{x_0^2}{A(t_0+1)}},$$

which hold by contradictory assumption.

Notice that, for any (t,x),

$$\partial_t \overline{g} - \frac{1}{2}\Delta \overline{g} = \overline{g}\left(\frac{x^2}{2A(t+1)^2}\left(1 - \frac{1}{A}\right) + \frac{1}{t+1}\left(-1 + \frac{1}{A}\right)\right).$$

Hence, using Proposition 5.1 and (5.4), we find, at (t_0, x_0),

$$
\begin{aligned}
&\partial_t \overline{g} - \frac{1}{2}\Delta \overline{g} - \overline{g}\,(E_g - R * g) \\
&\geq \overline{g}\left(\frac{x^2}{2A(t_0+1)^2}\left(1 - \frac{1}{A}\right) + \frac{1}{t_0+1}\left(-1 + \frac{1}{A}\right) - \frac{C}{(t_0+1)} + \frac{A e^{-\frac{2x_0^2}{A(t_0+1)}}}{C_R(t_0+1)}\right).
\end{aligned} \tag{5.5}
$$

where C is the implied constant in Proposition 5.1.

Increasing A if necessary, we have

$$
A > \max\{2, C_R(C+1)e^{8(C+1)}\}. \tag{5.6}
$$

We consider first the case when $|x_0|^2 \geq 4(C+1)A(t_0+1)$. Then (5.5) becomes

$$
\partial_t \overline{g} - \frac{1}{2}\Delta \overline{g} - \overline{g}\,(E_g - \overline{g}) > \overline{g}\left(\frac{2(C+1)}{t_0+1} - \frac{1}{t_0+1} - \frac{C}{(t_0+1)}\right) \geq 0.
$$

On the other hand, if $|x_0|^2 \leq 4(C+1)A(t_0+1)$, then (5.5) becomes

$$
\partial_t \overline{g} - \frac{1}{2}\Delta \overline{g} - \overline{g}\,(E_g - \overline{g}) > \overline{g}\left(-\frac{1}{t_0+1} - \frac{C}{(t_0+1)} + \frac{A}{C_R(t_0+1)}e^{-8(C+1)}\right) \geq 0.
$$

where the last line follows from (5.6). Thus, (5.3) follows from the two cases above. This finishes the proof of (5.1).

The proof of the upper bound on M_g follows from the fact that $g \leq \overline{g}$, while the upper bound on E_g is the content of Proposition 5.1. The proof of the lower bounds on the moments and on the L^2- and L^∞-norms of $g(t,\cdot)$ follows exactly as in the proof of Theorem 1.1 using the bounds established above. As such, we omit the details. The proof is, thus, finished. □

5.1.2. *The upper bound on g.* We now establish the key upper bound on E_g on which the previous section depends. We use classical methods based on the Nash inequality to order to establish the $t^{-d/2}$ decay of E_g; however, the Nash inequality must be slightly adapted to our macroscopic quantities E_g and D_g. We state this updated Nash inequality here, its proof is left until after the proof of Proposition 5.1.

Lemma 5.2. *Let $h \in H^1(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ and suppose that R satisfies (1.4) or $R = \delta$. Then*

$$
E_h^{1+2/d} \lesssim \|h\|_1^{4/d} D_h.
$$

Proof of Proposition 5.1. Applying the convolved Nash inequality from Lemma 5.2, we find that

$$
E_g^{1+2/d} \lesssim \|g\|_1^{4/d} D_g = D_g.
$$

Using this inequality in (2.12), we have

$$
\dot{E}_g = -D_g - 2\int g(R*g - E_g)^2\,dx \leq -D_g \lesssim -E_g^{1+2/d}. \tag{5.7}
$$

Integrating this in time, we find

$$
E_g(t) \lesssim \left(t + E_g(0)^{-2/d}\right)^{-d/2}, \tag{5.8}
$$

which concludes the proof. □

We now prove Lemma 5.2. The proof given is almost exactly as in the classical case; however, the new $\hat{R}$ terms in the Fourier transform must be addressed.

Proof of Lemma 5.2. We note that the case $R = \delta$ is the standard Nash inequality; hence we omit its proof and focus only on the case when R is continuous and satisfies (1.4). We use the Fourier transform: for any function h, we denote its Fourier transform by $\hat{h}$. We begin with Plancherel's identity

$$E_h = \langle h, R * h \rangle = \int \hat{h}\overline{\widehat{R * h}}\, d\xi = \int |\hat{h}|^2 \overline{\hat{R}}\, d\xi.$$

Notice that, due to the form of R in terms of ϕ (see (1.4)), we have $\hat{R} = \hat{\phi}^2$ and since ϕ is assumed to be even, it actually implies $\hat{R} = |\hat{\phi}|^2$. In other words, $\hat{R}$ is real valued and non-negative (this is not surprising, because R comes from the covariance function of a Gaussian process so it is positive definite).

Hence, for L to be chosen, we have

$$\begin{aligned} E_h &\leq \int_{B_L} |\hat{h}|^2 \hat{R}\, d\xi + \int_{B_L^c} \frac{|\xi|^2}{L^2} |\hat{h}|^2 \hat{R}\, d\xi \lesssim L^d \|\hat{h}\|_\infty^2 \|\hat{R}\|_\infty + \int_{B_L^c} \frac{|\xi|^2}{L^2} |\hat{h}|^2 \hat{R}\, d\xi \\ &\leq L^d \|h\|_1^2 + \frac{1}{L^2} \langle \nabla h, R * \nabla h \rangle = L^d \|h\|_1^2 + \frac{1}{L^2} D_h. \end{aligned}$$

The proof is then finished by choosing $L^{d+2} = D_h / \|h\|_1^2$. □

5.2. **Gaussian and non-Gaussian self-similar dynamics.** The goal of this section is to prove Theorem 1.6. Recall that we assumed the convolution kernel $R = \delta$. In order to attack this problem, we begin with a few transformations of the function g. We use self-similar variables here; that is, we define

$$G(\tau, y) = e^{\frac{d}{2}\tau} g\left(e^\tau, e^{\tau/2} y\right) \tag{5.9}$$

and find that

$$G_\tau = \frac{d}{2} G + \frac{1}{2} \Delta G + G e^{-\frac{d-2}{2}\tau} \left(\|G\|_2^2 - G\right) + \frac{y}{2} \cdot \nabla G. \tag{5.10}$$

First, we note that, in the above equation, the linear operator $\frac{1}{2}\Delta + \frac{y}{2} \cdot \nabla + \frac{d}{2}$ is actually the adjoint of $\frac{1}{2}\Delta - \frac{y}{2} \cdot \nabla$, which is the generator of an Ornstein–Uhlenbeck process with the standard Gaussian invariant density $(2\pi)^{-d/2} \exp(-|y|^2/2)$. Thus, without the nonlinear term, the above equation is actually the Fokker-Planck equation for the Ornstein-Uhlenbeck process which converges to its invariant density. We now see the reason for the different behavior in $d = 2$: when $d \geq 3$, the nonlinear terms are lower order terms decaying exponentially in τ, hence we get a Gaussian behavior as expected, while when $d = 2$, the nonlinear terms are $O(1)$ since, in these variables, Theorem 1.5 yields

$$\|G\|_2^2, \|G\|_\infty \approx 1. \tag{5.11}$$

5.2.1. *Decay to a Gaussian in higher dimensions $d \geq 3$.* We now use the above change of variables to obtain the convergence to a Gaussian; that is, we prove Theorem 1.6 (i). First, we make a few reductions. Up to shifting in time, we may assume that, for $\tau_0 \geq 0$, $g(e^{\tau_0}, x) = g_0(x)$, which yields

$$G(\tau_0, y) = g_0(e^{\tau_0/2} y) \leq e^{\frac{d\tau_0}{2}} A e^{-\frac{e^{\tau_0} y^2}{B}} \qquad \text{for all } y \in \mathbb{R}^d. \tag{5.12}$$

Hence, up to increasing τ_0 and increasing A, we may assume that

$$G(\tau_0, y) \leq A e^{-\frac{y^2}{2}} \qquad \text{for all } y \in \mathbb{R}^d.$$

Summing up the previous reductions, we assume that G solves

$$\begin{cases} G_\tau = \frac{1}{2}\Delta G + \frac{y}{2} \cdot \nabla G + \frac{d}{2} G + e^{-\frac{d-2}{2}\tau} G \left(\|G\|_2^2 - G\right) & \text{in } (\tau_0, \infty) \times \mathbb{R}^d, \\ G = G_0 \leq A e^{-\frac{y^2}{2}} & \text{on } \{\tau_0\} \times \mathbb{R}^d. \end{cases} \tag{5.13}$$

The above is not self-adjoint and, thus, not amenable to spectral analysis. Hence, we define a new function

$$W(\tau,y) = \frac{G(\tau,y)}{\psi_0(y)}, \qquad \text{where } \psi_0(y) = \frac{1}{Z}e^{-\frac{y^2}{4}}$$

with $Z = (2\pi)^{d/4}$ is a normalization constant chosen so that $\|\psi_0\|_2 = 1$. It is clear that ψ_0^2 is the standard Gaussian density. We note that, due to the bound on G_0 in (5.13), we have that $W_0 := G_0/\psi_0 \in L^2$.

We notice that

$$W_\tau = \frac{1}{2}\Delta W + W\left(\frac{d}{4} - \frac{|y|^2}{8}\right) + We^{-\frac{d-2}{2}\tau}\left(\|G\|_2^2 - G\right) \tag{5.14}$$

For simplicity, we write the linear operator

$$M = -\frac{1}{2}\Delta + \left(\frac{|y|^2}{8} - \frac{d}{4}\right).$$

We understand the behavior of G through the properties of M. First, note that M is an unbounded, symmetric operator on L^2. For each multi-index $\alpha \in \mathbb{N}_0^d$, let

$$\psi_\alpha = \psi_0^{-1}\partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}\psi_0^2 = H_\alpha(y)\psi_0(y),$$

where H_α is the α Hermite polynomial that is implicitly defined above. It is easy to check that

$$M\psi_\alpha = \frac{|\alpha|}{2}\psi_\alpha.$$

In addition, it is well-known[e] that ψ_α form an orthogonal basis of L^2. In particular, we conclude that if $\langle\psi,\psi_0\rangle = 0$, then

$$\langle M\psi,\psi\rangle \geq \frac{1}{2}\|\psi\|_2^2. \tag{5.15}$$

We are now in a position to complete the proof of Theorem 1.6 (i).

Proof of Theorem 1.6 (i). We write

$$W_\perp(\tau,y) = W(\tau,y) - \psi_0(y).$$

Notice that

$$\langle W_\perp(\tau),\psi_0\rangle = \langle W(\tau),\psi_0\rangle - \langle\psi_0,\psi_0\rangle = \int G(\tau,y)\,dy - 1 = 0.$$

The proof proceeds by showing that $W_\perp \to 0$ using this orthogonality.

Multiplying (5.14) by $W_\perp$, integrating, and noticing that $\langle W, W_\perp\rangle = \|W_\perp\|_2^2$ by orthogonality, yields

$$\frac{1}{2}\partial_\tau\|W_\perp\|_2^2 = -\langle MW, W_\perp\rangle + e^{-\frac{d-2}{2}\tau}\langle W\left(\|G\|_2^2 - G\right), W_\perp\rangle.$$

Next, using that $MW = M\psi_0 + MW_\perp = MW_\perp$ and (5.15), yields

$$\begin{aligned}\frac{1}{2}\partial_\tau\|W_\perp\|_2^2 &= -\langle MW_\perp, W_\perp\rangle + e^{-\frac{d-2}{2}\tau}\langle W\left(\|G\|_2^2 - G\right), W_\perp\rangle\\ &\leq -\frac{1}{2}\|W_\perp\|_2^2 + e^{-\frac{d-2}{2}\tau}\langle W\left(\|G\|_2^2 - G\right), W_\perp\rangle.\end{aligned} \tag{5.16}$$

Using the bounds in Theorem 1.5, we find

$$\langle W(\|G\|_2^2 - G), W_\perp\rangle \lesssim \|W\|_2\|W_\perp\|_2 \leq (1 + \|W_\perp\|_2)\,\|W_\perp\|_2.$$

[e] This is usually stated in the following way: the set of (rescaled) Hermite polynomials H_α form a basis of the weighted space $L^2(\psi_0^2)$. This is, however, equivalent to our statement.

Hence, (5.16) becomes, for some $C > 0$,

$$\partial_\tau \|W_\perp\|_2^2 + \left(1 - Ce^{-\frac{d-2}{2}\tau}\right)\|W_\perp\|_2^2 \lesssim e^{-\frac{(d-2)}{2}\tau}\|W_\perp\|_2.$$

Solving this differential inequality yields

$$\|W_\perp\|_2 \lesssim R_d(\tau)$$

where we define

$$R_d(\tau) = \begin{cases} (\tau+1)e^{-\tau/2} & \text{if } d = 3, \\ e^{-\tau/2} & \text{if } d \geq 4. \end{cases}$$

Returning to G, we find

$$\int e^{\frac{y^2}{2}}\left(G(\tau,y) - \psi_0^2\right)^2 dy \lesssim R_d(\tau)^2.$$

Using parabolic regularity theory, it is standard to conclude, for any $\sigma < 1/4$,

$$\|e^{\sigma y^2}\left(G(\tau) - \psi_0^2\right)\|_\infty \lesssim R_d(\tau),$$

which, after returning to the original variables, concludes the proof. □

5.2.2. *A non-Gaussian steady state in two dimensions.* In the following, we use R as the variable for the radius of a ball, which is not to be confused with the convolution kernel (the kernel is fixed to be δ in this section). We construct a steady solution of the self-similar problem (5.13) when $d = 2$ that is not the Gaussian ψ_0^2 from the previous subsection. The construction occurs in multiple steps. First, we replace the $\|G\|_2^2$ in (5.10) with a constant term E to "localize" the equation. For any $E > 0$ and $R \gtrsim 1/\sqrt{E}$, we construct radial (rotationally symmetric) steady solutions G_R of the localized equation on B_R with Dirichlet boundary conditions on ∂B_R. Second, we show that, choosing $E = \mathcal{E}_R$ well guarantees that $\int_{B_R} G_R(y)dy = 1$. Finally, we show that, in the limit $R \to \infty$, $\mathcal{E}_R - \|G_R\|_2^2 \to 0$ and that G_R converges to a steady solution G of (5.10) on $\mathbb{R}^2$.

Constructing a steady solution of the localized problem on a ball.

Lemma 5.3. *Fix $E > 0$ and $R \geq \max\{4, 20/\sqrt{E}\}$. There exists a radial function $\underline{G}_{E,R} : B_R \to [0, E/2]$ such that*

$$\begin{cases} \frac{1}{2}\Delta\underline{G}_{E,R} + \frac{y}{2}\cdot\nabla\underline{G}_{E,R} + \underline{G}_{E,R}(1 + E - \underline{G}_{E,R}) \geq 0 & \text{in } B_R, \\ \underline{G}_{E,R} = 0 & \text{on } \partial B_R, \end{cases} \tag{5.17}$$

and

$$\int_{B_R} \underline{G}_{E,R}dy \gtrsim E.$$

Proof. Let $\phi : [0, R] \to \mathbb{R}$ be a C^2 cut-off function such that

$$\begin{aligned} &(i)\ \ 0 \leq \phi \leq 1,\ \phi'(R) = \phi(R) = 0, \qquad (ii)\ \ \phi \equiv 1 \text{ on } \left[0, \frac{R}{3}\right], \\ &(iii)\ \ -\frac{20}{R^2} \leq \phi'',\ -\frac{10}{R} \leq \phi',\ \text{and } \phi(y) \geq 1/2 \text{ on } \left[\frac{R}{3}, \frac{2R}{3}\right], \text{ and} \\ &(iv)\ \ \phi'' \geq \frac{2}{R^2},\ \phi'(r) \geq -\frac{10}{R^2}(R - r) \text{ on } \left[\frac{2R}{3}, R\right]. \end{aligned} \tag{5.18}$$

Let

$$\underline{G}_{E,R}(y) = \frac{E}{2}e^{-\frac{y^2}{2}}\phi(|y|). \tag{5.19}$$

Notice that, by (5.18).(i), $\underline{G}_{E,R} \le E/2$. Using polar coordinates, we find

$$\begin{aligned} -\frac{1}{2}\Delta\underline{G}_{E,R} - \frac{y}{2}\cdot\nabla\underline{G}_{E,R} - \underline{G}_{E,R}(1+E-\underline{G}_{E,R}) \\ = -\frac{Ee^{-\frac{y^2}{2}}}{2}\left(\frac{1}{2}\phi'' + \frac{1}{2r}\phi' + \left(E - \frac{E}{2}e^{-\frac{r^2}{2}}\phi\right)\phi\right). \end{aligned} \tag{5.20}$$

It is clear that the last line is non-positive when $r \in [0, R/3]$ since $\phi \equiv 1$ on this set. When $r \in [R/3, 2R/3]$, we deduce, from (5.18).(iii), that

$$-\left(\frac{1}{2}\phi'' + \frac{1}{2r}\phi' + \left(E - \frac{E}{2}e^{-\frac{r^2}{2}}\phi\right)\phi\right) \le \frac{10}{R^2} + \frac{15}{R^2} - \frac{E}{2} = \frac{E}{2R^2}\left(\frac{50}{E} - R^2\right).$$

Since, $R \ge \max\{4, 20/\sqrt{E}\}$, this is non-positive. Hence, the right hand side of (5.20) is non-positive.

Finally consider the case when $r \in [2R/3, R]$. First notice that, due to (5.18).(i) and (iv), we have

$$\phi(y) \ge \frac{1}{R^2}(R-y)^2.$$

Using this lower bound, as well as (5.18).(iv) again, yields

$$-\left(\frac{1}{2}\phi'' + \frac{1}{2r}\phi' + \left(E - \frac{E}{2}e^{-\frac{r^2}{2}}\phi\right)\phi\right) \le -\frac{1}{R^2} + \frac{3}{4R}\frac{10}{R^2}(R-y) - \frac{E}{2}\frac{(R-y)^2}{R^2}. \tag{5.21}$$

We now use Young's inequality and then the fact that $R \ge 20/\sqrt{E}$ to find

$$\frac{3}{4R}\frac{10}{R^2}(R-y) \le \frac{225}{8ER^4} + \frac{E}{2}\frac{(R-y)^2}{R^2} \le \frac{225}{3200R^2} + \frac{E}{2}\frac{(R-y)^2}{R^2}.$$

Plugging this into (5.21) implies that the right hand side of (5.20) is non-positive on $[2R/3, R]$.

Hence, in all cases, the right hand side of (5.20) is non-positive, which implies that $\underline{G}_{E,R}$ is a subsolution; that is, it satisfies (5.17), as claimed. In addition, the lower bound on the integral of $\underline{G}_{E,R}$ is clear by (5.18) and (5.19). □

We now use $\underline{G}_{E,R}$ to construct a radial solution to the local problem on B_R.

Proposition 5.4. *Suppose that $E > 0$ and $R \ge \max\{4, 20/\sqrt{E}\}$. There exists a radial function $G_{E,R} : B_R \to [0, 1+E)$ of*

$$\begin{cases} 0 = \frac{1}{2}\Delta G_{E,R} + \frac{y}{2}\cdot\nabla G_{E,R} + (1+E-G_{E,R})G_{E,R} & \text{in } B_R, \\ G_{E,R} = 0 & \text{on } \partial B_R, \end{cases} \tag{5.22}$$

such that $\int_{B_R} G_{E,R}dy \gtrsim E$. This is the unique nontrivial solution of (5.22).

Proof. Let H be the solution of

$$\begin{cases} H_t = \frac{1}{2}\Delta H + \frac{y}{2}\cdot\nabla H + (1+E-H)H & \text{in } (0,\infty)\times B_R, \\ H = \underline{G}_{E,R} & \text{on } \{0\}\times B_R, \\ H = 0 & \text{on } [0,\infty)\times\partial B_R, \end{cases} \tag{5.23}$$

where $\underline{G}_{E,R}$ is from Lemma 5.3. The comparison principle immediately yields that $H \le 1+E$.

We claim that $H_t \ge 0$ for all $t > 0$. Since $\underline{G}_{E,R}$ satisfies (5.17), then $H_t(0,\cdot) \ge 0$. In addition, differentiating (5.23) in time yields a parabolic equation for H_t that enjoys the comparison principle and of which 0 is a solution. We conclude that $\min_{t,y} H_t(t,y) \ge 0$ by applying the comparison principle to H_t and 0.

Since, for all y, $H(t,y)$ is increasing in t, there exists $G_{E,R}(y)$ such that $H(t,y) \to G_{E,R}(y)$ as $t \to \infty$. In addition, we have that $H(t,\cdot) \le G_{E,R}$ for all t. Finally we point out that $G_{E,R} \le 1+E$ since $H \le 1+E$.

We also note that, by parabolic regularity theory, for any $\alpha \in (0,1)$, there exists $C > 0$, depending only on α and E, such that

$$\|H\|_{C^{2+\alpha}_{\text{parabolic}}([0,\infty)\times B_R)} \le C, \tag{5.24}$$

where $C^{2+\alpha}_{\text{parabolic}}$ is the standard parabolic Hölder space.

We claim that $\|\partial_t H(t_n)\|_{L^2(B_R)} \to 0$ along some subsequence $t_n \to \infty$. If not, then there exists $\delta > 0$ and $t_0 > 0$ such that, for all $t \ge t_0$, $\|\partial_t H(t)\|_2 \ge \delta$. Using (5.24) and the nonnegativity of $\partial_t H$, we find, for all $t \ge t_0$,

$$\delta^2 \le \int_{B_R} |\partial_t H(t,y)|^2 dy \le C \int_{B_R} \partial_t H(t,y) dy.$$

Integrating this and using that $0 \le H \le G_{E,R} \le 1+E$, we find, for any $T > 0$,

$$\begin{aligned}\frac{\delta^2 T}{C} &\le \int_{t_0}^{t_0+T} \int_{B_R} \partial_t H(t,y)dydt = \int_{t_0}^{t_0+T} \partial_t \int_{B_R} H(t,y)dydt \\ &= \int_{B_R} H(t_0+T,y)dy - \int_{B_R} H(t_0,y)dy \le \int_{B_R} G_{E,R}(y)dy \le \pi R^2(1+E).\end{aligned}$$

Taking $T \to \infty$ yields a contradiction. Hence, there exists a sequence $t_n \to \infty$ as $n \to \infty$ such that $\|\partial_t H(t_n)\|_2 \to 0$.

Up to taking a subsequence, the bounds in (5.24) and the compactness of $C^{2,\alpha}_{\text{parabolic}}$ in $C^2_{\text{parabolic}}$, imply that $H(t_n) \to G_{E,R}$ in $C^2_{\text{parabolic}}$. In addition, the resulting convergence of $\partial_t H(t_n)$ in L^∞ and its convergence to zero in L^2 implies that $\partial_t H(t_n) \to 0$ in L^∞. We conclude that

$$0 = \frac{1}{2}\Delta G_{E,R} + \frac{y}{2}\cdot\nabla G_{E,R} + (1+E-G_{E,R})\, G_{E,R}.$$

To conclude the proof, we check the various properties of $G_{E,R}$. First, the nonnegativity of $G_{E,R}$ follows from the fact that H is increasing in time and $H(0,\cdot) = \underline{G}_{E,R} \ge 0$.

Second, parabolic regularity theory implies that H is uniformly (in time) small near ∂B_R, which implies that $G_{E,R} = 0$ on ∂B_R.

Third, recall the earlier observation that $G_{E,R} \le 1+E$. The strong maximum principle applied to (5.22) implies that this inequality is strict.

Fourth, the lower bound on the integral of $G_{E,R}$ follows from the lower bound on $\underline{G}_{E,R} = H(0,\cdot)$ and the fact that H is increasing.

Finally, H is radial at $t=0$ by construction of $\underline{G}_{E,R}$. Let M be any rotation matrix and define $H_M(t,y) = H(t,My)$. It is easy to see that H_M solves (5.23). Thus, by the uniqueness of solutions of parabolic equations, we find $H_M = H$. We conclude that $H(t,\cdot)$ is radial for all t, from which it follows that $G_{E,R}$ is radial.

The last step is to check the uniqueness of $G_{E,R}$. We drop the E and R subscripts for ease. Suppose that H is another nontrivial solution (5.22). Define $G_A(y) = AG(x)$. It is easy to verify that

$$0 = \frac{1}{2}\Delta G_A + \frac{y}{2}\cdot\nabla G_A + \left(1+E-\frac{1}{A}G_A\right)G_A.$$

The Hopf maximum principle implies that the outward point normal derivative of G is negative on ∂B_R. Hence, we have that $G_A > H$ for all A sufficiently large. We

define
$$A_0 = \inf\{A \geq 1 : G_A \geq H\}.$$
If $A_0 = 1$, we conclude that $G \geq H$, which is our goal. If not, let $\Psi(y) = G_{A_0}(y) - H(y)$ for all y. We then find that
$$-H\left(H - \frac{G_{A_0}}{A_0}\right) = \frac{1}{2}\Delta\Psi + \frac{y}{2}\cdot\nabla\Psi + \left(1 + E - \frac{1}{A_0}G_{A_0}\right)\Psi.$$

By the choice of A_0, we either have that there exists $y_0 \in B_R$ such that $\Psi(y_0) = 0$ or there exists $y_0 \in \partial B_R(0)$ such that $y_0 \cdot \nabla G_{A_0}(y_0) = y_0 \cdot \nabla H(y_0)$.

Consider the first case. Let Σ be the maximal open connected component of $B_R \cap \{G_{A_0} < A_0 H\}$ containing y_0. Then $\Psi > 0$ on $\partial\Sigma$ and Ψ satisfies
$$0 \geq \frac{1}{2}\Delta\Psi + \frac{y}{2}\cdot\nabla\Psi + \left(1 + E - \frac{1}{A}G_{A_0}\right)\Psi \qquad \text{on } \Sigma.$$
The strong maximum principle implies that $\Psi > 0$ on Σ, which contradicts the fact that $\Psi(y_0) = 0$.

The second case proceeds similarly, except using the Hopf lemma to conclude that $y_0 \cdot \nabla\Psi(y_0) < 0$ to obtain a contradiction. We omit the details.

Above we showed that $G \leq H$. The argument used to establish this used nothing about G from its construction; we only used that it is nontrivial. Thus, an identical argument implies that $H \leq G$ as well, which yields $G = H$. Hence, non-trivial solutions of (5.22) are unique. This concludes the proof. □

We show that $G_{E,R}$ decays exponentially away from the origin independently of R. We require this to show that G, the limiting object, is exponentially decaying as stated in Theorem 1.6.(ii) and in order to establish a relationship between the sizes of E and $\int G_{E,R}\,dy$ in the sequel.

Lemma 5.5. *For any E and R as in Proposition 5.4,*
$$G_{E,R}(y) \leq (1+E)e^{4(1+E) - \frac{y^2}{4}} \qquad \textit{for all } |y| \geq 4\sqrt{1+E}.$$

Proof. For any $A \geq 0$, let $\psi_A(y) = Ae^{-\frac{y^2}{4}}$. We first claim that, for $|y| \geq 4\sqrt{1+E}$,
$$-\frac{1}{2}\Delta\psi - \frac{y}{2}\cdot\nabla\psi - (1+E)\,\psi > 0. \tag{5.25}$$
To this end, we compute:
$$\begin{aligned}-\frac{1}{2}\Delta\psi_A - \frac{y}{2}\cdot\nabla\psi_A - (1+E)\psi_A &= -\frac{1}{2}\left(\frac{y^2}{4}\psi_A - \psi_A\right) - \frac{y}{2}\cdot\left(-\frac{y}{2}\psi_A\right) - (1+E)\psi_A \\ &= \frac{1}{8}\psi_A\left(y^2 - 2(2+4E)\right).\end{aligned}$$
Hence, if $y^2 \geq 16(1+E)$, then we conclude (5.25).

If $A \geq (1+E)e^{R^2/4}$, then $\psi_A \geq 1 + E > G_{E,R}$ (recall the upper bound on $G_{E,R}$ from Proposition 5.4). Thus, let
$$A_0 = \inf\left\{A > 0 : \psi_A \geq G_{E,R} \text{ on } |y| \in [4\sqrt{1+E}, R]\right\}$$
is well-defined. We claim that $A_0 \leq (1+E)\,e^{4(1+E)}$. We argue by contradiction assuming that $A_0 > (1+E)e^{4(1+E)}$.

By continuity, there exists y_0 such that $|y_0| \in [4\sqrt{1+E}, R]$ such that $\psi_{A_0}(y_0) = G_{E,R}(y_0)$. Since
$$\psi_{A_0}(4\sqrt{1+E}) = A_0 e^{-4(1+E)} > (1+E) > G_{E,R} \quad \text{and} \quad \psi_{A_0}(R) > 0 = G_{E,R}(R),$$

it must be that $|y_0| \in (4\sqrt{1+E}, R)$. In addition, by construction, y_0 is the location of a minimum of zero of $\psi_{A_0} - G_{E,R}$. Hence $\Delta(\psi_{A_0} - G_{E,R}) \geq 0$, $\nabla(\psi_{A_0} - G_{E,R}) = 0$, and $\psi_{A_0}(y_0) = G_{E,R}(y_0)$. Hence, at y_0,

$$\begin{aligned} 0 &\geq -\frac{1}{2}\Delta\left(\psi_{A_0} - G_{E,R}\right) \\ &> \left(\frac{y}{2}\cdot\nabla\psi_{A_0} + (1+E)\psi_{A_0}\right) - \left(\frac{y}{2}\cdot\nabla G_{E,R} + (1+E-G_{E,R})G_{E,R}\right) \\ &= G_{E,R}^2 > 0. \end{aligned}$$

Here we used (5.22) and (5.25) to obtain the second inequality. This is clearly a contradiction, so we conclude that $A_0 \leq (1+E)e^{4(1+E)}$. It follows that, for all $|y| \in [4\sqrt{1+E}, R]$,

$$G_{E,R}(y) \leq (1+E)e^{4(1+E)}e^{-\frac{y^2}{4}},$$

which concludes the proof. □

Next we show that there exists E such that $G_{E,R}$ has L^1-norm one.

Lemma 5.6. *There exists $\mathcal{E}_R$[f] ≈ 1, depending only on R, such that*

$$\int_{B_R} G_{\mathcal{E}_R,R}(y)dy = 1.$$

Proof. To establish this we use the continuity of solutions of elliptic equations with respect to their coefficients. With this in mind, we need only find E_1 and E_2 such that

$$\int_{B_R} G_{E_1,R}(y)dy \leq 1 \qquad \text{and} \qquad \int_{B_R} G_{E_2,R}(y) \geq 1.$$

The second inequality follows from Proposition 5.4, after taking $E_2 \gtrsim 1$.

To find the first inequality, integrate (5.22) over B_R to find

$$-\frac{1}{2}\int_{\partial B_R}\frac{y}{|y|}\cdot\nabla G_{E_1,R}dy = E_1\int_{B_R} G_{E_1,R}dy - \int_{B_R} G_{E_1,R}^2 dy. \tag{5.26}$$

As G is positive in B_R and zero on ∂B_R, we find that the left hand side is nonnegative. Hence,

$$\int_{B_R} G_{E_1,R}^2 dy \leq E\int_{B_R} G_{E_1,R}dy$$

We wish to estimate the integral on the right hand side above. Let $L > 4\sqrt{1+E_1}$ be a constant chosen in the sequel. Then, using Hölder's inequality and Lemma 5.5, we obtain

$$\begin{aligned} \int_{B_R} G_{E_1,R}dy &\leq \int_{B_L} G_{E_1,R}dy + \int_{B_R\setminus B_L}(1+E_1)e^{4(1+E_1)-\frac{y^2}{4}}dy \\ &\lesssim L\left(\int_{B_L} G_{E_1,R}^2 dy\right)^{1/2} + L^2(1+E_1)e^{4(1+E_1)-\frac{L^2}{4}} \\ &\leq L\left(\int_{B_R} G_{E_1,R}^2 dy\right)^{1/2} + L^2(1+E_1)e^{4(1+E_1)-\frac{L^2}{4}} \\ &\leq L\left(E_1\int_{B_R} G_{E_1,R}dy\right)^{1/2} + L^2(1+E_1)e^{4(1+E_1)-\frac{L^2}{4}}. \end{aligned}$$

Choosing L sufficiently large and then E_1 sufficiently small independent of R, we find

$$\int_{B_R} G_{E_1,R}dy < 1.$$

[f] We make a slight change in convention here using the italicized "E" in order to avoid clashing notation with E_f for the squared L^2-norm of a function f.

This concludes the proof. □

Using Lemma 5.6, we let $G_R = G_{\mathcal{E}_R,R}$ denote the solution of (5.22) with mass one. We now show that $\mathcal{E}_R \approx \|G_R\|_2^2$ as $R \to \infty$.

Lemma 5.7. *As R tends to ∞, $|\mathcal{E}_R - \|G_R\|_2^2| \to 0$.*

Proof. Using (5.26), Lemma 5.6, and the fact that G_R is radial, it is sufficient to show that

$$-\frac{1}{2}\int_{\partial B_R} \frac{y}{|y|}\cdot\nabla G_R(y)dy = -\pi R (G_R)_r(R) \to 0 \tag{5.27}$$

Let $W = e^{y^2/4}G_R$ and, as in (5.14),

$$0 = \frac{1}{2}\Delta W + W\left(\frac{1}{2} - \frac{y^2}{4} + \mathcal{E}_R - G_R\right).$$

Abusing notation and changing to polar coordinates, we find

$$0 = \frac{1}{2}W_{rr} + \frac{1}{2r}W_r + W\left(\frac{1}{2} - \frac{r^2}{4} + \mathcal{E}_R - G_R\right)$$

We note that the reason for the change to working with W is to work with an equation whose first order term is bounded uniformly regardless of R. Hence, we can apply L^2 estimates for solutions of elliptic equations up to the boundary (see, e.g., [13, Theorem 8.12]), we find

$$\|W\|_{H^2([R-1,R])} \lesssim \left\|\left(\frac{1}{2} - \frac{r^2}{4} + \mathcal{E}_R - G_R\right)W\right\|_{L^2([R-2,R])} + \|W\|_{L^2([R-2,R])}.$$

Using Lemma 5.5 to bound G_R and, thus, W from above and Lemma 5.6 to bound $\mathcal{E}_R$ from above, we see that W is uniformly bounded from above on $[R-2,R]$ as long as R is sufficiently large. We deduce that

$$\|W\|_{H^2([R-1,R])} \lesssim R^2.$$

Using the Sobolev embedding theorem and the relationship between G_R and W, we find

$$\begin{aligned}\|G_R\|_{C^1([R-1,R])} &\lesssim Re^{-\frac{(R-1)^2}{4}}\|W\|_{C^1([R-1,R])}\\ &\lesssim Re^{-\frac{(R-1)^2}{4}}\|W\|_{H^2([R-1,R])} \lesssim R^3 e^{-\frac{(R-1)^2}{4}}.\end{aligned}$$

This establishes (5.27), which concludes the proof. □

We now finish the construction of the steady state G.

Proof of Theorem 1.6.(ii). From Proposition 5.4, Lemma 5.5, and Lemma 5.6 we have that

$$G_R \lesssim e^{-\frac{y^2}{4}}.$$

Using a similar argument as we did in the conclusion of Lemma 5.7 along with the Schauder estimates for elliptic equations, we find that, for any $\alpha \in (0,1)$,

$$\|e^{y^2/5}G_R\|_{C^{2,\alpha}} \lesssim 1.$$

We thus find a subsequence $R_n \to \infty$ as $n \to \infty$, and $G \in C^{2,\alpha}$ such that $G_{R_n} \to G$ uniformly in C^2, and, due to the decay in y, in L^1 and L^2 as well. We conclude that $\int G\,dy = 1$ and $\|G_{R_n}\|_2^2 \to \|G\|_2^2$. From Lemma 5.7, we further have that $E_{R_n} \to \|G\|_2^2$.

Using all conclusions from the above, we find that G is a radial function satisfying

$$0 = \Delta G + \frac{y}{2}\cdot\nabla G + G(1 + \|G\|_2^2 - G),$$

which concludes the proof. □

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