The random heat equation in dimensions three and higher: the homogenization viewpoint

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Abstract

We consider the stochastic heat equation $\partial_s u = \frac{1}{2}\Delta u + (\beta V(s,y) - \lambda)u$, driven by a smooth space-time stationary Gaussian random field $V(s,y)$, in dimensions $d \geq 3$, with an initial condition $u(0,x) = u_0(\varepsilon x)$. It is known that the diffusively rescaled solution $u^\varepsilon(t,x) = u(\varepsilon^{-2}t, \varepsilon^{-1}x)$ converges weakly to a scalar multiple of the solution $\bar{u}$ of a homogenized heat equation with an effective diffusivity a, and that fluctuations converge (again, in a weak sense) to the solution of the Edwards-Wilkinson equation with an effective noise strength ν. In this paper, we derive a pointwise approximation $w^\varepsilon(t,x) = \bar{u}(t,x)\Psi^\varepsilon(t,x) + \varepsilon u_1^\varepsilon(t,x)$, where $\Psi^\varepsilon(t,x) = \Psi(t/\varepsilon^2, x/\varepsilon)$, Ψ is a solution of the SHE with constant initial conditions, and u_1 is an explicit corrector. We show that $\Psi(t,x)$ converges to a stationary process $\tilde{\Psi}(t,x)$ as $t \to \infty$, that $w^\varepsilon(t,x)$ converges pointwise (in L^1) to $u^\varepsilon(t,x)$ as $\varepsilon \to 0$, and that $\varepsilon^{-d/2+1}(u^\varepsilon - w^\varepsilon)$ converges weakly to 0 for fixed t. As a consequence, we derive new representations of the diffusivity a and effective noise strength ν. Our approach uses a Markov chain in the space of trajectories introduced in [13], as well as tools from homogenization theory. The corrector $u_1^\varepsilon(t,x)$ is constructed using a seemingly new approximation scheme on "long but not too long time intervals".

1 Introduction

We consider the long time and large space behavior of the solutions $u(s,y)$ of the random heat equation with slowly varying initial conditions

$$\partial_s u = \frac{1}{2}\Delta u + (\beta V(s,y) - \lambda)u, \tag{1.1}$$

$$u(0,y) = u_0(\varepsilon y), \tag{1.2}$$

with $y \in \mathbb{R}^d, d \geq 3$. The potential $V(s,y)$ is a smooth space-time homogeneous mean-zero Gaussian random field with a finite correlation length. We assume it has the form

$$V(s,y) = \int_{\mathbb{R}^{d+1}} \mu(s-s')\nu(y-y')\,\mathrm{d}W(s',y'), \tag{1.3}$$

where μ and ν are nonnegative functions of compact support, such that ν is isotropic and

$$\operatorname{supp}\mu \subset [0,1], \quad \operatorname{supp}\nu \subset \{y \mid |y| \leq 1/2\}.$$

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The covariance function of $V(s,y)$ has the form

$$R(s,y) := \mathbb{E}(V(s+s',y+y')V(s',y')) = \int_{\mathbb{R}} \mu(s+t)\mu(t)\,\mathrm{d}t \int_{\mathbb{R}^d} \nu(y+z)\nu(z)\,\mathrm{d}z.$$

The constant λ in (1.1) ensures that the solutions of (1.1) -(1.2) do not grow exponentially as $t \to +\infty$ – otherwise, one would need to rescale them by a simple exponential in time factor. The small parameter $\varepsilon \ll 1$ measures the ratio of the typical scale of variation of the initial condition and the correlation length of the random potential. As we are interested in the long time behavior of the solution to (1.1)-(1.2) , we consider its macroscopic rescaling:

$$u^\varepsilon(t,x) = u(\varepsilon^{-2}t, \varepsilon^{-1}x),$$

that satisfies the correspondingly rescaled problem

$$\begin{aligned} &\partial_t u^\varepsilon = \frac{1}{2}\Delta u^\varepsilon + \frac{1}{\varepsilon^2}\Big(\beta V\big(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\big) - \lambda\Big)u^\varepsilon \\ &u^\varepsilon(0,x) = u_0(x). \end{aligned} \tag{1.4}$$

It was shown in [13, 14], and also in [15] at the level of the expectation, that there exists $\beta_0 > 0$ so that, if $0 < \beta < \beta_0$, there exists λ depending on β, μ, ν, and $\tilde{c} > 0$, so that

$$v^\varepsilon(t,x) = \tilde{c}u^\varepsilon(t,x) \tag{1.5}$$

converges weakly as $\varepsilon \to 0$ to the solution $\overline{u}(t,x)$ to the homogenized problem

$$\begin{aligned} \partial_t \overline{u} &= \frac{1}{2}a\Delta\overline{u} \\ \overline{u}(0,x) &= u_0(x), \end{aligned} \tag{1.6}$$

with an effective diffusivity $a > 0$. It was also shown that the fluctuation

$$\frac{1}{\varepsilon^{d/2-1}}(v^\varepsilon(t,x) - \mathbb{E}v^\varepsilon(t,x)), \tag{1.7}$$

converges in law and weakly in space as $\varepsilon \to 0$ to the solution $\mathscr{U}$ of the Edwards-Wilkinson equation

$$\begin{aligned} \partial_t \mathscr{U}(t,x) &= \frac{1}{2}a\Delta\mathscr{U}(t,x) + \beta\nu\overline{u}(t,x)\dot{W}(t,x) \\ \mathscr{U}(0,x) &= 0, \end{aligned} \tag{1.8}$$

with an effective noise strength $\nu > 0$.

The results of [13, 14] concern weak convergence, that is, after integration against a macroscopic test function. In the present paper, we seek to understand the microscopic behavior of the solutions, somewhat in the spirit of the classical random homogenization theory, and explain how the microscopic behavior leads to the macroscopic results in [13, 14]. We are also interested in a more explicit interpretation of the macroscopic parameters: the renormalization constant $\lambda(\beta)$, the effective diffusivity a in (1.6), the renormalization constant $\tilde{c}$ in (1.5) and the effective noise ν in (1.8), also in terms of the classical objects of the homogenization approach to PDEs with random coefficients. We mention that as this paper was being written, we learned of the very interesting recent paper [6], where a limit theorem for local fluctuations for the solution u^ε in the special case when the random potential $V(t,x)$ is white in time and $u_0 = 1$, was obtained. This result is complementary to ours, and the methods of proof are quite different. We also mention that the restriction to dimension $d \geq 3$ is crucial: for $d = 2$ the behavior is different, see in particular [4] and [5].

As is standard in the PDE homogenization theory, it is natural to introduce fast variables and consider a formal asymptotic expansion for the solutions u^ε to (1.4) in the form

$$u^\varepsilon(t,x) = u^{(0)}(t,x,\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}) + \varepsilon u^{(1)}(t,x,\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}) + \varepsilon^2 u^{(2)}(t,x,\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}) + \cdots , \tag{1.9}$$

There are two known issues that often arise in such expansions – first, the existence of stationary correctors is difficult to prove, and may actually be false, and, second, higher order correctors may have slower decaying correlations. This means that, after integration against a test function, all terms could contribute to the same order, and including more correctors may not improve the expansion as far as the weak approximation is concerned. Some relevant discussions on the random fluctuations in elliptic homogenization can be found in [9, 12]. In the present case, it is easy to see that the leading order term in (1.9) should have the form

$$u^{(0)}(t,x,\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}) = \overline{u}(t,x)\Psi(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}), \tag{1.10}$$

where $\Psi(s,y)$ is a solution to (1.1). The corrector $\Psi(s,y)$ is "universal" in the sense that it does not depend on the initial condition $u_0(x)$ in (1.2). As we have mentioned, ideally, one would want $\Psi(s,y)$ to be a stationary solution to (1.1) and capture the behavior of u^ε. However, as we will see, such a choice would lead to the aforementioned "large" macroscopic errors, after integration against a test function. Instead, we will let $\Psi(s,y)$ be the solution to the same problem with a constant initial condition:

$$\begin{aligned} &\partial_s\Psi(s,y) = \frac{1}{2}\Delta\Psi(s,y) + (\beta V(s,y) - \lambda)\Psi(s,y) \\ &\Psi(0,y) = 1, \end{aligned} \tag{1.11}$$

and let $\overline{u}(t,x)$ be a function of the macroscopic variables that will end up being the solution to the homogenized problem (1.6).

Existence of a stationary solution and the leading order term in the expansion

Our first result gives an explanation for the choice of $\lambda = \lambda(\beta)$: if λ is chosen appropriately, then $\Psi(s,y)$ approaches a global in time stationary solution $\widetilde{\Psi}(s,y)$ as $s \to \infty$. Thus, it is reasonable to take $\Psi^\varepsilon(t,x) = \Psi(t/\varepsilon^2, x/\varepsilon)$ as a proxy for the stationary solution in the leading order term for the asymptotic expansion (1.9) – note that both are equally "universal" in the sense that they do not depend on the initial condition $u_0(x)$ for (1.1).

Theorem 1.1. *There is a $\beta_0 > 0$ so that for all $0 \leq \beta < \beta_0$, there is a $\lambda = \lambda(\beta) > 0$ and a space-time stationary random function $\widetilde{\Psi}(s,y) > 0$ that solves*

$$\partial_s\widetilde{\Psi}(s,y) = \frac{1}{2}\Delta\widetilde{\Psi}(s,y) + (\beta V(s,y) - \lambda)\widetilde{\Psi}(s,y), \tag{1.12}$$

and for any $y \in \mathbb{R}^d$, we have

$$\lim_{s\to\infty} \mathbb{E}|\Psi(s,y) - \widetilde{\Psi}(s,y)|^2 = 0. \tag{1.13}$$

Throughout the paper, we will always assume that $\lambda = \lambda(\beta)$ is chosen as in the statement of Theorem 1.1. Theorem 1.1 can be seen as an extension of [16, Theorem 2.1] to the colored-noise setting, even though that result was formulated in different terms. Some of the other relevant results in the literature are [8] and [18], which establish existence of stationary solutions and convergence along subsequences in weighted L^2 spaces, also in the white-noise setting case. The proof of Theorem 1.1 is similar in spirit to that of [16, Theorem 2.1] but uses the framework of [13] to deal with the necessary renormalization parameter λ, and is presented in Section 3. For elliptic operators in divergence form, the existence of stationary correctors in high dimensions was studied in [2, 10, 11], and we refer to the recent monograph [1] for a more complete list of references.

As an application of the existence of the stationary solution, we will show in Section 4 that the effective noise strength ν in (1.8), which has a complicated expression given in [13, (5.6)], has a more intuitive expression in terms of the stationary solution. To set the notation, let

$$G_a(t,x) = (2\pi a t)^{-d/2}\exp(-|x|^2/(2at))$$

be the Green function for the heat equation with diffusivity a, and note that there exists a constant $\bar{c}$ so that

$$\int_0^\infty \int_{\mathbb{R}^d} G_a(r,z)G_a(r,z+x)\,\mathrm{d}z\,\mathrm{d}r = \frac{\bar{c}}{a|x|^{d-2}}$$

Theorem 1.2. *The effective noise strength ν in (1.8) has the expression*

$$\nu^2 = \frac{a\lim_{\varepsilon\to 0}\int\int g(x)g(\tilde{x})\left(\frac{1}{\varepsilon^{d-2}}\operatorname{Cov}\left(\widetilde{\Psi}\left(0,\frac{x}{\varepsilon}\right),\widetilde{\Psi}\left(0,\frac{\tilde{x}}{\varepsilon}\right)\right)\right)\,\mathrm{d}x\,\mathrm{d}\tilde{x}}{\bar{c}\beta^2 e^{2\alpha_\infty}\int\int g(x)g(\tilde{x})|x-\tilde{x}|^{2-d}\,\mathrm{d}x\,\mathrm{d}\tilde{x}} \tag{1.14}$$

for any test function $g\in\mathcal{C}_c^\infty(\mathbb{R}^d)$. The deterministic constant α_∞ is defined in (2.3) below.

Theorem 1.2 is a weak version of the asymptotics

$$\operatorname{Cov}(\widetilde{\Psi}(0,0),\widetilde{\Psi}(0,y)) \sim \frac{\bar{c}\beta^2\nu^2 \mathrm{e}^{2\alpha_\infty}}{a|y|^{d-2}},\quad |y|\gg 1,$$

so that the effective noise constant in the Edwards-Wilkinson equation (1.8) is directly related to the decay of the covariance of the stationary solution, in the spirit of the central limit theorem applied to integration over a large number of microscopic boxes, taking into account the correlation on various such boxes.

Going back to the expansion (1.9), the leading order term in (1.10) is justified by the following microscopic convergence result.

Theorem 1.3. *Suppose that $0\le\beta<\beta_0$, and set $\Psi^\varepsilon(t,x)=\Psi(t/\varepsilon^2,x/\varepsilon)$. If $u_0\in C_c^\infty(\mathbb{R}^d)$, then*

$$\lim_{\varepsilon\to 0}\mathbb{E}|u^\varepsilon(t,x)-\overline{u}(t,x)\Psi^\varepsilon(t,x)|^2 = 0. \tag{1.15}$$

One can now see how the renormalization constant $\tilde{c}$ in the non-divergent renormalization (1.5) shows up: $\Psi(s,y)\to\tilde{\Psi}(s,y)$ during the initial time layer on the "microscopically large" time scale. However, even though $\Psi(0,y)\equiv 1$, $\mathbb{E}(\tilde{\Psi}(s,y))$ is not necessarily equal to one, hence the need for the renormalization to ensure convergence to the effective diffusion equation (1.6) with the initial condition $u_0(x)$ – and not $\tilde{c}u_0(x)$. In light of Theorems 1.1 and 1.3, one may wonder if one should be using $\tilde{\Psi}(t/\varepsilon^2,x/\varepsilon)$ rather than $\Psi(t/\varepsilon^2,x/\varepsilon)$ in the approximation (1.15) – we will see below that the answer is that $\Psi(t/\varepsilon^2,x/\varepsilon)$ improves the weak error.

A higher order approximation

In order to obtain higher order corrections in the asymptotic expansion, if we plug (1.9) into (1.4) and group terms by the powers of ε, we obtain the following equations for u_1 and u_2:

$$\partial_s u_1(t,x,s,y) = \frac{1}{2}\Delta_y u_1(t,x,s,y) + (\beta V(s,y)-\lambda)u_1(t,x,s,y) + \nabla_y\Psi(s,y)\cdot\nabla_x\overline{u}(t,x), \tag{1.16}$$

and

$$\begin{aligned}\partial_s u_2(t,x,s,y) &= \frac{1}{2}\Delta_y u_2(t,x,s,y) + (\beta V(s,y)-\lambda)u_2(t,x,s,y) + \nabla_y\cdot\nabla_x u_1(t,x,s,y)\\ &\quad + \frac{1}{2}(1-a)\Psi(s,y)\Delta_x\overline{u}(t,x).\end{aligned} \tag{1.17}$$

As we will show in Section 5, the effective diffusivity a can be recovered from a formal solvability condition for (1.17) to have a solution u_2 that is stationary in the fast variables – a rather standard situation in the homogenization theory. However, here, as stationary correctors are not expected to exist in low dimensions, a justification of this expression requires a construction of approximate correctors and passage to a large time limit, similar to the "large box" limit in the elliptic homogenization theory. In particular, Theorem 5.1 below provides a computational tool to evaluate the effective diffusivity in purely PDE terms.

Our last result concerns the connection between the local expansion (1.9) and the weak approximation of the solution. As we have mentioned, typically, the leading order terms in such expansions in stochastic homogenization only provide a local approximation, while control of the the weak error (after integration against a test function) requires extra terms. This is partly because the higher the order of the corrector, the slower the spatial decay of its covariance function, leading to accumulation of errors from terms of all orders. We circumvent this issue by borrowing some ideas reminiscent of the "straight line" approximation of trajectories on "long but not too long" time scales in random finite-dimensional models of particles in a random velocity fields or random forces. If we look at (1.16) for each macroscopic $t>0$ and $x\in\mathbb{R}^d$ fixed, as an evolution problem in s, we would have a "complete separation of scales" factorization

$$u_1(t,x,s,y) = \sum_{k=1}^{d}\zeta^{(k)}(s,y)\frac{\partial\overline{u}(t,x)}{\partial x_k}, \tag{1.18}$$

where $\zeta(s,y)$ is a solution to the microscopic problem

$$\partial_s\zeta^{(k)} = \frac{1}{2}\Delta\zeta^{(k)} + (\beta V(s,y)-\lambda)\zeta^{(k)} + \frac{\partial\Psi(s,y)}{\partial y_k}, \tag{1.19}$$

defined for all $s>0$ and $y\in\mathbb{R}^d$. Instead of using (1.18) directly, we consider "microscopically long but macroscopically short" time intervals in s of the size $\varepsilon^{-\gamma}$, with some $\gamma\in(1,2)$. Accordingly, for each $j\geq 1$, let $\theta_j^{(k)}(s,y)$, with $1\leq k\leq d$, be the solution to

$$\begin{aligned}&\partial_s\theta_j^{(k)} = \frac{1}{2}\Delta_y\theta_j^{(k)} + (\beta V(s,y)-\lambda)\theta_j + \frac{\partial\Psi(s,y)}{\partial y_k}, \quad s>\varepsilon^{-\gamma}(j-1),\\ &\theta_j^{(k)}(\varepsilon^{-\gamma}(j-1),y) = 0.\end{aligned} \tag{1.20}$$

Then, define $u_{1;j}(s,y)$ as the solution to

$$\begin{aligned}&\partial_s u_{1;j} = \frac{1}{2}\Delta u_{1;j} + (\beta V(s,y)-\lambda)u_{1;j}, \quad s>\varepsilon^{-\gamma}j,\\ &u_{1;j}(\varepsilon^{-\gamma}j,y) = \sum_{k=1}^{d}\theta_j^{(k)}(\varepsilon^{-\gamma}j,y)\frac{\partial\overline{u}(\varepsilon^{2-\gamma}j,\varepsilon y)}{\partial x_k},\end{aligned} \tag{1.21}$$

and finally put

$$u_1^{\varepsilon}(t,x) = \sum_{j=1}^{[\varepsilon^{\gamma-2}t]} u_{1;j}(\varepsilon^{-2}t,\varepsilon^{-1}x) + \theta_{[\varepsilon^{\gamma-2}t]+1}(\varepsilon^{-2}t,\varepsilon^{-1}x)\cdot\nabla\overline{u}(t,x). \tag{1.22}$$

This is similar to putting $s = t/\varepsilon^2$, $y = x/\varepsilon$ in the formal PDE (1.16), except that rather than continuously multiplying the forcing by $\nabla\overline{u}$, the multiplication by $\nabla\overline{u}$ happens only at discrete times. With this definition of u_1^ε, we have a weak convergence theorem for the fluctuations:

Theorem 1.4. *Suppose that $0 \le \beta < \beta_0$, and let $g \in \mathcal{C}_c^\infty(\mathbb{R}^d)$. For any $\zeta < (1-\gamma/2) \wedge (\gamma - 1)$, there exists $C > 0$ so that*

$$\operatorname{Var}\left(\varepsilon^{-d/2+1}\int g(x)\left[u^\varepsilon(t,x) - \Psi^\varepsilon(t,x)\overline{u}(t,x) - \varepsilon u_1^\varepsilon(t,x)\right]\mathrm{d}x\right) \le C\varepsilon^{2\zeta}. \tag{1.23}$$

The optimal bound in Theorem 1.4 is achieved when $\gamma = 4/3$, in which case ζ is required to be less than $1/3$.

We note that it would be hopeless to get a convergence-of-fluctuations result like Theorem 1.4 even with an error of size $\varepsilon^{d/2-1}$ as in (1.7)–(1.8), using only the first term of the expansion as in Theorem 1.3. This is because at that scale, [13] gives different central limit theorem statements for u and for $\Psi\overline{u}$: the rescaled and renormalized fluctuations of u converge to a solution of the SPDE

$$\partial_t \mathscr{U}(t,x) = \frac{1}{2}a\Delta\mathscr{U}(t,x) + \beta\nu\overline{u}(t,x)\dot{W}(t,x), \tag{1.24}$$

while the rescaled and renormalized fluctuations of Ψ converge to a solution of the SPDE

$$\partial_t\psi(t,x) = \frac{1}{2}a\Delta\mathscr{U}(t,x) + \beta\nu\dot{W}(t,x), \tag{1.25}$$

and so the rescaled and renormalized fluctuations of $\Psi\overline{u}$ converge to a solution of the SPDE

$$\begin{aligned}\partial_t(\psi\overline{u})(t,x) &= \frac{1}{2}a\overline{u}(t,x)\Delta\psi(t,x) + \beta\nu\overline{u}(t,x)\dot{W}(t,x) + \frac{1}{2}a\psi(t,x)\Delta\overline{u}(t,x)\\ &= \frac{1}{2}a\Delta(\psi\overline{u})(t,x) - a\nabla\psi(t,x)\cdot\nabla\overline{u}(t,x) + \beta\nu\overline{u}(t,x)\dot{W}(t,x).\end{aligned} \tag{1.26}$$

The limiting SPDEs (1.24) and (1.26) are not the same, so some extra corrector is needed. It is also important to note that Theorem 1.4 holds precisely because we take the leading order term with $\Psi(s,y)$ rather than the stationary solution $\tilde{\Psi}(s,y)$. It is a combination of the "correct" leading order and the specific construction of u_1^ε that leads to the expansion capturing correctly the macroscopic error after integration against a test function.

Let us explain intuitively how the definitions (1.20)–(1.22) come from (1.16). They sit midway between two natural ways of interpreting (1.16). On one hand, (1.16), for fixed x and t, can be solved as in (1.18)-(1.19). However, defining the corrector u_1 by (1.18) and evaluating at $s = t/\varepsilon^2$ and $y = x/\varepsilon$ does not seem to yield a good convergence result, because $\nabla_x\overline{u}(\tau,x)$ is not constant on the time scale from $\tau = 0$ to $\tau = \varepsilon^2 s = t$. Instead, (1.20)–(1.22) gives a way to solve the corrector problem on shorter time intervals and sum up the contributions from each time interval, so that an appropriate value of $\nabla_x\overline{u}(\tau,x)$ is used for each time interval. On the other hand, (1.16) could be solved by plugging $t = \varepsilon^2 s$, $x = \varepsilon y$ into (1.16), yielding the PDE

$$\partial_s u_1(s,y) = \frac{1}{2}\Delta_y u_1(s,y) + (\beta V(s,y) - \lambda)u_1(s,y) + \nabla_y\Psi(s,y)\cdot\nabla_x\overline{u}(\varepsilon^2 s,\varepsilon y). \tag{1.27}$$

However, we do not obtain a convergence result along the lines of Theorem 1.4 for this definition of u_1 either. This is because the Feynman–Kac formula that arises from the solution to (1.27) involves the behavior of the Markov chain of [13] (reviewed in Section 2 below) on *short* time scales, while the limits appear to arise from the averaged behavior of the Markov chain on *long* time scales.

The delay in multiplying by $\nabla_x \overline{u}$ introduced by the updates on the time scale $\varepsilon^{-\gamma}$ allows the short-time fluctuations to be averaged out, leaving only the effect of the long-time scales, which allows us to see the limiting behavior. Once again, this is not unrelated to the strategy in the proofs of the convergence of particle in random velocity fields and the random acceleration problem to the corresponding diffusive limits.

Organization of the paper. Our results all rely on the Feynman–Kac formula, and in particular a certain Markov chain representing a tilted Brownian path introduced in [13]. Thus we review the relevant results from that work in Section 2. Section 2 also contains additional facts about the tilted Markov chain that will be necessary in our proofs. In Section 3 we establish the existence of the stationary solution $\widetilde{\Psi}$ (Theorem 1.1). The next two sections are devoted to the parameters a and ν obtained in [13]: in Section 4 we prove Theorem 1.2 regarding the effective noise strength ν, and in Section 5 we show how the effective diffusivity can be recovered from the formal asymptotic expansion. Finally, in the last two sections we establish our convergence results for the formal asymptotic expansion: the strong convergence Theorem 1.3 in Section 6, and the weak convergence of fluctuations Theorem 1.4 in Section 7.

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2 The Markov chain for the tilted paths

Preliminaries and basic facts

In this section, we recall some facts on the Markov chain for the tilted path from [13] that appears naturally in the Feynman-Kac formula for the solutions to the random heat equation that we will use extensively in the rest of the paper. Let us introduce some notation. By $\mathbb{E}_B^y$ we denote expectation with respect to the probability measure in which $B = (B^1, \ldots, B^d)$ is a standard d-dimensional Brownian motion with $B_0 = y$. For any $\mathfrak{A} \subset \mathbb{R}$, we set

$$\mathscr{V}_{\mathfrak{A}}[B] = \int_{\mathfrak{A}} V(s-\tau, B_\tau)\,\mathrm{d}\tau. \tag{2.1}$$

We will often use the shorthand $\mathscr{V}_s = \mathscr{V}_{[0,s]}$. We define, for any $\mathfrak{A}, \widetilde{\mathfrak{A}} \subset \mathbb{R}$,

$$\mathscr{R}_{\mathfrak{A},\widetilde{\mathfrak{A}}}[B,\widetilde{B}] = \mathbb{E}\big(\mathscr{V}_{\mathfrak{A}}[B]\mathscr{V}_{\widetilde{\mathfrak{A}}}[\widetilde{B}]\big) = \int_{\widetilde{\mathfrak{A}}}\int_{\mathfrak{A}} R(\tau-\tilde{\tau}, B_\tau - \widetilde{B}_{\tilde{\tau}})\,\mathrm{d}\tau\,\mathrm{d}\tilde{\tau}. \tag{2.2}$$

We will often use the abbreviations $\mathscr{R}_{s,\tilde{s}} = \mathscr{R}_{[0,s],[0,\tilde{s}]}$, $\mathscr{R}_{\mathfrak{A}} = \mathscr{R}_{\mathfrak{A},\mathfrak{A}}$, and $\mathscr{R}_s = \mathscr{R}_{s,s}$. We will also abbreviate $\mathscr{R}_\bullet[B] = \mathscr{R}_\bullet[B,B]$, where the $\bullet$ can be replaced by any of the allowable subscripts for $\mathscr{R}$. The tilted measure $\widehat{\mathbb{P}}_{B;\bullet}^y$ is determined by

$$\widehat{\mathbb{E}}_{B;\bullet}^y \mathscr{F}[B] = \frac{1}{Z_\bullet}\mathbb{E}_B^y\Big[\mathscr{F}[B]\exp\Big\{\frac{1}{2}\beta^2\mathscr{R}_\bullet[B]\Big\}\Big], \quad Z_\bullet = \mathbb{E}_B^y \exp\Big\{\frac{1}{2}\beta^2\mathscr{R}_\bullet[B]\Big\},$$

for any measurable functional $\mathscr{F}$ on the space $\mathcal{C}([0,\infty);\mathbb{R}^d)$, where again the $\bullet$ can be replaced by any of the allowable subscripts for $\mathscr{R}$. We further define

$$\alpha_s = \log Z_s - \lambda s, \tag{2.3}$$

and note that, according to [13, Lemma A.1] and its proof, there exists a unique $\lambda = \lambda(\beta)$ such that

$$|\alpha_s - \alpha_\infty| \le C\mathrm{e}^{-cs}. \tag{2.4}$$

for some $\alpha_\infty > 0$, $c > 0$ and $C > 0$. This is where the constant $\lambda(\beta)$ comes from. We fix this value for the rest of the paper. We denote by $\Xi_T = \{\omega \in \mathcal{C}([0,T]) \mid \omega(0) = 0\}$, and, given $W_i \in \Xi_{T_i}$, we define $[W_1, \dots, W_k] \in \Xi_T$, for $T = \sum T_i$ by the concatenation of the increments, as in [13].

The expectation with respect to the product measure $\widehat{\mathbb{P}}^y_{B;\bullet} \otimes \widehat{\mathbb{P}}^{\tilde{y}}_{\widetilde{B};\bullet}$ will be denoted by $\widehat{\mathbb{E}}^{y,\tilde{y}}_{B,\widetilde{B};\bullet}$.

Theorem 2.1 ([13])**.** *Let $T > 1$ and $N = [T] - 1$. Then there is a Markov chain*

$$w_0, w_1, \dots, w_N, w_{N+1}$$

such that $w_0 \in \Xi_{T-[T]}$ and $w_j \in \Xi_1$ for $1 \le j \le N+1$ and the transition probability

$$\hat{\pi}(w_j, w_{j+1}) = \mathrm{Law}(w_{j+1} \mid w_j)$$

does not depend on j for $j = 1, \dots, N-1$ and such that if we put $W = [w_0, \dots, w_{N+1}] \in \Xi_T$ then we have, for any bounded continuous function $\mathscr{F}$ on Ξ_T, that

$$\widehat{\mathbb{E}}_{B;T} \mathscr{F}[B] = \widetilde{\mathbb{E}}_W \left[\mathscr{F}[W] \mathscr{G}[w_N]\right], \tag{2.5}$$

where $\widetilde{\mathbb{E}}_W$ is expectation with respect to the measure in which W is obtained from the Markov chain, and $\mathscr{G} : \Xi_1 \to \mathbf{R}$ is bounded, measurable, even, and independent of T. Moreover, there is an auxiliary sequence of i.i.d. Bernoulli random variables η^W_j, $j = 1, 2, \dots$, with distribution not depending on T, so that $\mathrm{Law}(w_j \mid \eta_j = 1, w_i, i < j) = \overline{\pi}$, where $\overline{\pi}$ is the invariant measure of $\hat{\pi}$.

We will denote

$$\widetilde{\mathbb{E}}^y_W \mathscr{F}[W] = \widetilde{\mathbb{E}}_W \mathscr{F}[y + W]. \tag{2.6}$$

Define the stopping times $\sigma^W_0 = 0$, $\sigma^W_n = \min\{t \ge \sigma_{n-1} \mid \eta^W_t = 1\}$, and put, for $n \ge 0$,

$$\mathbf{W}^W_n = W_{\sigma^W_{n+1}} - W_{\sigma^W_n}.$$

Lemma 2.2 ([13, Lemma A.1])**.** *The family $\{\mathbf{W}^W_n\}_{n \ge 0}$ is a collection of independent, symmetric, mean-0 random variables with exponential tails. Moreover, $\mathbf{W}^W_1, \mathbf{W}^W_2, \dots$ are identically distributed.*

The above procedure can be applied to pairs of paths as well. Given two independent copies $W, \widetilde{W}$ of the Markov chain, define

$$\eta^{W,\widetilde{W}}_j = \eta^W_j \eta^{\widetilde{W}}_j,$$

and the stopping times

$$\sigma^{W,\widetilde{W}}_n = \begin{cases} 0 & n = 0 \\ \min\{t \ge \sigma_{n-1} \mid \eta^{W,\widetilde{W}}_t = 1\} & n \ge 1. \end{cases}$$

Put

$$\mathbf{W}^{W,\widetilde{W}}_n = W_{\sigma^{W,\widetilde{W}}_{n+1}} - W_{\sigma^{W,\widetilde{W}}_n}$$
$$\widetilde{\mathbf{W}}^{W,\widetilde{W}}_n = \widetilde{W}_{\sigma^{W,\widetilde{W}}_{n+1}} - \widetilde{W}_{\sigma^{W,\widetilde{W}}_n}.$$

Analogously to (2.6), we use the notation $\widetilde{\mathbb{P}}^{y,\tilde{y}}_{W,\widetilde{W}} = \widetilde{\mathbb{P}}^y_W \otimes \widetilde{\mathbb{P}}^{\tilde{y}}_{\widetilde{W}}$.

Corollary 2.3 ([13, p. 17])**.** *The family* $\{\mathbf{W}_n^{W,\widetilde{W}}\}_{n\geq 0}\cup\{\widetilde{\mathbf{W}}_n^{W,\widetilde{W}}\}_{n\geq 0}$ *is a collection of independent, symmetric, mean-*0 *random variables with exponential tails. Moreover,*

$$\mathbf{W}_1^{W,\widetilde{W}},\mathbf{W}_2^{W,\widetilde{W}},\ldots,\widetilde{\mathbf{W}}_1^{W,\widetilde{W}},\widetilde{\mathbf{W}}_2^{W,\widetilde{W}},\ldots$$

are all identically distributed.

Let us set

$$\kappa^W=\mathbb{P}(\eta_j=1),\quad \kappa^{W,\widetilde{W}}=\mathbb{P}(\eta_j^{W,\widetilde{W}}=1)=\mathbb{P}(\eta_j^W=1)^2. \tag{2.7}$$

The next proposition gives an expression for the effective diffusivity in (1.6) in terms of the Markov chain.

Proposition 2.4 ([13, Proposition 4.1])**.** *There is a diagonal* $d\times d$ *matrix*

$$\mathbf{a}=aI_{d\times d}=\kappa^W\widetilde{\mathbb{E}}_W[\mathbf{W}_n^W(\mathbf{W}_n^W)^\top] \tag{2.8}$$

so that as $\varepsilon\to 0$*, the process* $\{\varepsilon W_{\varepsilon^2\tau}\}_{0\leq\tau\leq t}$ *converges in distribution in* $\mathcal{C}([0,t])$ *(under* $\widetilde{\mathbb{P}}_W$*) to a Brownian motion with covariance matrix* $\mathbf{a}$*.*

Proposition 2.5 ([13, Corollary 4.4])**.** *If* $d\geq 3$*, there is a* $\beta_0>0$ *and a constant* $C<\infty$ *so that if* $0\leq\beta<\beta_0$ *then for any* $s\geq 0$*,* $y,\tilde{y}\in\mathbb{R}^d$*, we have*

$$\widetilde{\mathbb{E}}_{W,\widetilde{W}}^{y,\tilde{y}}\left[\exp\left\{\beta^2\mathscr{R}_{[s,\infty]}[W,\widetilde{W}]\right\}\mid\mathcal{F}_s\right]\leq C\quad a.s.$$

Here, $\mathcal{F}_s$ is the sigma algebra generated by the paths $W,\widetilde{W}$ up to time s; we emphasize that the constant C in Proposition 2.5 is deterministic.

We will require a slightly strengthened version of Proposition 2.5, which can be proved similarly.

Proposition 2.6. *If* $d\geq 3$*, there is a* $\beta_0>0$ *and a constant* $C<\infty$ *so that if* $0\leq\beta<\beta_0$ *then for all* $r,\tilde{r}>0$*, we have*

$$\widetilde{\mathbb{E}}_{W,\widetilde{W}}\left[\exp\left\{\beta^2\mathscr{R}_\infty[W,\widetilde{W}]\right\}\,\middle|\,W_r,\widetilde{W}_{\tilde{r}}\right]\leq C,\quad a.s.$$

In Proposition 2.6, the constant C is deterministic and the sigma-algebra $W_r,\widetilde{W}_{\tilde{r}}$ is the one generated by the paths $W,\widetilde{W}$ up to times $r,\tilde{r}$ respectively.

We also need some estimates from [13] on various error terms.

Lemma 2.7 ([13, (4.30)])**.** *There is a constant* C_{v} *so that*

$$\widetilde{\mathbb{E}}_W\left|\varepsilon W_{\varepsilon^{-2}t_2}-\varepsilon W_{\varepsilon^{-2}t_1}\right|^2\leq C_{\mathrm{v}}(t_2-t_1). \tag{2.9}$$

Lemma 2.8 ([13, Lemma A.3])**.** *For any* $\chi>0$*, there are constants* $0<c,C<\infty$ *so that if* $F_T:\Xi_T\to\mathbf{R}$ *is a family of uniformly bounded functions for each* T *fixed, and* $\{S_n\}$*,* $\{T_n\}$ *are sequences of real numbers such that* $S_n,T_n,S_n-T_n\to\infty$*, then*

$$\begin{aligned}&\left|\widetilde{\mathbb{E}}_W F_{T_n}(W\restriction_{[0,T_n]})-\widetilde{\mathbb{E}}_W F_{T_n}(W\restriction_{[0,T_n]})\mathscr{G}(w_{S_n})\right|\\&\leq C\left(\widetilde{\mathbb{E}}_W\left(F_{T_n}(W\restriction_{[0,T_n]})\right)^\chi\right)^{1/\chi}\exp\left\{-c(T_n\wedge(S_n-T_n))\right\}.\end{aligned}$$

Here, $\mathscr{G}$ is as in Theorem 2.1.

Remark 2.9. The rate of convergence is not stated explicitly in [13] but it comes from the proof there.

Lemma 2.10 ([13, Lemma A.2])**.** *We have constants* $0<c,\ C<\infty$ *so that*

$$\widetilde{\mathbb{P}}_{W,\widetilde{W}}^{x,\tilde{x}}\left[\max_{r,\tilde{r}\in[\sigma_n,\sigma_{n+2}]}\left(|W_r-W_{\sigma_n^{W,\widetilde{W}}}|+|\widetilde{W}_{\tilde{r}}-\widetilde{W}_{\sigma_n^{W,\widetilde{W}}}|\right)>a\right]\leq C\mathrm{e}^{-ca}.$$

Estimates on path intersections

Finally, we prove a fact about the tilted Markov chain that will be essential for us: that two paths started at points at distance of order ε^{-1} get close to each other with probability ε^{d-2}.

Proposition 2.11. *There is a constant C so that*

$$\widetilde{\mathbb{P}}^{x,\tilde{x}}_{W,\widetilde{W}}\left[\inf_{\substack{r,\tilde{r}>0\\|r-\tilde{r}|\le 1}}\left|W_r-\widetilde{W}_{\tilde{r}}\right|\le 1\right]\le\frac{C}{|x-\tilde{x}|^{d-2}}.$$

In order to prove Proposition 2.11, we first prove the same result just at regeneration times. For the rest of this section, to economize on notation we will put $\sigma_n=\sigma_n^{W,\widetilde{W}}$.

Lemma 2.12. *We have*

$$\widetilde{\mathbb{P}}^{x,\tilde{x}}_{W,\widetilde{W}}\left[\inf_{n\ge 0}\left|W_{\sigma_n}-\widetilde{W}_{\sigma_n}\right|\le A\right]\le\frac{A^{d-2}}{|x-\tilde{x}|^{d-2}}.$$

Proof. Let

$$X_n=W_{\sigma_n}-\widetilde{W}_{\sigma_n},$$

and set

$$q(z)=\frac{1}{(|z|\vee A)^{d-2}}.$$

For any $z\in\mathbb{R}^d$ with $|z|\ge A$ and any $M>0$, if we let $\mathrm{d}S$ denote the surface measure on $\{|\tilde{z}-z|=M\}$, then we have

$$⨍_{|\tilde{z}-z|=M}q(\tilde{z})\,\mathrm{d}S(\tilde{z})\le ⨍_{|\tilde{z}-z|=M}\frac{1}{|\tilde{z}|^{d-2}}\,\mathrm{d}S(\tilde{z})\le\frac{1}{|z|^{d-2}}=q(z)\tag{2.10}$$

by the mean-value inequality for superharmonic functions, as $z\mapsto|z|^{-d+2}$ is superharmonic. Let ω be the smallest n such that $|X_n|\le A$, or ∞ if $|X_n|>A$ for all A. Since the distribution of X_n-X_{n-1} is radially symmetric for each $n\ge 1$, (2.10) means that the sequence $(q(X_{n\wedge\omega}))$ is a supermartingale. By the optional stopping theorem, for any n we have

$$\frac{1}{|x-\tilde{x}|^{d-2}}=q(X_0)\ge\widetilde{\mathbb{E}}_{W,\widetilde{W}}q(X_{n\wedge\omega})\ge\frac{1}{A^{d-2}}\widetilde{\mathbb{P}}_{W,\widetilde{W}}(\omega\le n).$$

Therefore, we have

$$\widetilde{\mathbb{P}}_{W,\widetilde{W}}(\omega<\infty)\le\frac{A^{d-2}}{|x-\tilde{x}|^{d-2}}$$

by Fatou's lemma. □

Proof of Proposition 2.11. Let

$$B_n=\max_{r,\tilde{r}\in[\sigma_n,\sigma_{n+2}]}\left(|W_r-W_{\sigma_n^{W,\tilde{W}}}|+|\widetilde{W}_{\tilde{r}}-\widetilde{W}_{\sigma_n^{W,\widetilde{W}}}|\right)$$

and

$$\omega_M=\inf\left\{n\ge 0\;:\;|W_{\sigma_n}-\widetilde{W}_{\sigma_n}|\le 2^M\right\}.$$

We have

$$\Big\{\inf_{|r-\tilde r|\le 1}\Big|W_r-\widetilde W_{\tilde r}\Big|\le 1\Big\}\subseteq\bigcup_{M=0}^{\infty}\bigcup_{n=0}^{\infty}\Big(\Big\{\Big|W_{\sigma_n}-\widetilde W_{\sigma_n}\Big|\le 2^M\Big\}\cap\Big\{B_n\ge 2^{M-1}-1\Big\}\Big)$$
$$\subseteq\bigcup_{M=0}^{\infty}\left[\{\omega_M<\infty\}\cap\left(\bigcup_{n=\omega_M}^{\infty}\Big(\Big\{\Big|W_{\sigma_n}-\widetilde W_{\sigma_n}\Big|\le 2^M\Big\}\cap\Big\{B_n\ge 2^{M-1}-1\Big\}\Big)\right)\right]. \tag{2.11}$$

Therefore, we can estimate (abbreviating $\mathbb{P}=\widetilde{\mathbb{P}}^{x,\tilde x}_{W,\widetilde W}$ and letting the constant C change from line to line)

$$\begin{aligned}
&\mathbb{P}\Big[\inf_{|r-\tilde r|\le 1}|W_r-\widetilde W_{\tilde r}|\le 1\Big]\le\sum_{M,\ell=0}^{\infty}\mathbb{P}\left[\omega_M=\ell\right]\sum_{n=\ell}^{\infty}\mathbb{P}\left[|W_{\sigma_n}-\widetilde W_{\sigma_n}|\le 2^M\Big|\,\omega_M=\ell\right]\mathbb{P}\Big[B_n\ge 2^{M-1}-1\Big]\\
&\le C\sum_{M=0}^{\infty}\mathrm{e}^{-c(2^{M-1}-1)}\sum_{\ell=0}^{\infty}\mathbb{P}\left[\omega_M=\ell\right]\sum_{n=\ell}^{\infty}\frac{2^{Md}}{(n-\ell+1)^{d/2}}=C\sum_{M=0}^{\infty}\mathrm{e}^{-c(2^{M-1}-1)+CMd}\mathbb{P}\left[\omega_M<\infty\right]\\
&\le C\sum_{M=0}^{\infty}\mathrm{e}^{-c(2^{M-1}-1)+CMd}\cdot\frac{2^{(d-2)M}}{|x-\tilde x|^{d-2}}\le\frac{C}{|x-\tilde x|^{d-2}},
\end{aligned} \tag{2.12}$$

where the first inequality is by (2.11), the second is by Lemma 2.10 and a local central limit theorem ([17] as applied in [13, (4.36)]), and the third is by Lemma 2.12. □

We will also need a slightly different version of the bound in Proposition 2.11:

Proposition 2.13. *There is a constant C so that*

$$\widetilde{\mathbb{P}}^{x,\tilde x}_{W,\widetilde W}\Big[\inf_{r,\tilde r>s,|r-\tilde r|\le 1}\Big|W_r-\widetilde W_{\tilde r}\Big|\le 1\Big]\le Cs^{-(d/2)+1}.$$

To show this, we first need to establish another simple lemma.

Lemma 2.14. *Suppose that $T_1,\ldots,T_J$ are iid geometric random variables taking values $n=1,2,\ldots,$ omitting $n=0$. There exists $c>0$ so that*

$$\mathbb{P}\Big(\sum_{j=1}^{J}T_j>2J\mathbb{E}T_i\Big)\le\mathrm{e}^{-cJ}.$$

Proof. The result follows from large deviations estimates in the form of Cramer's theorem. We instead provide a direct proof based on Chebyshev's inequality. Let $m=\mathbb{E}T_i$. We have

$$\begin{aligned}
\mathbb{P}\Big(\sum_{j=1}^{J}T_j>2mJ\Big)&=\mathbb{P}\Big(\exp\Big\{\xi\sum_{j=1}^{J}T_j\Big\}>\exp\{2\xi mJ\}\Big)\le\frac{\mathbb{E}\exp\Big\{\xi\sum_{j=1}^{J}T_j\Big\}}{\exp\{2\xi mJ\}}=\left(\frac{\mathbb{E}\exp\{\xi T_j\}}{\exp\{2\xi m\}}\right)^J\\
&=\Big((m\mathrm{e}^{-\xi}-(m-1))\mathrm{e}^{2m\xi}\Big)^{-J}:=(\alpha(\xi))^{-J}.
\end{aligned} \tag{2.13}$$

As

$$\frac{d\alpha(\xi)}{d\xi}\Big|_{\xi=0}=\frac{d}{d\xi}\Big((m\mathrm{e}^{-\xi}-(m-1))\mathrm{e}^{2m\xi}\Big)\Big|_{\xi=0}=m>0, \tag{2.14}$$

there is a $\xi>0$ small enough so that $\alpha(\xi)>1$, which proves the lemma. □

Proof of Proposition 2.13. Recall the definition (2.7) of $\kappa^{W,\widetilde{W}}$ and put

$$n_0 = \frac{s}{2\kappa^{W,\widetilde{W}}}.$$

We can then estimate

$$\mathbb{P}\Big[\inf_{r,\tilde{r}>s,|r-\tilde{r}|\le 1} |W_r - \widetilde{W}_{\tilde{r}}| \le 1\Big] \le \mathbb{P}\Big[\inf_{r,\tilde{r}>\sigma_{n_0},|r-\tilde{r}|\le 1} \Big|W_r - \widetilde{W}_{\tilde{r}}\Big| \le 1\Big] + \mathbb{P}\left[\sigma_{n_0} \ge s\right].$$

By Lemma 2.14, we have that

$$\mathbb{P}\left[\sigma_{n_0} \ge s\right] \le \mathrm{e}^{-cn_0} \le Cs^{1-d/2},$$

so it suffices to show that

$$\mathbb{P}\Big[\inf_{r,\tilde{r}>\sigma_{n_0},|r-\tilde{r}|\le 1} \Big|W_r - \widetilde{W}_{\tilde{r}}\Big| \le 1\Big] \le Cn_0^{1-d/2}.$$

Define

$$B_k = \max_{r,\tilde{r}\in[\sigma_k,\sigma_{k+2}]} \Big(|W_r - W_{\sigma_k(s)}| + |\widetilde{W}_{\tilde{r}} - \widetilde{W}_{\sigma_k(s)}|\Big),$$

then we have

$$\begin{aligned}\mathbb{P}\Big[\inf_{r,\tilde{r}>\sigma_{n_0},|r-\tilde{r}|\le 1} \Big|W_r - \widetilde{W}_{\tilde{r}}\Big| \le 1\Big] &\le \sum_{M=0}^{\infty}\sum_{k=n_0}^{\infty} \mathbb{P}\left[\Big|W_{\sigma_k} - \widetilde{W}_{\sigma_k}\Big| \le 2^M\right] \mathbb{P}\left[B_k \ge 2^{M-1}-1\right] \\ &\le C\sum_{M=0}^{\infty} \mathrm{e}^{-c(2^{M-1}-1)} \sum_{k=n_0}^{\infty} \frac{2^{Md}}{k^{d/2}} = Cn_0^{1-d/2}\sum_{M=0}^{\infty} \mathrm{e}^{-c(2^{M-1}-1)+CMd} \le Cn_0^{1-d/2},\end{aligned}$$

where the second inequality uses the local limit theorem of [17]. □

3 Existence of a stationary solution

In this section we prove Theorem 1.1. The strategy is typical for such problems: we consider the Cauchy problem on the time interval $s \in (-S, \infty)$ and pass to the limit $S \to +\infty$, obtaining a global in time solution to the problem that satisfies appropriate uniform bounds, provided that the Lyapunov exponent $\lambda(\beta)$ is chosen so that (2.4) holds. To this end, put

$$\Psi(s,y;S) = \begin{cases} 1 & s \le -S, \\ \mathbb{E}_B^y \exp\left\{\beta\mathscr{V}_{s+S}[B] - \lambda(s+S)\right\} & s > -S, \end{cases} \tag{3.1}$$

where we recall that

$$\mathscr{V}_{s+S}[B] = \int_0^{s+S} V(s-\tau, B_\tau)d\tau.$$

By the Feynman–Kac formula, Ψ solves the Cauchy problem

$$\begin{aligned}&\partial_s\Psi(s,y;S) = \frac{1}{2}\Delta\Psi(s,y) + \beta V(s,y)\Psi(s,y) - \lambda\Psi(s,y), \quad s > -S \\ &\Psi(s,y;S) = 1, \quad s \le -S.\end{aligned} \tag{3.2}$$

We first prove a preliminary lemma. Recall the notation $\mathscr{R}$, see (2.2).

Lemma 3.1. *There exists a constant $C<\infty$ so that for all β sufficiently small, the following holds. If $s\le s'\le\tilde s\le\tilde s'$, then*

$$\widetilde{\mathbb{E}}^{y,\tilde y}_{W,\widetilde W}\left|\exp\left\{\beta^2\mathscr{R}_{\tilde s,\tilde s'}[W,\widetilde W]\right\}-\exp\left\{\beta^2\mathscr{R}_{s,s'}[W,\widetilde W]\right\}\right|\le C(s-1)^{1-d/2}. \tag{3.3}$$

Proof. We have

$$\begin{aligned}&\widetilde{\mathbb{E}}^{y,\tilde y}_{W,\widetilde W}\left|\exp\left\{\beta^2\mathscr{R}_{\tilde s,\tilde s'}[W,\widetilde W]\right\}-\exp\left\{\beta^2\mathscr{R}_{s,s'}[W,\widetilde W]\right\}\right|\\ &\quad\le\widetilde{\mathbb{E}}^{y,\tilde y}_{W,\widetilde W}\left|\exp\left\{\beta^2\mathscr{R}_{\infty}[W,\widetilde W]\right\}-\exp\left\{\beta^2\mathscr{R}_{s}[W,\widetilde W]\right\}\right|\\ &\quad\le\widetilde{\mathbb{E}}^{y,\tilde y}_{W,\widetilde W}\exp\left\{\beta^2\mathscr{R}_{\infty}[W,\widetilde W]\right\}\mathbf{1}_{\mathscr{R}_\infty[W,\widetilde W]\neq\mathscr{R}_s[W,\widetilde W]}\\ &\quad\le\widetilde{\mathbb{E}}^{y,\tilde y}_{W,\widetilde W}\exp\left\{\beta^2\mathscr{R}_{\infty}[W,\widetilde W]\right\}\mathbf{1}\{(\exists r,\tilde r\ge s-1)\ |W_r-\widetilde W_{\tilde r}|\le 1\}.\end{aligned} \tag{3.4}$$

On the event $\{\mathscr{R}_\infty[W,\widetilde W]\neq\mathscr{R}_s[W,\widetilde W]\}$, let $\tau<\tilde\tau$ be the first pair of times after $s-1$ such that $|\tau-\tilde\tau|\le 1$ and $|W_\tau-\widetilde W_{\tilde\tau}|\le 1$. Then, we have

$$\begin{aligned}&\widetilde{\mathbb{E}}^{y,\tilde y}_{W,\widetilde W}\left|\exp\left\{\beta^2\mathscr{R}_{\tilde s,\tilde s'}[W,\widetilde W]\right\}-\exp\left\{\beta^2\mathscr{R}_{s,s'}[W,\widetilde W]\right\}\right|\\ &\quad\le\int_{s-1}^{\infty}\int_r^{r+1}\widetilde{\mathbb{E}}^{y,\tilde y}_{W,\widetilde W}\left[\exp\left\{\beta^2\mathscr{R}_{\infty}[W,\widetilde W]\right\}\mid\tau=r,\tilde\tau=\tilde r\right]\mathrm{d}\mathbb{P}(\tau=r,\tilde\tau=\tilde r)\\ &\quad\le C\widetilde{\mathbb{P}}^{y,\tilde y}_{W,\widetilde W}\left[(\exists r,\tilde r\ge s-1)\ |W_r-\widetilde W_{\tilde r}|\le 1\right]\\ &\quad\le C(s-1)^{1-d/2},\end{aligned} \tag{3.5}$$

where the second inequality is by Proposition 2.6 and the last is by Proposition 2.13. □

The next proposition is a key step to show the convergence of the family $\Psi(s,y;S)$ as $S\to+\infty$.

Proposition 3.2. *If β is sufficiently small then, for $0\le S_1\le S_2$, we have*

$$\mathbb{E}\left(\Psi(0,y;S_2)-\Psi(0,y;S_1)\right)^2\le CS_1^{-d/2+1},$$

Proof. Without loss of generality, we consider $y=0$. We have, with α_s as in (2.3) and $\mathscr{V}$ as in (2.1),

$$\mathbb{E}\mathbb{E}_{B,\widetilde B}\exp\left\{\beta\mathscr{V}_{S_1}[B]-\lambda S_1+\beta\mathscr{V}_{S_2}[\widetilde B]-\lambda S_2\right\}=\mathrm{e}^{\alpha_{S_1}+\alpha_{S_2}}\widehat{\mathbb{E}}_{B;S_1}\widehat{\mathbb{E}}_{\widetilde B;S_2}\exp\left\{\beta^2\mathscr{R}_{S_1,S_2}[B,\widetilde B]\right\}.$$

Therefore, we have

$$\begin{aligned}\mathbb{E}\left(\Psi(0,0;S_2)-\Psi(0,0;S_1)\right)^2&=\mathbb{E}\left(\mathbb{E}_B\left[\exp\{\beta\mathscr{V}_{S_2}[B]-\lambda S_2\}-\exp\{\beta\mathscr{V}_{S_2}[B]-\lambda S_1\}\right]\right)^2\\ &=\mathrm{e}^{2\alpha_{S_2}}\widehat{\mathbb{E}}_{B,\widetilde B;S_2}\exp\left\{\beta^2\mathscr{R}_{S_2}[B,\widetilde B]\right\}-2\mathrm{e}^{\alpha_{S_2}+\alpha_{S_1}}\widehat{\mathbb{E}}_{B;S_2}\widehat{\mathbb{E}}_{\widetilde B;S_1}\exp\left\{\beta^2\mathscr{R}_{S_2,S_1}[B,\widetilde B]\right\}\\ &\quad+\mathrm{e}^{2\alpha_{S_1}}\widehat{\mathbb{E}}_{B,\widetilde B;S_1}\exp\left\{\beta^2\mathscr{R}_{S_1}[B,\widetilde B]\right\},\end{aligned} \tag{3.6}$$

which can be re-written as

$$\begin{aligned}\mathbb{E}\left(\Psi(0,0;S_2)-\Psi(0,0;S_1)\right)^2&=\widetilde{\mathbb{E}}_{W,\widetilde W}\Big[\mathrm{e}^{2\alpha_{S_2}}\exp\left\{\beta^2\mathscr{R}_{S_2}[W,\widetilde W]\right\}\mathscr{G}[w_{[S_2]-1}]\mathscr{G}[\tilde w_{[S_2]-1}]\\ &\quad-2\mathrm{e}^{\alpha_{S_2}+\alpha_{S_1}}\exp\left\{\beta^2\mathscr{R}_{S_2,S_1}[W,\widetilde W]\right\}\mathscr{G}[w_{[S_2]-1}]\mathscr{G}[\tilde w_{[S_1]-1}]\\ &\quad+\mathrm{e}^{2\alpha_{S_1}}\exp\left\{\beta^2\mathscr{R}_{S_1}[W,\widetilde W]\right\}\mathscr{G}[w_{[S_1]-1}]\mathscr{G}[\tilde w_{[S_1]-1}]\Big].\end{aligned} \tag{3.7}$$

However, for any $T_1 < T_2$ we can write

$$
\begin{aligned}
&\widetilde{\mathbb{E}}_{W,\widetilde{W}} \mathrm{e}^{\alpha_{T_2}+\alpha_{T_1}} \exp\left\{\beta^2 \mathscr{R}_{T_2,T_1}[W,\widetilde{W}]\right\} \mathscr{G}[w_{[T_2]-1}]\mathscr{G}[\tilde{w}_{[T_1]-1}] \\
&\qquad = \widetilde{\mathbb{E}}_{W,\widetilde{W}} \mathrm{e}^{2\alpha_\infty} \exp\left\{\beta^2 \mathscr{R}_{0.9T_2,0.9T_1}[W,\widetilde{W}]\right\} \\
&\qquad + \widetilde{\mathbb{E}}_{W,\widetilde{W}} \mathrm{e}^{2\alpha_\infty} \exp\left\{\beta^2 \mathscr{R}_{0.9T_2,0.9T_1}[W,\widetilde{W}]\right\} \left(\mathscr{G}[w_{[T_2]-1}]\mathscr{G}[\tilde{w}_{[T_1]-1}] - 1\right) \\
&\qquad + \widetilde{\mathbb{E}}_{W,\widetilde{W}} \mathrm{e}^{2\alpha_\infty} \left(\exp\left\{\beta^2 \mathscr{R}_{T_2,T_1}[W,\widetilde{W}]\right\} - \exp\left\{\beta^2 \mathscr{R}_{0.9T_2,0.9T_1}[W,\widetilde{W}]\right\}\right) \mathscr{G}[w_{[T_2]-1}]\mathscr{G}[\tilde{w}_{[T_1]-1}] \\
&\qquad + \widetilde{\mathbb{E}}_{W,\widetilde{W}} \left(\mathrm{e}^{\alpha_{T_2}+\alpha_{T_1}} - \mathrm{e}^{2\alpha_\infty}\right) \exp\left\{\beta^2 \mathscr{R}_{T_2,T_1}[W,\widetilde{W}]\right\} \mathscr{G}[w_{[T_2]-1}]\mathscr{G}[\tilde{w}_{[T_1]-1}]. \qquad (3.8)
\end{aligned}
$$

Furthermore, (2.4) together with Lemma 3.1 and Proposition 2.6 imply that the last term in (3.8) can be estimated as

$$
\lim_{T_1,T_2\to\infty} T_2^{d/2-1} \widetilde{\mathbb{E}}_{W,\widetilde{W}} \left(\mathrm{e}^{\alpha_{T_2}+\alpha_{T_1}} - \mathrm{e}^{2\alpha_\infty}\right) \exp\left\{\beta^2 \mathscr{R}_{T_2,T_1}[W,\widetilde{W}]\right\} \mathscr{G}[w_{[T_2]-1}]\mathscr{G}[\tilde{w}_{[T_1]-1}] = 0, \qquad (3.9)
$$

and the third term in (3.8) can be bounded using Lemma 3.1 as

$$
\begin{aligned}
&\limsup_{T_1,T_2\to\infty} T_1^{\frac{d}{2}-1} \left| \widetilde{\mathbb{E}}_{W,\widetilde{W}} \mathrm{e}^{2\alpha_\infty} \left[\exp\{\beta^2 \mathscr{R}_{T_2,T_1}[W,\widetilde{W}]\} - \exp\{\beta^2 \mathscr{R}_{0.9T_2,0.9T_1}[W,\widetilde{W}]\}\right] \mathscr{G}[w_{[T_2]-1}]\mathscr{G}[\tilde{w}_{[T_1]-1}] \right| \\
&\qquad \le \mathrm{e}^{2\alpha_\infty} \|\mathscr{G}\|_\infty^2 (T_1 \wedge T_2)^{d/2-1} \widetilde{\mathbb{E}}_{W,\widetilde{W}} \left(\exp\left\{\beta^2 \mathscr{R}_{T_2,T_1}[W,\widetilde{W}]\right\} - \exp\left\{\beta^2 \mathscr{R}_{0.9T_2,0.9T_1}[W,\widetilde{W}]\right\}\right) < \infty. \qquad (3.10)
\end{aligned}
$$

For the second term in (3.8) we can use Lemma 2.8 to get

$$
\limsup_{T_1,T_2\to\infty} T_1^{d/2-1} \widetilde{\mathbb{E}}_{W,\widetilde{W}} \mathrm{e}^{2\alpha_\infty} \exp\left\{\beta^2 \mathscr{R}_{0.9T_2,0.9T_1}[W,\widetilde{W}]\right\} \left(\mathscr{G}[w_{[T_2]-1}]\mathscr{G}[\tilde{w}_{[T_1]-1}] - 1\right) = 0. \qquad (3.11)
$$

Finally, we have

$$
\begin{aligned}
\limsup_{T_1,T_2\to\infty} T_1^{d/2-1} \widetilde{\mathbb{E}}_{W,\widetilde{W}} \Big[&\exp\left\{\beta^2 \mathscr{R}_{0.9T_2}[W,\widetilde{W}]\right\} - 2\exp\left\{\beta^2 \mathscr{R}_{0.9T_2,0.9T_1}[W,\widetilde{W}]\right\} \\
&+ \exp\left\{\beta^2 \mathscr{R}_{0.9T_1}[W,\widetilde{W}]\right\}\Big] < \infty, \qquad (3.12)
\end{aligned}
$$

also by Lemma 3.1. Substituting (3.8), (3.9), (3.10), (3.11), and (3.12) into (3.7), we see that

$$
\mathbb{E}\left(\Psi(0,y;S_2) - \Psi(0,y;S_1)\right)^2 \le C S_1^{-d/2+1},
$$

as desired. □

Theorem 1.1 is an easy consequence of this proposition.

Proof of Theorem 1.1. For a positive weight $w \in L^1(\mathbb{R}^d)$, consider the weighted space $L^2_w(\mathbb{R}^d)$, with the inner product

$$
\langle f, g\rangle_{L^2_w(\mathbb{R}^d)} = \int f(y)\overline{g(y)} w(y)\,\mathrm{d}y
$$

Then, by Proposition 3.2 and stationarity of $V(s,y)$ in time, we have

$$
\mathbb{E}\|\Psi(s,\cdot;S_1) - \Psi(s,\cdot;S_2)\|^2_{L^2_w(\mathbb{R}^d)} = \int \mathbb{E}\left|\Psi(0,y;s+S_1) - \Psi(0,y;s+S_2)\right|^2 w(y)\,\mathrm{d}y \to 0, \qquad (3.13)
$$

as $S_1, S_2 \to +\infty$, by the dominated convergence theorem, and uniformly in s on compact sets, as $\mathbb{E}\,|\Psi(s,y;S)|^2$ is bounded uniformly in s, y, S due to the choice of $\lambda(\beta)$. Hence, the family $\Psi(s,y;S)$ converges locally uniformly in s and in $L^2(\Omega; L^2_w(\mathbb{R}^d))$ to a limit $\widetilde{\Psi}(s,y)$. The stationarity of $\widetilde{\Psi}$ is standard.

To prove (1.13), we use an argument similar to the above one: note that the solution $\Psi(s,y)$ to (1.11) is stationary in y, and so is $\widetilde{\Psi}(s,y)$, thus for any $y \in \mathbb{R}^d$ fixed we have

$$\begin{aligned}
&\mathbb{E}|\Psi(s,y) - \widetilde{\Psi}(s,y)|^2 \int w(y')\,\mathrm{d}y' = \int \mathbb{E}|\Psi(s,y) - \widetilde{\Psi}(s,y)|^2 w(y)\,\mathrm{d}y \\
&= \lim_{S\to+\infty} \int \mathbb{E}\,|\Psi(s,y) - \Psi(s,y;S)|^2\, w(y)\,\mathrm{d}y = \lim_{S\to+\infty} \int \mathbb{E}\,|\Psi(0,y;s) - \Psi(0,y;S+s)|^2\, w(y)\,\mathrm{d}y \\
&\le \frac{C}{s^{1-d/2}} \to 0 \text{ as } s \to +\infty,
\end{aligned} \tag{3.14}$$

also by Proposition 3.2. We used stationarity of $V(s,y)$ in the last equality above. The convergence of Ψ to $\widetilde{\Psi}$ locally in $L^2(\Omega; L^2_w(\mathbb{R}^d))$ implies that $\widetilde{\Psi}$ satisfies (1.12) in a weak sense almost surely, and therefore, by standard parabolic regularity, it satisfies the latter almost surely. □

We record the covariance of the stationary solution for completeness.

Corollary 3.3. *We have*

$$\mathbb{E}[\widetilde{\Psi}\,(s,y)\,\widetilde{\Psi}\,(s,\tilde{y})] = \mathrm{e}^{2\alpha_\infty} \widetilde{\mathbb{E}}^y_W \widetilde{\mathbb{E}}^{\tilde{y}}_{\widetilde{W}} \exp\left\{\beta^2 \mathscr{R}_\infty[W,\widetilde{W}]\right\}.$$

4 The effective noise strength

In this section, we explain how the effective noise strength parameter ν in (1.8) arises from the stationary solution $\widetilde{\Psi}$ and prove Theorem 1.2.

Lemma 4.1. *If β is sufficiently small and $g \in C_c^\infty(\mathbb{R}^d)$, then we have*

$$\lim_{t\to\infty} \left| \mathrm{Var}\left(\frac{1}{\varepsilon^{d/2-1}} \int g(x)\Psi\left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right)\,\mathrm{d}x\right) - \mathrm{Var}\left(\frac{1}{\varepsilon^{d/2-1}} \int g(x)\widetilde{\Psi}\left(0, \frac{x}{\varepsilon}\right)\,\mathrm{d}x\right)\right| = 0,$$

uniformly in ε.

Proof. We have

$$\begin{aligned}
&\left|\sqrt{\mathrm{Var}\left(\frac{1}{\varepsilon^{d/2-1}}\int g(x)\Psi\Big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\Big)\,\mathrm{d}x\right)} - \sqrt{\mathrm{Var}\left(\frac{1}{\varepsilon^{d/2-1}}\int g(x)\widetilde{\Psi}\Big(0,\frac{x}{\varepsilon}\Big)\,\mathrm{d}x\right)}\right| \\
&= \left|\sqrt{\mathrm{Var}\left(\frac{1}{\varepsilon^{d/2-1}}\int g(x)\Psi\Big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\Big)\,\mathrm{d}x\right)} - \sqrt{\mathrm{Var}\left(\frac{1}{\varepsilon^{d/2-1}}\int g(x)\widetilde{\Psi}\Big(t,\frac{x}{\varepsilon}\Big)\,\mathrm{d}x\right)}\right| \\
&\le \frac{1}{\varepsilon^{d/2-1}}\sqrt{\mathbb{E}\left|\int g(x)\Psi\Big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\Big)\,\mathrm{d}x - \int g(x)\widetilde{\Psi}\Big(t,\frac{x}{\varepsilon}\Big)\,\mathrm{d}x\right|^2} \\
&\le \frac{1}{\varepsilon^{d/2-1}}\int |g(x)|\sqrt{\mathbb{E}\left[\Psi\Big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\Big) - \widetilde{\Psi}\Big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\Big)\right]^2}\,\mathrm{d}x \le \frac{C}{\varepsilon^{d/2-1}}\left(\frac{\varepsilon^2}{t}\right)^{\frac{d-2}{4}} \to 0,
\end{aligned} \tag{4.1}$$

uniformly in ε, as $t \to \infty$, where the first equality is due to the stationarity of $\widetilde{\Psi}$, the first inequality due to the triangle inequality, the second due to the Cauchy-Schwarz inequality, and the last inequality by estimates on $\Psi(t,x)$ as in Proposition 3.2. □

We recall from [13] that

$$\lim_{\varepsilon\to 0}\mathrm{Var}\Big(\frac{e^{-\alpha_{t/\varepsilon^2}}}{\varepsilon^{d/2-1}}\int g(x)\Psi\Big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\Big)\,\mathrm{d}x\Big)=\mathrm{Var}\Big(\int g(x)\psi(t,x)\,\mathrm{d}x\Big), \tag{4.2}$$

where ψ is the solution to the Edwards-Wilkinson stochastic partial differential equation

$$\begin{aligned}&\partial_t\psi(t,x)=\frac{1}{2}a\Delta\psi(t,x)+\beta\nu\dot{W}(t,x),\quad t>0,\ x\in\mathbb{R}^d,\\&\psi(0,x)=0,\end{aligned} \tag{4.3}$$

which is simply (1.8) with $\overline{u}\equiv 1$.

Lemma 4.2. *We have*

$$\lim_{t\to\infty}\mathrm{Var}\left(\int g(x)\psi(t,x)\,\mathrm{d}x\right)=\beta^2\nu^2\int_0^\infty\int|\overline{g}(r,x)|^2\,\mathrm{d}x\,\mathrm{d}r, \tag{4.4}$$

where $\overline{g}(t,x)$ is the solution of

$$\begin{aligned}&\partial_t\overline{g}(t,x)=\frac{1}{2}a\Delta\overline{g}(t,x),\quad t>0,\ x\in\mathbb{R}^d,\\&\overline{g}(0,x)=g(x).\end{aligned} \tag{4.5}$$

Proof. As in [13, (3.16)], we have

$$\begin{aligned}\mathrm{Var}\left(\int g(x)\psi(t,x)\,\mathrm{d}x\right)&=\beta^2\nu^2\int_0^t\int|\overline{g}(t-r,x)|^2\,\mathrm{d}x\,\mathrm{d}r=\beta^2\nu^2\int_0^t\int|\overline{g}(r,x)|^2\,\mathrm{d}x\,\mathrm{d}r\\&\to\beta^2\nu^2\int_0^\infty\int|\overline{g}(r,x)|^2\,\mathrm{d}x\,\mathrm{d}r,\end{aligned} \tag{4.6}$$

as $t\to\infty$ by the monotone convergence theorem. □

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Given $\delta>0$, by Lemmas 4.1 and 4.2, we can choose t large enough so that

$$\left|\mathrm{Var}\left(\int g(x)\psi(t,x)\,\mathrm{d}x\right)-\beta^2\nu^2\int_0^\infty\int|\overline{g}(r,x)|^2\,\mathrm{d}x\,\mathrm{d}r\right|<\frac{\delta}{3} \tag{4.7}$$

and

$$\left|\mathrm{Var}\Big(\frac{1}{\varepsilon^{d/2-1}}\int g(x)\Psi\Big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\Big)\,\mathrm{d}x\Big)-\mathrm{Var}\Big(\frac{1}{\varepsilon^{d/2-1}}\int g(x)\widetilde{\Psi}\Big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\Big)\,\mathrm{d}x\Big)\right|<\frac{\delta}{3}, \tag{4.8}$$

uniformly in ε. Then by (4.2) we can choose ε so small that

$$\left|\mathrm{Var}\Big(\frac{e^{-\alpha_{t/\varepsilon^2}}}{\varepsilon^{d/2-1}}\int g(x)\Psi\Big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\Big)\,\mathrm{d}x\Big)-\mathrm{Var}\Big(\int g(x)\psi(t,x)\,\mathrm{d}x\Big)\right|<\frac{\delta}{3}. \tag{4.9}$$

Using the triangle inequality applied to (4.7), (4.8), (4.9), and (2.4), we obtain

$$\lim_{\varepsilon\to 0}\mathrm{Var}\left(\frac{1}{\varepsilon^{d/2-1}}\int g(x)\widetilde{\Psi}\Big(0,\frac{x}{\varepsilon}\Big)\,\mathrm{d}x\right)=e^{2\alpha_\infty}\beta^2\nu^2\int_0^\infty\int|\overline{g}(r,x)|^2\,\mathrm{d}x\,\mathrm{d}r, \tag{4.10}$$

so

$$e^{2\alpha_\infty}\nu^2\beta^2 = \Big(\int_0^\infty\int |\overline{g}(r,x)|^2\,\mathrm{d}x\,\mathrm{d}r\Big)^{-1}\lim_{\varepsilon\to 0}\int\int g(x)g(\tilde{x})\Big(\frac{1}{\varepsilon^{d-2}}\,\mathrm{Cov}\,\Big(\widetilde{\Psi}\Big(0,\frac{x}{\varepsilon}\Big),\widetilde{\Psi}\Big(0,\frac{\tilde{x}}{\varepsilon}\Big)\Big)\Big)\,\mathrm{d}x\,\mathrm{d}\tilde{x}, \tag{4.11}$$

and thus

$$\begin{aligned}\frac{1}{\nu^2\beta^2e^{2\alpha_\infty}}\lim_{\varepsilon\to 0}\int\int g(x)g(\tilde{x})\left(\frac{1}{\varepsilon^{d-2}}\,\mathrm{Cov}\left(\widetilde{\Psi}\Big(0,\frac{x}{\varepsilon}\Big),\widetilde{\Psi}\Big(0,\frac{\tilde{x}}{\varepsilon}\Big)\right)\right)\mathrm{d}x\,\mathrm{d}\tilde{x} &= \int_0^\infty\int |\overline{g}(r,z)|^2\,\mathrm{d}z\,\mathrm{d}r\\ &= \int\int\Big(\int_0^\infty\int G_a(r,z-x)G_a(r,z-\tilde{x})\,\mathrm{d}z\,\mathrm{d}r\Big)g(x)g(\tilde{x})\,\mathrm{d}x\,\mathrm{d}\tilde{x}.\end{aligned} \tag{4.12}$$

This means that

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{d-2}}\widetilde{\mathbb{E}}_W^{\varepsilon^{-1}x}\widetilde{\mathbb{E}}_{\widetilde{W}}^{\varepsilon^{-1}\tilde{x}}\exp\Big\{\beta^2\mathscr{R}_\infty[W,\widetilde{W}]\Big\} &= \nu^2\beta^2e^{2\alpha_\infty}\int_0^\infty\int G_a(r,z-x)G_a(r,z-\tilde{x})\,\mathrm{d}z\,\mathrm{d}r\\ &= \frac{\nu^2\beta^2e^{2\alpha_\infty}\bar{c}}{a|x-\tilde{x}|^{d-2}}\end{aligned}$$

in the sense of distributions, and moreover that

$$\nu^2 = \frac{\bar{c}\lim_{\varepsilon\to 0}\int\int g(x)g(\tilde{x})\left(\frac{1}{\varepsilon^{d-2}}\,\mathrm{Cov}\left(\widetilde{\Psi}\Big(0,\frac{x}{\varepsilon}\Big),\widetilde{\Psi}\Big(0,\frac{\tilde{x}}{\varepsilon}\Big)\right)\right)\mathrm{d}x\,\mathrm{d}\tilde{x}}{a\beta^2e^{2\alpha_\infty}\int\int|x-\tilde{x}|^{2-d}g(x)g(\tilde{x})\,\mathrm{d}x\,\mathrm{d}\tilde{x}}.$$

□

5 The effective diffusivity

In this section we will show how to recover the effective diffusivity a from the asymptotic expansion (1.9).

5.1 The solvability condition

We first explain how the effective diffusivity can be formally recovered from the homogenization correctors – at the moment we disregard the question of the existence of such correctors and proceed on a purely formal level. We start with the equations (1.16)–(1.17) for the terms u_1 and u_2 in the formal asymptotic expansion (1.9) for $u^\varepsilon(t,x)$. We will replace Ψ in the right side of these equations by the stationary solution $\widetilde{\Psi}$, so our formal starting point is

$$\partial_s u_1(t,x,s,y) = \frac{1}{2}\Delta_y u_1(t,x,s,y) + (\beta V(s,y)-\lambda)u_1(t,x,s,y) + \nabla_y\widetilde{\Psi}(s,y)\cdot\nabla_x\overline{u}(t,x) \tag{5.1}$$

and

$$\begin{aligned}\partial_s u_2(t,x,s,y) = {}&\frac{1}{2}\Delta_y u_2(t,x,s,y) + (\beta V(s,y)-\lambda)u_2(t,x,s,y) + \nabla_y\cdot\nabla_x u_1(t,x,s,y)\\ &+\frac{1}{2}(1-a)\widetilde{\Psi}(s,y)\Delta_x\overline{u}(t,x).\end{aligned} \tag{5.2}$$

We can now formally decompose the solution to (5.1) as

$$u_1(t,x,s,y) = \widetilde{\boldsymbol{\omega}}(s,y)\cdot\nabla_x\overline{u}(t,x), \tag{5.3}$$

where $\widetilde{\boldsymbol{\omega}}(s,y) = (\widetilde{\omega}^{(1)}(s,y),\dots,\widetilde{\omega}^{(d)}(s,y))$ is a time/space stationary solution to

$$\partial_s \widetilde{\omega}^{(k)} = \frac{1}{2}\Delta_y \widetilde{\omega}^{(k)} + (\beta V(s,y) - \lambda)\widetilde{\omega}^{(k)} + \frac{\partial \widetilde{\Psi}(s,y)}{\partial y_k}. \tag{5.4}$$

We should note that unlike the random heat equation (1.12), the forced equation (5.4) may not have stationary solutions in all $d \geq 3$. Nevertheless, this computation will give us an idea on how the effective diffusivity can be approximated. By Theorem 1.1, applied with time reversed, or, equivalently, to the random heat equation with potential $V(-s,y)$, we also have a stationary solution $\widetilde{\Phi}$ to the equation

$$-\partial_s \widetilde{\Phi} = \frac{1}{2}\Delta \widetilde{\Phi} + \beta V(s,y)\widetilde{\Phi} - \lambda \widetilde{\Phi}. \tag{5.5}$$

Multiplying (5.2) by $\widetilde{\Phi}$ and using (5.3) and (5.5) gives

$$\begin{aligned} \partial_s(\widetilde{\Phi}(s,y)u_2(t,x,s,y)) &= \frac{1}{2}\widetilde{\Phi}(s,y)\Delta_y u_2(t,x,s,y) - \frac{1}{2}u_2(t,x,s,y)\Delta\widetilde{\Phi}(s,y) \\ &\quad + \widetilde{\Phi}(s,y)\,\mathrm{tr}\,(\nabla_y \widetilde{\boldsymbol{\omega}} \cdot \mathrm{Hess}\,\overline{u}) + \frac{1}{2}(1-a)\widetilde{\Phi}(s,y)\widetilde{\Psi}(s,y)\Delta_x \overline{u}(t,x). \end{aligned} \tag{5.6}$$

The assumed stationarity of u_2 in s and the stationarity of $\widetilde{\Phi}$ in s imply that the expectation of the left-hand side is 0. Stationarity of u_2 in y, on the other hand, implies that

$$\mathbb{E}\Big[\widetilde{\Phi}(s,y)\Delta_y u_2(t,x,s,y) - u_2(t,x,s,y)\Delta\widetilde{\Phi}(s,y)\Big] = 0.$$

Therefore, taking the expectation in (5.6) gives

$$\mathbb{E}\widetilde{\Phi}(s,y)\left[\mathrm{tr}\,(\nabla_y \widetilde{\boldsymbol{\omega}} \cdot \mathrm{Hess}\,\overline{u}) + \frac{1}{2}(1-a)\widetilde{\Psi}(s,y)\Delta_x \overline{u}(t,x)\right] = 0. \tag{5.7}$$

Due to the isotropic assumption, we have

$$\mathbb{E}\widetilde{\Phi}(s,y)\nabla_y \widetilde{\boldsymbol{\omega}} = \frac{1}{d}\,\mathrm{tr}\left(\mathbb{E}\widetilde{\Phi}(s,y)\nabla_y \widetilde{\boldsymbol{\omega}}\right)\cdot I_{d\times d} = \frac{1}{d}\mathbb{E}\widetilde{\Phi}(s,y)\left(\nabla_y \cdot \widetilde{\boldsymbol{\omega}}\right)\cdot I_{d\times d},$$

and thus

$$\begin{aligned} 0 &= \mathbb{E}\widetilde{\Phi}(s,y)\left[\mathrm{tr}\,(\nabla_y \widetilde{\boldsymbol{\omega}} \cdot \mathrm{Hess}\,\overline{u}) + \frac{1}{2}(1-a)\widetilde{\Psi}(s,y)\Delta_x \overline{u}(t,x)\right] \\ &= \mathbb{E}\widetilde{\Phi}(s,y)\left[\frac{1}{d}\nabla_y \cdot \widetilde{\boldsymbol{\omega}} + \frac{1}{2}(1-a)\widetilde{\Psi}(s,y)\right]\Delta_x \overline{u}(t,x), \end{aligned} \tag{5.8}$$

leading to

$$a = 1 + \frac{2}{d}\frac{\mathbb{E}[\widetilde{\Phi}(s,y)\nabla_y \cdot \widetilde{\boldsymbol{\omega}}(s,y)]}{\mathbb{E}[\widetilde{\Phi}(s,y)\widetilde{\Psi}(s,y)]}. \tag{5.9}$$

As we have not proved that a stationary corrector $\widetilde{\boldsymbol{\omega}}(s,y)$ exists, expression (5.9) is purely formal. In the next section, we will explain how an approximate version of $\widetilde{\boldsymbol{\omega}}$ can be used to justify the computation leading to (5.9).

5.2 An approximation of the effective diffusivity

In this section, we will show how approximate correctors can be used in the right side of (5.9) to provide a good approximation of the effective diffusivity. Instead of trying to build a stationary solution to the corrector equation (5.4) we take $S > 0$ and consider the solution $\boldsymbol{\omega}(s,y;S)$ of the

Cauchy problem for (5.4), with the stationary forcing coming from $\widetilde{\Psi}(s,y)$ replaced by its approximation $\Psi(s,y;S)$ used to construct $\widetilde{\Psi}(s,y)$:

$$\partial_s\omega^{(k)} = \frac{1}{2}\Delta_y\omega^{(k)} + (\beta V(s,y)-\lambda)\omega^{(k)} + \frac{\partial\Psi(s,y;S)}{\partial y_k}, \quad s > -S, \tag{5.10}$$

and with the initial condition

$$\boldsymbol{\omega}(-S,y;S) = 0 \quad y\in\mathbb{R}^d. \tag{5.11}$$

The solution is given by the Feynman-Kac formula

$$\boldsymbol{\omega}(s,y;S) = \mathbb{E}_B^y\left[\int_0^{s+S}\exp\left(\beta\int_0^r V(s-\tau,B_\tau)\,\mathrm{d}\tau-\lambda r\right)\nabla\Psi(s-r,B_r;S)\,\mathrm{d}r\right]. \tag{5.12}$$

We also define $\Psi(s,y;S)$ as in (3.1)–(3.2) and similarly consider an approximation $\Phi(s,y;T)$ of the stationary solution $\widetilde{\Phi}(s,y)$ to the backward random heat equation (5.5),

$$\Phi(s,y;T) = \mathbb{E}_B^y\exp\left\{\beta\mathscr{V}_{[s-T,0]}[B]-\lambda(T-s)\right\}, \quad s<T,$$

which satisfies (5.5) with the terminal condition

$$\Phi(T,y;T) = 1.$$

We recall that

$$\mathscr{V}_{[s-T,0]}[B] = \int_{s-T}^0 V(s-\tau,B_\tau)d\tau$$

where we interpret B as a two-sided Brownian motion. We define an approximate version of (5.9)

$$a_{S,T}(s,y) = 1+\frac{2}{d}\frac{\mathbb{E}[\Phi(s,y;T)\nabla_y\cdot\boldsymbol{\omega}(s,y;S)]}{\mathbb{E}[\Phi(s,y;T)\Psi(s,y;S)]}. \tag{5.13}$$

The next theorem, which is the main result of this section, shows that the "large S,T" limit of (5.13) agrees with the effective diffusivity (2.8) established in [13].

Theorem 5.1. *Let a be the effective diffusivity defined by (2.8). Then we have, for each $s\in\mathbb{R}$, and $y\in\mathbb{R}^d$,*

$$\lim_{\substack{S\to\infty\\T\to\infty}} a_{S,T}(s,y) = a.$$

We note that if a stationary $\widetilde{\boldsymbol{\omega}}$ given by

$$\widetilde{\boldsymbol{\omega}}(s,y) = \lim_{S\to\infty}\boldsymbol{\omega}(s,y;S)$$

exists, then Theorem 5.1 verifies the formal expression (5.9).

We will set $s=0$ and $y=0$ in the proof of Theorem 5.1, without loss of generality. In the course of the proof, we will denote by $H(x)$ the standard Heaviside function and also use its approximation

$$H_\gamma(x) = \begin{cases} 0 & x\le 0;\\ \gamma^{-1}x & 0<x<\gamma;\\ 1 & x\ge\gamma, \end{cases}$$

as well as $J(x)=xH(x)$. We begin by rewriting the Feynman–Kac formula for the numerator of the right side of (5.13).

Lemma 5.2. *We have*

$$\mathbb{E}\left[\Phi(0,0;T)(\nabla_y\cdot\boldsymbol{\omega})(0,0;S)\right]$$
$$=\lim_{\gamma\downarrow 0}\nabla_\eta\big|_{\eta=0}\cdot\nabla_\xi\big|_{\xi=0}\mathbb{E}\mathbb{E}_B^0\exp\Big(\beta\int_{-T}^{S}V(-\tau,B_\tau+H_\gamma(\tau)\eta+J(\tau)\xi)\,\mathrm{d}\tau-\lambda(T+S)\Big).\quad(5.14)$$

Proof. From (5.12), we can compute, using the Feynman-Kac forumla for $\Psi(s,y;S)$:

$$\begin{aligned}\boldsymbol{\omega}(0,y;S)&=\mathbb{E}_B^y\int_0^S\exp\Big(\beta\int_0^r V(-\tau,B_\tau)\,\mathrm{d}\tau-\lambda r\Big)\nabla\Psi(-r,B_r;S)\,\mathrm{d}r\\&=\nabla_\xi\big|_{\xi=0}\mathbb{E}_B^y\int_0^S\exp\Big(\beta\int_0^S V(-\tau,B_\tau+H(\tau-r)\xi)\,\mathrm{d}\tau-\lambda S\Big)\,\mathrm{d}r.\end{aligned}\quad(5.15)$$

One check by explicit differentiation of both expressions that the right side of (5.15) can be re-written as

$$\boldsymbol{\omega}(0,y;S)=\nabla_\xi\big|_{\xi=0}\mathbb{E}_B^y\exp\Big(\beta\int_0^S V(-\tau,B_\tau+\tau\xi)\,\mathrm{d}\tau-\lambda S)\Big).\quad(5.16)$$

Similarly, we can write

$$(\nabla_y\cdot\boldsymbol{\omega})(0,0;S)=\nabla_\eta\big|_{\eta=0}\cdot\nabla_\xi\big|_{\xi=0}\mathbb{E}_B^0\exp\Big(\beta\int_0^S V(-\tau,B_\tau+\eta+\tau\xi)\,\mathrm{d}\tau-\lambda S)\Big).$$

Using the Feynman-Kac formula for $\Phi(s,y;T)$, we obtain

$$\Phi(0,0;T)(\nabla_y\cdot\boldsymbol{\omega})(0,0;S)=\nabla_\eta\big|_{\eta=0}\cdot\nabla_\xi\big|_{\xi=0}\mathbb{E}_B^0\exp\Big(\beta\int_{-T}^S V(-\tau,B_\tau+H(\tau)\eta+J(\tau)\xi)\,\mathrm{d}\tau-\lambda(T+S)\Big).\quad(5.17)$$

Taking the expectation above gives

$$\mathbb{E}\left[\Phi(0,0;T)(\nabla\cdot\boldsymbol{\omega})(0,0;S)\right]=\nabla_\eta\big|_{\eta=0}\cdot\nabla_\xi\big|_{\xi=0}\mathrm{e}^{-\lambda(T+S)}\mathbb{E}_B^0\exp\left(\frac{1}{2}\beta^2\mathscr{R}_{[-T,S]}[B+H\eta+J\xi]\right),\quad(5.18)$$

with the notation as explained below (2.2). This expression is almost (5.14) except we need to add the regularization of H by H_γ in (5.18). To do this, we write out the gradients in the right side of (5.14). Define $\delta f(\tau,\tilde\tau)=f(\tau)-f(\tilde\tau)$. Then we have, for all $\gamma\geq 0$,

$$\nabla_\eta\big|_{\eta=0}\cdot\nabla_\xi\big|_{\xi=0}\mathrm{e}^{-\lambda(T+S)}\mathbb{E}_B^y\exp\Big(\beta^2\mathscr{R}_{[s-T,s-S]}[B+H_\gamma\eta+J\xi]\Big)\quad(5.19)$$
$$=\beta^2\mathrm{e}^{-\lambda(T-S)}\mathbb{E}_B^y\left(g_{1;\gamma}(B)+g_{2;\gamma}(B)\cdot g_{3;\gamma}(B)\right)\exp\Big(\beta^2\mathscr{R}_{[-T,S]}[B]\Big),\quad(5.20)$$

where, because the support of $R(s)$ is in $[-1,1]$, we have

$$\begin{aligned}g_{1;\gamma}[B]&=\iint\limits_{[-2,2]^2}\delta H_\gamma(\tau,\tilde\tau)\delta J(\tau,\tilde\tau)\Delta R(\tau-\tilde\tau,\delta B(\tau,\tilde\tau))\,\mathrm{d}\tau\,\mathrm{d}\tilde\tau\\g_{2;\gamma}[B]&=\iint\limits_{[-S,-T]^2}\delta J(\tau,\tilde\tau)\nabla R(\tau-\tilde\tau,\delta B(\tau,\tilde\tau))\,\mathrm{d}\tau\,\mathrm{d}\tilde\tau\\g_{3;\gamma}[B]&=\iint\limits_{[-2,2]^2}\delta H_\gamma(\tau,\tilde\tau)\nabla R(\tau-\tilde\tau,\delta B(\tau,\tilde\tau))\,\mathrm{d}\tau\,\mathrm{d}\tilde\tau.\end{aligned}$$

The bounded convergence theorem then implies that

$$\lim_{\gamma\downarrow 0}\nabla_\eta\big|_{\eta=0}\cdot\nabla_\xi\big|_{\xi=0}\mathrm{e}^{-\lambda(T+S)}\mathbb{E}_B^y\exp\left(\beta^2\mathscr{R}_{[-T,S]}[B+H_\gamma\eta+J\xi]\right)$$
$$=\nabla_\eta\big|_{\eta=0}\cdot\nabla_\xi\big|_{\xi=0}\mathrm{e}^{-\lambda(T+S)}\mathbb{E}_B^y\exp\left(\beta^2\mathscr{R}_{[-T,S]}[B+H\eta+J\xi]\right). \qquad \square$$

The reason why the γ-regularization in (5.14) is useful is clear from the next lemma.

Lemma 5.3. *We have*

$$a_{S,T}(0,0)=1+2\lim_{\gamma\downarrow 0}\widehat{\mathbb{E}}^0_{B;[-T,S]}\left(\frac{1}{\gamma d}B_S\cdot B_\gamma-1\right). \tag{5.21}$$

Proof. We use the Girsanov formula, writing

$$\begin{aligned}&\nabla_\eta\big|_{\eta=0}\cdot\nabla_\xi\big|_{\xi=0}\mathbb{E}_B^y\exp\left(\beta\int_{-T}^S V(-\tau,B_\tau+H_\gamma(\tau)\eta+J(\tau)\xi)\,\mathrm{d}\tau-\lambda(T+S)\right)\\&=\nabla_\eta\big|_{\eta=0}\cdot\nabla_\xi\big|_{\xi=0}\mathbb{E}_B^y\Big[\exp\left(\beta\int_{-T}^S V(-\tau,B_\tau)\,\mathrm{d}\tau\right)\\&\times\exp\Big(-\lambda(T+S)+\frac{1}{\gamma}\langle B_\gamma,\eta\rangle-\frac{1}{2\gamma}|\eta|^2-\langle\xi,\eta\rangle+\langle B_S,\xi\rangle-\frac12|\xi|^2S\Big)\Big]\\&=\mathbb{E}_B^y(\frac{1}{\gamma}B_S\cdot B_\gamma-d)\exp\left(\beta\int_{-T}^S V(-\tau,B_\tau)\,\mathrm{d}\tau-\lambda(T+S)\right).\end{aligned}$$

Passing to the limit as $\gamma\downarrow 0$, and taking expectations shows that

$$\mathbb{E}\left[\Phi(0,0;T)(\nabla\cdot\boldsymbol{\omega})(0,0;S)\right]=\mathrm{e}^{-\lambda(T+S)}\lim_{\gamma\downarrow 0}\mathbb{E}^0_B\left(\gamma^{-1}B_S\cdot B_\gamma-d\right)\exp\left\{\beta^2\mathscr{R}_{[-T,S]}[B]\right\}. \tag{5.22}$$

Finally, for the denominator of (5.13) we have

$$\Phi(0,0;T)\Psi(0,0;S)=\mathbb{E}^0_B\exp\left\{\beta\mathscr{V}_{[-T,S]}[B]-\lambda(T+S)\right\},$$

where $\mathscr{V}_{[-T,S]}[B]=\int_{-T}^S V(-\tau,B_\tau)d\tau$, so

$$\mathbb{E}\Phi(0,0;T)\Psi(0,0;S)=\mathrm{e}^{-\lambda(T+S)}\mathbb{E}^0_B\exp\left\{\beta^2\mathscr{R}_{[-T,S]}[B]\right\}. \tag{5.23}$$

Dividing (5.22) by (5.23) and plugging into (5.13) yields (5.21). $\square$

Lemma 5.4. *We have*

$$\lim_{\gamma\downarrow 0}\frac{1}{\gamma d}\widehat{\mathbb{E}}^0_{B;[-T,S]}|B_\gamma|^2=1,$$

uniformly in S and T.

Proof. We have

$$\begin{aligned}\widehat{\mathbb{E}}^0_{B;-T,S}|B_\gamma|^2-\mathbb{E}^0_B|B_\gamma|^2&=\mathbb{E}^0_B|B_\gamma|^2\left(\frac{1}{Z_{-T,S}}\exp\{\frac12\beta^2\mathscr{R}_{[-T,S]}[B]\}-1\right)\\&=\frac{1}{Z_{-T,S}}\mathbb{E}^0_B|B_\gamma|^2\left(\exp\{\frac12\beta^2\mathscr{R}_{[-T,S]}[B]\}-\exp\{\frac12\beta^2\mathscr{R}_{[-T,S]}[\widetilde{B}]\}\right),\end{aligned}$$

where $\widetilde{B}$ is a Brownian motion whose increments on $[-T,0]$ and $[\gamma,S]$ are identical to those of B and whose increments on $[0,\gamma]$ are independent of those of B. (The second equality is because $\mathscr{R}_{[-T,S]}[\widetilde{B}]$ is independent of B_γ.) This means that

$$\begin{aligned}\left|\widehat{\mathbb{E}}^0_{B;-T,S}|B_\gamma|^2-\mathbb{E}^0_B|B_\gamma|^2\right| &= \frac{\mathbb{E}^0_B\left(\exp\{\beta^2\mathscr{R}_{[-T,0]}[B]\}+\exp\{\beta^2\mathscr{R}_{[\gamma,S]}[B]\}\right)}{Z_{-T,S}}\\ &\quad\times\mathbb{E}^0_B|B_\gamma|^2\left|\exp\left\{2\beta^2\int_{-1}^{\gamma}\int_{\tau\vee 0}^{1}R(\tau-\tilde{\tau},B_\tau-B_{\tilde{\tau}})\,\mathrm{d}\tilde{\tau}\,\mathrm{d}\tau\right\}\right.\\ &\qquad\left.-\exp\left\{2\beta^2\int_{-1}^{\gamma}\int_{\tau\vee 0}^{1}R(\tau-\tilde{\tau},\widetilde{B}_\tau-\widetilde{B}_{\tilde{\tau}})\,\mathrm{d}\tilde{\tau}\,\mathrm{d}\tau\right\}\right|\\ &\le C(\mathbb{E}^0_B|B_\gamma|^4)^{1/2}(\mathbb{E}^0_B(\exp\{4\beta^2\max_{0\le s\le\gamma}|B_s-\widetilde{B}_s|\}-1)^2)^{1/2}\\ &\le C\gamma^2,\end{aligned}$$

where C is a constant that may depend on β and R. Since $\mathbb{E}^0_B|B_\gamma|^2=\gamma d$, this proves the lemma. ☐

Corollary 5.5. *We have*

$$a_{S,T}(0,0)=\lim_{\gamma\downarrow 0}a_{S,T;\gamma}, \tag{5.24}$$

where

$$a_{S,T;\gamma}=1+\frac{2}{d\gamma}\widehat{\mathbb{E}}^0_{B;[-T,S]}(B_S-B_\gamma)\cdot(B_\gamma-B_0).$$

Proof. This is a simple consequence of Lemma 5.3 and Lemma 5.4. ☐

Lemma 5.6. *The limit*

$$\lim_{\substack{T\to\infty\\ S\to\infty}}a_{S,T}(0,0) \tag{5.25}$$

exists.

Proof. We have, for any $\tau_1<\tau_2<\tau_3<\tau_4\le\tau_5$,

$$\begin{aligned}\widehat{\mathbb{E}}_{B;\tau_5}(B_{\tau_4}-B_{\tau_3})\cdot(B_{\tau_2}-B_{\tau_1}) &= \widetilde{\mathbb{E}}_W(W_{\tau_4}-W_{\tau_3})\cdot(W_{\tau_2}-W_{\tau_1})\mathscr{G}(W_{[\tau_5]-1})\\ &= \widetilde{\mathbb{E}}_W(W_{\tau_4\wedge\sigma}-W_{\tau_3\wedge\sigma})\cdot(W_{\tau_2}-W_{\tau_1}),\end{aligned} \tag{5.26}$$

where σ is the first regeneration time after τ_4 and the second equality comes from the fact that $\mathscr{G}$ is even and the increments of W after a regeneration time are isotropic. This makes it clear that there are constants $0<c$, $C<\infty$ so that

$$\widehat{\mathbb{E}}_{B;\tau_5}(B_{\tau_4}-B_{\tau_3})\cdot(B_{\tau_2}-B_{\tau_1})\le C\mathrm{e}^{-c(\tau_3-\tau_2)}. \tag{5.27}$$

Then it follows from Corollary 5.5 that $a_{S,T}$ is Cauchy in S, and thus the limit (5.25) exists. ☐

Now we prove Theorem 5.1.

Proof of Theorem 5.1. We have, from (2.8),

$$a=\lim_{U\to\infty}\frac{1}{dU}\widetilde{\mathbb{E}}_W\left(W_{3U}-W_0\right)\cdot\left(W_{2U}-W_U\right)=\lim_{U\to\infty}\frac{1}{dU}\widehat{\mathbb{E}}^0_{B;3U}\left(B_{3U}-B_0\right)\cdot\left(B_{2U}-B_U\right), \tag{5.28}$$

where the second equality is by (5.27) and Lemma 2.8. Define

$$\tau_j^{(\gamma)}=(U+j\gamma)\wedge 2U$$

and note that

$$B_{2U} - B_U = \sum_{j=1}^{\lceil U/\gamma \rceil - 1} (B_{\tau_{j+1}^{(\gamma)}} - B_{\tau_j^{(\gamma)}}).$$

Substituting this into (5.28) yields

$$\begin{aligned}
a &= \lim_{U\to\infty} \frac{1}{dU} \lim_{\gamma\downarrow 0} \widehat{\mathbb{E}}^0_{B;3U}(B_{3U} - B_0) \cdot \sum_{j=0}^{\lceil U/\gamma \rceil - 1} (B_{\tau_{j+1}^{(\gamma)}} - B_{\tau_j^{(\gamma)}}) \\
&= \lim_{U\to\infty} \frac{1}{dU} \lim_{\gamma\downarrow 0} \sum_{j=0}^{\lceil U/\gamma \rceil - 1} \widehat{\mathbb{E}}^0_{B;3U} \Big((B_{3U} - B_{\tau_{j+1}^{(\gamma)}}) + (B_{\tau_{j+1}^{(\gamma)}} - B_{\tau_j^{(\gamma)}}) + (B_{\tau_j^{(\gamma)}} - B_0) \Big) \cdot (B_{\tau_{j+1}^{(\gamma)}} - B_{\tau_j^{(\gamma)}})
\end{aligned}$$

Now by Lemma 5.4, we have

$$\lim_{U\to\infty} \frac{1}{dU} \lim_{\gamma\downarrow 0} \sum_{j=0}^{\lceil U/\gamma \rceil - 1} \widehat{\mathbb{E}}^0_{B;3U}(B_{\tau_{j+1}^{(\gamma)}} - B_{\tau_j^{(\gamma)}}) \cdot (B_{\tau_{j+1}^{(\gamma)}} - B_{\tau_j^{(\gamma)}}) = 1.$$

Moreover, we have by (5.13) that

$$\begin{aligned}
\widehat{\mathbb{E}}^0_{B;3U}(B_{3U} - B_{\tau_{j+1}^{(\gamma)}}) \cdot (B_{\tau_{j+1}^{(\gamma)}} - B_{\tau_j^{(\gamma)}}) &= \frac{\gamma d}{2}(a_{3U-\tau_j^{(\gamma)}, \tau_j^{(\gamma)}; \tau_{j+1}^{(\gamma)} - \tau_j^{(\gamma)}} - 1) \\
\widehat{\mathbb{E}}^0_{B;3U}(B_{\tau_j^{(\gamma)}} - B_0) \cdot (B_{\tau_{j+1}^{(\gamma)}} - B_{\tau_j^{(\gamma)}}) &= \frac{\gamma d}{2}(a_{\tau_{j+1}^{(\gamma)}, 3U-\tau_{j+1}^{(\gamma)}; \tau_{j+1}^{(\gamma)} - \tau_j^{(\gamma)}} - 1).
\end{aligned}$$

Therefore,

$$\begin{aligned}
a &= 1 + \lim_{U\to\infty} \frac{1}{dU} \lim_{\gamma\downarrow 0} \sum_{j=0}^{\lceil U/\gamma \rceil - 1} \Big(\frac{\gamma d}{2}(a_{3U-\tau_j^{(\gamma)}, \tau_j^{(\gamma)}; \tau_{j+1}^{(\gamma)} - \tau_j^{(\gamma)}} - 1) + \frac{\gamma d}{2}(a_{\tau_{j+1}^{(\gamma)}, 3U-\tau_{j+1}^{(\gamma)}; \tau_{j+1}^{(\gamma)} - \tau_j^{(\gamma)}} - 1) \Big) \\
&= \lim_{U\to\infty} \frac{1}{U} \lim_{\gamma\downarrow 0} \frac{\gamma}{2} \sum_{j=0}^{\lceil U/\gamma \rceil - 1} (a_{3U-\tau_j^{(\gamma)}, \tau_j^{(\gamma)}; \tau_{j+1}^{(\gamma)} - \tau_j^{(\gamma)}} + a_{\tau_{j+1}^{(\gamma)}, 3U-\tau_{j+1}^{(\gamma)}; \tau_{j+1}^{(\gamma)} - \tau_j^{(\gamma)}}) \\
&= \lim_{\substack{T\to\infty \\ S\to\infty}} a_{S,T}(0,0),
\end{aligned}$$

where the last inequality is by Lemma 5.6. □

6 Convergence of the leading-order term

In this section, we prove Theorem 1.3. We begin by deriving an expression for the error in (1.15) using the Feynman–Kac formula. We use the normalization

$$\hat{u}_0(\omega) = \int \mathrm{e}^{-\mathrm{i}\langle \omega, x\rangle} u_0(x) \frac{\mathrm{d}x}{(2\pi)^d}, \quad u_0(x) = \int \mathrm{e}^{\mathrm{i}\langle \omega, x\rangle} \hat{u}_0(\omega)\, \mathrm{d}\omega$$

for the Fourier transform of a function $u_0(x) \in C_c^\infty(\mathbb{R}^d)$. In this section, $\omega, \tilde{\omega}$ denote Fourier variables and not the function $\boldsymbol{\omega}$ from Section 5.

Proposition 6.1. *We have that*

$$\mathbb{E}\, |u^\varepsilon(t,x) - \Psi^\varepsilon(t,x)\overline{u}(t,x)|^2 = e^{2\alpha_{t/\varepsilon^2}} \int\int \mathrm{e}^{\mathrm{i}\langle \omega+\tilde{\omega}, x\rangle} \hat{u}_0(\omega)\hat{u}_0(\tilde{\omega}) \widehat{\mathbb{E}}_{B,\varepsilon^{-2}t} \widehat{\mathbb{E}}_{\tilde{B},\varepsilon^{-2}t} \mathscr{A}^\varepsilon_{t;\omega,\tilde{\omega}}[B,\tilde{B}]\, \mathrm{d}\omega\, \mathrm{d}\tilde{\omega}, \tag{6.1}$$

where

$$\mathscr{A}^{\varepsilon}_{t;\omega,\tilde\omega}[B,\tilde B] = \exp\left\{\beta^2\mathscr{R}_{\varepsilon^{-2}t}[B,\tilde B]\right\}\mathscr{E}^{\varepsilon}_{t,\omega}[B]\mathscr{E}^{\varepsilon}_{t,\tilde\omega}[\tilde B] \tag{6.2}$$

$$\mathscr{E}^{\varepsilon}_{t,\omega}[B] = e^{\langle \mathrm{i}\omega,\varepsilon(B_{\varepsilon^{-2}t}-B_0)\rangle} - e^{-\frac{1}{2}at\langle\omega,\omega\rangle}. \tag{6.3}$$

Proof. We start with the Feynman–Kac formula for (1.1):

$$u^{\varepsilon}(t,x) = \mathbb{E}^{\varepsilon^{-1}x}_{B}\exp\left\{\beta\mathscr{V}_{\varepsilon^{-2}t}[B]-\lambda\varepsilon^{-2}t\right\}u_0(\varepsilon B_{\varepsilon^{-2}t}), \tag{6.4}$$

and note that

$$u_0(\varepsilon B_{\varepsilon^{-2}t}) = \int \mathrm{e}^{\mathrm{i}\langle\omega,\varepsilon B_{\varepsilon^{-2}t}\rangle}\hat u_0(\omega)\,\mathrm{d}\omega, \qquad \overline{u}(t,x) = \int \mathrm{e}^{\mathrm{i}\langle\omega,x\rangle-\frac{1}{2}at\langle\omega,\omega\rangle}\hat u_0(\omega)\,\mathrm{d}\omega,$$

so, if $B_0=\varepsilon^{-1}x$, then

$$u_0(\varepsilon B_{\varepsilon^{-2}t}) - \overline{u}(t,x) = \int \mathrm{e}^{\mathrm{i}\langle\omega,x\rangle}\mathscr{E}^{\varepsilon}_{t,\omega}[B]\hat u_0(\omega)\,\mathrm{d}\omega. \tag{6.5}$$

The Feynman–Kac formula also shows that

$$\Psi^{\varepsilon}(t,x) = \mathbb{E}^{\varepsilon^{-1}x}_{B}\exp\left\{\beta\mathscr{V}_{\varepsilon^{-2}t}[B]-\lambda\varepsilon^{-2}t\right\}. \tag{6.6}$$

Combining (6.4), (6.5), and (6.6) yields

$$u^{\varepsilon}(t,x)-\Psi^{\varepsilon}(t,x)\overline{u}(t,x) = \mathbb{E}^{\varepsilon^{-1}x}_{B}\exp\left\{\beta\mathscr{V}_{\varepsilon^{-2}t}[B]-\lambda\varepsilon^{-2}t\right\}\int e^{\mathrm{i}\langle\omega,x\rangle}\mathscr{E}^{\varepsilon}_{t,\omega}[B]\hat u_0(\omega)\,\mathrm{d}\omega.$$

We finish the proof of the lemma by simply computing the second moment:

$$\begin{aligned}\mathbb{E}(u^{\varepsilon}(t,x)-\Psi^{\varepsilon}(t,x)\overline{u}(t,x))^2 &= \mathbb{E}\Big(\mathbb{E}^{\varepsilon^{-1}x}_{B}\exp\left\{\beta\mathscr{V}_{\varepsilon^{-2}t}[B]-\lambda\varepsilon^{-2}t\right\}\int e^{\mathrm{i}\langle\omega,x\rangle}\mathscr{E}^{\varepsilon}_{t,\omega}[B]\hat u_0(\omega)\,\mathrm{d}\omega\Big)^2\\ &= \int\int \mathrm{e}^{\mathrm{i}\langle\omega+\tilde\omega,x\rangle}\hat u_0(\omega)\hat u_0(\tilde\omega)\mathbb{E}^{\varepsilon^{-1}x}_{B}\mathbb{E}^{\varepsilon^{-1}x}_{\tilde B}\mathbb{E}\exp\left\{\mathscr{V}_{\varepsilon^{-2}t}[B]+\mathscr{V}_{\varepsilon^{-2}t}[\tilde B]-2\lambda\varepsilon^{-2}t\right\}\mathscr{E}^{\varepsilon}_{t,\omega}[B]\mathscr{E}^{\varepsilon}_{t,\tilde\omega}[\tilde B]\,\mathrm{d}\omega\,\mathrm{d}\tilde\omega\\ &= e^{2\alpha_{\varepsilon^{-2}t}}\int\int \mathrm{e}^{\mathrm{i}\langle\omega+\tilde\omega,x\rangle}\hat u_0(\omega)\hat u_0(\tilde\omega)\widehat{\mathbb{E}}_{B,\varepsilon^{-2}t}\widehat{\mathbb{E}}_{\tilde B,\varepsilon^{-2}t}\mathscr{A}^{\varepsilon}_{t;\omega,\tilde\omega}[B,\tilde B]\,\mathrm{d}\omega\,\mathrm{d}\tilde\omega.\end{aligned}$$

□

To prove Theorem 1.3, we will bound the expression in the right side of (6.1) using the techniques of [13] recalled in Section 2. The proof will be very similar to that of [13, Lemma 5.2]. The key idea is that, with high probability, the only significant contributions to $\exp\left\{\beta^2\mathscr{R}_{\varepsilon^{-2}t}[B,\tilde B]\right\}$ come from times close to 0. Therefore, B and $\widetilde B$ in (6.2) are "nearly" independent. Moreover, the expectation of (6.3) is approximately 0 since the Markov chain has effective diffusivity a.

Our first lemma is that the correction $\mathscr{G}$ on the end of the Markov chain that appears in (2.5) does not matter.

Lemma 6.2. *We have*

$$\lim_{\varepsilon\to 0}\left|\widehat{\mathbb{E}}_{B,\tilde B;\varepsilon^{-2}t}\mathscr{A}^{\varepsilon}_{t;\omega,\tilde\omega}[B,\tilde B]-\widetilde{\mathbb{E}}_{W,\tilde W}\mathscr{A}^{\varepsilon}_{t;\omega,\tilde\omega}[W,\tilde W]\right| = 0. \tag{6.7}$$

As this lemma is a technical point, we defer its proof to the end of the section. We note that

$$\frac{\partial^2}{\partial r \partial \tilde{r}} \exp\left\{\beta^2 \mathscr{R}_{r,\tilde{r}}[W,\tilde{W}]\right\} = \frac{\partial}{\partial r}\left[\left(\beta^2 \int_0^r R(\tau - \tilde{r}, W_\tau - \tilde{W}_{\tilde{r}})\,\mathrm{d}\tau\right) \exp\left\{\beta^2 \mathscr{R}_{r,\tilde{r}}[W,\tilde{W}]\right\}\right]$$
$$= \mathscr{Q}_{r,\tilde{r}}[W,\tilde{W}] \exp\left\{\beta^2 \mathscr{R}_{r,\tilde{r}}[W,\tilde{W}]\right\}, \tag{6.8}$$

where

$$\mathscr{Q}_{r,\tilde{r}}[W,\tilde{W}] = \beta^2 R(r-\tilde{r}, W_r - \tilde{W}_{\tilde{r}}) + \beta^4 \int_{\tilde{r}-2}^r R(\tau - \tilde{r}, W_\tau - \tilde{W}_{\tilde{r}})\,\mathrm{d}\tau \int_{r-2}^{\tilde{r}} R(r-\tilde{\tau}, W_r - \tilde{W}_{\tilde{\tau}})\,\mathrm{d}\tilde{\tau}. \tag{6.9}$$

We note that

$$\mathscr{Q}_{r,\tilde{r}}[W,\tilde{W}] \geq 0 \tag{6.10}$$

almost surely, as the functions $\mu(s)$ and $\nu(y)$ in (1.3) are non-negative. We can now write

$$\widetilde{\mathbb{E}}_{W,\tilde{W}} \mathscr{A}^\varepsilon_{t;\omega,\tilde{\omega}}[W,\tilde{W}] = \widetilde{\mathbb{E}}_{W,\tilde{W}} \mathscr{E}^\varepsilon_{t,\omega}[W,\tilde{W}] \exp\left\{\beta^2 \mathscr{R}_{\varepsilon^{-2}t}[W,\tilde{W}]\right\}$$
$$= \int_0^{\varepsilon^{-2}t}\int_0^{\varepsilon^{-2}t} \widetilde{\mathbb{E}}_{W,\tilde{W}} \mathscr{E}^\varepsilon_{t;\omega,\tilde{\omega}}[W,\tilde{W}] \mathscr{Q}_{r,\tilde{r}}[W,\tilde{W}] \exp\left\{\beta^2 \mathscr{R}_{r,\tilde{r}}[W,\tilde{W}]\right\} \mathrm{d}r\,\mathrm{d}\tilde{r}, \tag{6.11}$$

with the shorthand

$$\mathscr{E}^\varepsilon_{t;\omega,\tilde{\omega}}[W,\tilde{W}] = \mathscr{E}^\varepsilon_{t,\omega}[W]\mathscr{E}^\varepsilon_{t,\tilde{\omega}}[\tilde{W}].$$

The next lemma gives an estimate for the contribution to the integral (6.11) from each $r, \tilde{r}$. The key point is that, if B is a Brownian motion with diffusivity σ^2, then $\exp\{\mathrm{i}\langle \omega, B_t\rangle + \frac{1}{2}t\sigma^2|\omega|^2\}$ is a martingale. Since W is converging to a Brownian motion with diffusivity a, the contribution to the integrand in (6.11) from $\mathscr{E}^\varepsilon_{t;\omega,\tilde{\omega}}[W,\tilde{W}] - \mathscr{E}^\varepsilon_{r;\omega,\tilde{\omega}}[W,\tilde{W}]$ should be small.

Lemma 6.3. *For fixed $r, \tilde{r} \geq 0$, we have*

$$\lim_{\varepsilon\to 0} \widetilde{\mathbb{E}}_{W,\tilde{W}} \mathscr{E}^\varepsilon_{t;\omega,\tilde{\omega}}[W,\tilde{W}] \mathscr{Q}_{r,\tilde{r}}[W,\tilde{W}] \exp\left\{\beta^2 \mathscr{R}_{r,\tilde{r}}[W,\tilde{W}]\right\} = 0, \tag{6.12}$$

Proof. To simplify the notation, in this proof we will avoid writing the dependencies on r and $\tilde{r}$, treating them as fixed. Using the objects of Theorem 2.1, we will put $\sigma_j = \sigma_j^{W,\tilde{W}}$ and $\kappa = \kappa^{W,\tilde{W}}$, and let $j_0 = \min\{j \geq 0 \mid \sigma_j \geq r \vee \tilde{r}\}$, and the σ-algebra $\mathcal{F}_{j_0}$ be generated by the collection of random variables

$$\{\eta_n^{W,\tilde{W}} \mid n < \sigma_{j_0}\} \cup \{w_n \mid n < \sigma_{j_0}\} \cup \{\tilde{w}_n \mid n < \sigma_{j_0}\}.$$

We note that

$$\mathscr{Q}_{r,\tilde{r}}[W,\tilde{W}] \exp\left\{\beta^2 \mathscr{R}_{r,\tilde{r}}[W,\tilde{W}]\right\} \in \mathcal{F}_{j_0}.$$

Therefore, we have

$$\widetilde{\mathbb{E}}_{W,\tilde{W}} \mathscr{E}^\varepsilon_{t;\omega,\tilde{\omega}}[W,\tilde{W}] \mathscr{Q}_{r,\tilde{r}}[W,\tilde{W}] \exp\left\{\beta^2 \mathscr{R}_{r,\tilde{r}}[W,\tilde{W}]\right\}$$
$$= \widetilde{\mathbb{E}}_{W,\tilde{W}}\left(\widetilde{\mathbb{E}}_{W,\tilde{W}}\left[\mathscr{E}^\varepsilon_{t;\omega,\tilde{\omega}}[W,\tilde{W}] \,\middle|\, \mathcal{F}_{j_0}\right] \exp\left\{\beta^2 \mathscr{R}_{r,\tilde{r}}[W,\tilde{W}]\right\} \mathscr{Q}_{r,\tilde{r}}[W,\tilde{W}]\right)$$
$$= \widetilde{\mathbb{E}}_{W,\tilde{W}}\left(e^{\mathrm{i}\langle \omega, \varepsilon W_{j_0}\rangle} \widetilde{\mathbb{E}}_W\left[e^{\mathrm{i}\langle \omega, \varepsilon(W_{\varepsilon^{-2}t} - W_{j_0})\rangle} \,\middle|\, \mathcal{F}_{j_0}\right] - e^{-\frac{1}{2}at|\omega|^2}\right)$$
$$\times \left(e^{\mathrm{i}\langle \tilde{\omega}, \varepsilon \tilde{W}_{j_0}\rangle} \widetilde{\mathbb{E}}_W\left[e^{\mathrm{i}\langle \tilde{\omega}, \varepsilon(\tilde{W}_{\varepsilon^{-2}t} - \tilde{W}_{j_0})\rangle} \,\middle|\, \mathcal{F}_{j_0}\right] - e^{-\frac{1}{2}at|\tilde{\omega}|^2}\right) \mathscr{Q}_{r,\tilde{r}}[W,\tilde{W}] \exp\left\{\beta^2 \mathscr{R}_{r,\tilde{r}}[W,\tilde{W}]\right\}. \tag{6.13}$$

Observe that

$$\widetilde{\mathbb{E}}_W \left[e^{\mathrm{i}\langle \omega, \varepsilon(W_{\varepsilon^{-2}t} - W_{j_0})\rangle} \,\middle|\, \mathcal{F}_{j_0} \right] = \widetilde{\mathbb{E}}_W \left[e^{\mathrm{i}\langle \omega, \varepsilon(W_{\varepsilon^{-2}t} - W_{j_0})\rangle} \,\middle|\, j_0 \right] \to e^{-\frac{1}{2}at|\omega|^2}$$

almost surely as $\varepsilon \to 0$ by Proposition 2.4, and similarly for $\widetilde{\mathbb{E}}_W \left[\mathrm{e}^{\mathrm{i}\langle \tilde{\omega}, \varepsilon(\tilde{W}_{\varepsilon^{-2}t} - \tilde{W}_{j_0})\rangle} \,\middle|\, \mathcal{F}_{j_0} \right]$. In addition, we have

$$e^{\mathrm{i}\langle \tilde{\omega}, \varepsilon \tilde{W}_{j_0}\rangle} \to 1$$

almost surely. The statement of the lemma then follows from the bounded convergence theorem applied to (6.13). □

Lemma 6.4. *We have*

$$\lim_{\varepsilon \to 0} \widetilde{\mathbb{E}}_{W,\tilde{W}} \mathscr{A}^{\varepsilon}_{t;\omega,\tilde{\omega}}[W, \tilde{W}] = 0.$$

Proof. We have, using (6.10), that

$$\left| \widetilde{\mathbb{E}}_{W,\tilde{W}} \mathscr{E}^{\varepsilon}_{t;\omega,\tilde{\omega}}[W, \tilde{W}] \mathscr{Q}_{r,\tilde{r}}[W, \tilde{W}] \exp\left\{ \beta^2 \mathscr{R}_{r,\tilde{r}}[W, \tilde{W}] \right\} \right| \le 4 \widetilde{\mathbb{E}}_{W,\tilde{W}} \mathscr{Q}_{r,\tilde{r}}[W, \tilde{W}] \exp\left\{ \beta^2 \mathscr{R}_{r,\tilde{r}}[W, \tilde{W}] \right\},$$

and by (6.8) that

$$\begin{aligned} \int_0^{\tilde{q}} \int_0^{q} \widetilde{\mathbb{E}}_{W,\tilde{W}} \mathscr{Q}_{r,\tilde{r}}[W, \tilde{W}] \exp\left\{ \beta^2 \mathscr{R}_{r,\tilde{r}}[W, \tilde{W}] \right\} \mathrm{d}r \, \mathrm{d}\tilde{r} \\ = \widetilde{\mathbb{E}}_{W,\tilde{W}} \exp\left\{ \beta^2 \mathscr{R}_{q,\tilde{q}}[W, \tilde{W}] \right\} \le \widetilde{\mathbb{E}}_{W,\tilde{W}} \exp\left\{ \beta^2 \mathscr{R}_{\infty}[W, \tilde{W}] \right\} < \infty, \end{aligned}$$

where the last inequality is by Proposition 2.5. The dominated convergence theorem applied to the integral (6.11), in light of the pointwise convergence (6.12), then implies the result. □

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. Combining Lemma 6.2 and Lemma 6.4, we see that the integrand in (6.1) converges pointwise to 0 as $\varepsilon \to 0$. On the other hand, as long as $\beta < \beta_0$, there is a constant C so that

$$\left| \widehat{\mathbb{E}}_{B,\tilde{B};\varepsilon^{-2}t} \mathscr{A}^{\varepsilon}_{t;\omega,\tilde{\omega}} \right| \le C$$

independently of $\varepsilon, \omega, \tilde{\omega}$. As $u_0 \in C_c^{\infty}(\mathbb{R}^d)$, the dominated convergence theorem and (2.4) then imply that

$$|u^{\varepsilon}(t, x) - \Psi^{\varepsilon}(t, x)\overline{u}(t, x)| \to 0$$

as $\varepsilon \to 0$. □

It remains to prove Lemma 6.2.

Proof of Lemma 6.2. We have

$$\begin{aligned} \widehat{\mathbb{E}}_{B,\tilde{B};\varepsilon^{-2}t} \mathscr{A}^{\varepsilon}_{t;\omega,\tilde{\omega}}[B, \tilde{B}] &= \widehat{\mathbb{E}}_{B,\tilde{B};\varepsilon^{-2}t} \mathscr{E}^{\varepsilon}_{t;\omega,\tilde{\omega}}[B, \tilde{B}] \exp\left\{ \beta^2 \mathscr{R}_{\varepsilon^{-2}t}[B, \tilde{B}] \right\} \\ &= \widetilde{\mathbb{E}}_{W,\tilde{W}} \mathscr{G}[w_{[\varepsilon^{-2}t]}] \mathscr{G}[\tilde{w}_{[\varepsilon^{-2}t]}] \mathscr{E}^{\varepsilon}_{t;\omega,\tilde{\omega}}[W, \tilde{W}] \exp\left\{ \beta^2 \mathscr{R}_{\varepsilon^{-2}t}[W, \tilde{W}] \right\}. \end{aligned} \tag{6.14}$$

Let $\gamma \in (0,2)$ be arbitrary, then

$$\begin{aligned}\widetilde{\mathbb{E}}_{W,\tilde{W}} &\left|\mathscr{E}^{\varepsilon}_{t;\omega,\tilde{\omega}}[W,\tilde{W}] \exp\left\{\beta^2 \mathscr{R}_{\varepsilon^{-2}t}[W,\tilde{W}]\right\} - \mathscr{E}^{\varepsilon}_{t-\varepsilon^{\gamma};\omega,\tilde{\omega}}[W,\tilde{W}] \exp\left\{\beta^2 \mathscr{R}_{\varepsilon^{-2}(t-\varepsilon^{\gamma})}[W,\tilde{W}]\right\}\right| \\ &\le \widetilde{\mathbb{E}}_{W,\tilde{W}} \left|\mathscr{E}^{\varepsilon}_{t;\omega,\tilde{\omega}}[W,\tilde{W}] - \mathscr{E}^{\varepsilon}_{t-\varepsilon^{\gamma};\omega,\tilde{\omega}}[W,\tilde{W}]\right| \exp\left\{\beta^2 \mathscr{R}_{\varepsilon^{-2}t}[W,\tilde{W}]\right\} \\ &\qquad + \widetilde{\mathbb{E}}_{W,\tilde{W}} \left|\mathscr{E}^{\varepsilon}_{t-\varepsilon^{\gamma};\omega,\tilde{\omega}}[W,\tilde{W}]\right| \left|\exp\left\{\beta^2 \mathscr{R}_{\varepsilon^{-2}t}[W,\tilde{W}]\right\} - \exp\left\{\beta^2 \mathscr{R}_{\varepsilon^{-2}(t-\varepsilon^{\gamma})}[W,\tilde{W}]\right\}\right|.\end{aligned} \tag{6.15}$$

Let's address the first term of (6.15). By (2.9), we have

$$\widetilde{\mathbb{E}}_W \left|\varepsilon W_{\varepsilon^{-2}t} - \varepsilon W_{\varepsilon^{-2}(t-\varepsilon^{\gamma})}\right|^2 \le C_{\mathrm{v}} \varepsilon^{\gamma},$$

which in particular means that

$$\lim_{\varepsilon\to 0} \left|\varepsilon W_{\varepsilon^{-2}t} - \varepsilon W_{\varepsilon^{-2}(t-\varepsilon^{\gamma})}\right| = 0 \tag{6.16}$$

in probability. The same statement holds for $\tilde{W}$. We then have, using Hölder's inequality,

$$\begin{aligned}\lim_{\varepsilon\to 0} \widetilde{\mathbb{E}}_{W,\tilde{W}} &\left|\mathscr{E}^{\varepsilon}_{t;\omega,\tilde{\omega}}[W,\tilde{W}] - \mathscr{E}^{\varepsilon}_{t-\varepsilon^{\gamma};\omega,\tilde{\omega}}[W,\tilde{W}]\right| \exp\left\{\beta^2 \mathscr{R}_{\varepsilon^{-2}t}[W,\tilde{W}]\right\} \\ &\le C_{\delta} \lim_{\varepsilon\to 0} \left(\widetilde{\mathbb{E}}_{W,\tilde{W}} \left|\mathscr{E}^{\varepsilon}_{t;\omega,\tilde{\omega}}[W,\tilde{W}] - \mathscr{E}^{\varepsilon}_{t-\varepsilon^{\gamma};\omega,\tilde{\omega}}[W,\tilde{W}]\right|^{\frac{1}{\delta}+1}\right)^{\frac{\delta}{1+\delta}} = 0,\end{aligned} \tag{6.17}$$

by Proposition 2.5 and the bounded convergence theorem in light of (6.16).

Finally, let us look at the second term of (6.15), which is easier. Here, we have

$$\begin{aligned}\lim_{\varepsilon\to 0} \widetilde{\mathbb{E}}_{W,\tilde{W}} &\left|\mathscr{E}^{\varepsilon}_{t-\varepsilon^{\gamma};\omega,\tilde{\omega}}[W,\tilde{W}]\right| \left|\exp\left\{\beta^2 \mathscr{R}_{\varepsilon^{-2}t}[W,\tilde{W}]\right\} - \exp\left\{\beta^2 \mathscr{R}_{\varepsilon^{-2}(t-\varepsilon^{\gamma})}[W,\tilde{W}]\right\}\right| \\ &\le 4 \lim_{\varepsilon\to 0} \widetilde{\mathbb{E}}_{W,\tilde{W}} \left|\exp\left\{\beta^2 \mathscr{R}_{\varepsilon^{-2}t}[W,\tilde{W}]\right\} - \exp\left\{\beta^2 \mathscr{R}_{\varepsilon^{-2}(t-\varepsilon^{\gamma})}[W,\tilde{W}]\right\}\right| = 0\end{aligned} \tag{6.18}$$

by the dominated convergence theorem, again in light of (2.9). Applying (6.17) and (6.18) to (6.15) implies that

$$\lim_{\varepsilon\to 0} \widehat{\mathbb{E}}_{W,\tilde{W}} \left|\mathscr{E}_{a,\omega,\varepsilon,t}[W,\tilde{W}] \exp\left\{\beta^2 \mathscr{R}_{\varepsilon^{-2}t}[W,\tilde{W}]\right\} - \mathscr{E}_{a,\omega,\varepsilon,t-\varepsilon^{\gamma}}[W,\tilde{W}] \exp\left\{\beta^2 \mathscr{R}_{\varepsilon^{-2}(t-\varepsilon^{\gamma})}[W,\tilde{W}]\right\}\right| = 0. \tag{6.19}$$

Combining (6.14), (6.19), and Lemma 2.8, and recalling that $\mathscr{G}$ is bounded above and away from zero, completes the proof of the lemma. □

7 The second term of the expansion

In this section we will prove Theorem 1.4. We first introduce some notation. Having fixed $\gamma \in (1,2)$, for each $s > 0$ we define a discrete set of times

$$r_k = \begin{cases} 0 & k = 0 \\ s - \varepsilon^{-\gamma}([\varepsilon^{\gamma} s] - (k-1)) & k > 0, \end{cases} \tag{7.1}$$

and set

$$\mathscr{I}^{\varepsilon}_t[B] = \sum_{k=0}^{K^{\varepsilon}_t} (\varepsilon B_{r_{k+1}} - \varepsilon B_{r_k}) \cdot \nabla \overline{u}(t - \varepsilon^2 r_k, \varepsilon B_{r_k}), \tag{7.2}$$

with

$$K_t^\varepsilon = [\varepsilon^{\gamma-2} t] \tag{7.3}$$

The next lemma gives a Feynman–Kac formula for the corrector u_1^ε defined in (1.22).

Lemma 7.1. *We have*

$$u_1^\varepsilon(t,x) = \frac{1}{\varepsilon}\mathbb{E}_B^{\varepsilon^{-1}x} \exp\left\{\mathscr{V}_{\varepsilon^{-2}t}[B] - \lambda\varepsilon^{-2}t\right\} \mathscr{I}_t^\varepsilon[B]. \tag{7.4}$$

Proof. The Feynman–Kac formula applied to (1.20), in the same way as in (5.16), gives the following expression for the solution $\theta_j(s,y)$ to that equation:

$$\begin{aligned}
\theta_j(s,y) &= \mathbb{E}_B^y \int_0^{s-\varepsilon^{-\gamma}(j-1)} \exp\left\{\int_0^r [\beta V(s-\tau, B_\tau) - \lambda]\, \mathrm{d}\tau\right\} \nabla\Psi(s-r, B_r)\, \mathrm{d}r \\
&= \mathbb{E}_B^y \nabla_\xi\big|_{\xi=0} \int_0^{s-\varepsilon^{-\gamma}(j-1)} \exp\left\{\int_0^s [\beta V(s-\tau, B_\tau + H(\tau-r)\xi) - \lambda]\, \mathrm{d}\tau\right\} \mathrm{d}r \\
&= \nabla_\xi\big|_{\xi=0} \mathbb{E}_B^y \exp\left\{\int_0^s [\beta V(s-\tau, B_\tau + (\tau\wedge(s-\varepsilon^{-\gamma}(j-1)))\xi) - \lambda]\, \mathrm{d}\tau\right\},
\end{aligned}$$

where $H(x)$ is the Heaviside function. The Girsanov formula yields

$$\begin{aligned}
\theta_j(s,y) &= \nabla_\xi\big|_{\xi=0} \mathbb{E}_B^y \exp\left\{\mathscr{V}_s[B] - \lambda s + (B_{s-\varepsilon^{-\gamma}(j-1)} - y)\cdot\xi - \frac{s-\varepsilon^{-\gamma}(j-1)}{2}|\xi|^2\right\} \\
&= \mathbb{E}_B^y (B_{s-\varepsilon^{-\gamma}(j-1)} - y) \exp\left\{\mathscr{V}_s[B] - \lambda s\right\}.
\end{aligned}$$

Given this expression for θ_j, we use it in (1.21) to write

$$\begin{aligned}
u_{1;j}(s,y) &= \mathbb{E}_B^y \exp\left\{\int_0^{s-\varepsilon^{-\gamma}j} [\beta V(s-\tau, B_\tau) - \lambda]\, \mathrm{d}\tau\right\} \theta_j(\varepsilon^{-\gamma}j, B_{s-\varepsilon^{-\gamma}j}) \cdot \nabla\overline{u}(\varepsilon^{2-\gamma}j, \varepsilon B_{s-\varepsilon^{-\gamma}j}) \\
&= \mathbb{E}_B^y (B_{s-\varepsilon^{-\gamma}(j-1)} - B_{s-\varepsilon^{-\gamma}j}) \cdot \nabla\overline{u}(\varepsilon^{2-\gamma}j, \varepsilon B_{s-\varepsilon^{-\gamma}j}) \exp\left\{\mathscr{V}_s[B] - \lambda s\right\}\cdot
\end{aligned}$$

Finally, by (1.22) we have

$$u_1^\varepsilon(t,x) = u_1(\varepsilon^{-2}t, \varepsilon^{-1}x)$$

where

$$\begin{aligned}
u_1(s,y) &= \sum_{j=1}^{[\varepsilon^\gamma s]} \mathbb{E}_B^y (B_{s-\varepsilon^{-\gamma}(j-1)} - B_{s-\varepsilon^{-\gamma}j}) \cdot \nabla\overline{u}(\varepsilon^{2-\gamma}j, \varepsilon B_{s-\varepsilon^{-\gamma}j}) \exp\left\{\mathscr{V}_s[B] - \lambda s\right\} \\
&\quad + \mathbb{E}_B^y (B_{s-\varepsilon^{-\gamma}\lfloor\varepsilon^\gamma s\rfloor} - y) \exp\left\{\mathscr{V}_s[B] - \lambda s\right\} \cdot \nabla\overline{u}(\varepsilon^2 s, \varepsilon y) \\
&= \mathbb{E}_B^y \exp\left\{\mathscr{V}_s[B] - \lambda s\right\} \sum_{k=0}^{[\varepsilon^\gamma s]} (B_{r_{k+1}} - B_{r_k}) \cdot \nabla\overline{u}(\varepsilon^2(s-r_k), \varepsilon B_{r_k}),
\end{aligned} \tag{7.5}$$

with r_k defined in (7.1). Rescaling (7.5) yields (7.4). □

Next, we consider the error term

$$q^\varepsilon(t,x) = u^\varepsilon(t,x) - \Psi^\varepsilon(t,x)\overline{u}(t,x) - \varepsilon u_1^\varepsilon(t,x).$$

Combining (6.4), (6.6), and (7.4) gives an expression

$$q^\varepsilon(t,x) = \mathbb{E}_B^{\varepsilon^{-1}x}\left[u_0(\varepsilon B_{\varepsilon^{-2}t}) - \overline{u}(t,x) - \mathscr{I}_t^\varepsilon[B]\right]\exp\left\{\mathscr{V}_{\varepsilon^{-2}t}[B] - \lambda\varepsilon^{-2}t\right\},$$

with its expectation given by

$$\mathbb{E}q^\varepsilon(t,x) = \exp\{\alpha_{t/\varepsilon^2}\}\widehat{\mathbb{E}}_{B;\varepsilon^{-2}t}^{\varepsilon^{-1}x}\left[u_0(\varepsilon B_{\varepsilon^{-2}t}) - \overline{u}(t,x) - \mathscr{I}_t^\varepsilon[B]\right].$$

Taking covariances, we obtain

$$\begin{aligned}
&\mathbb{E}q^\varepsilon(t,x)q^\varepsilon(t,\tilde{x}) - \mathbb{E}q^\varepsilon(t,x)\mathbb{E}q^\varepsilon(t,\tilde{x})\\
&\quad= \exp\{2\alpha_{t/\varepsilon^2}\}\widehat{\mathbb{E}}_{B,\tilde{B};\varepsilon^{-2}t}^{\varepsilon^{-1}x,\varepsilon^{-1}\tilde{x}}\left(u_0(\varepsilon B_{\varepsilon^{-2}t}) - \overline{u}(t,x) - \mathscr{I}_t^\varepsilon[B]\right)\left(u_0(\varepsilon\widetilde{B}_{\varepsilon^{-2}t}) - \overline{u}(t,\tilde{x}) - \mathscr{I}_t^\varepsilon[\widetilde{B}]\right)\\
&\quad\times\left(\exp\left\{\beta^2\mathscr{R}_{\varepsilon^{-2}t}[B,\tilde{B}]\right\} - 1\right)\\
&\quad= \exp\{2\alpha_{t/\varepsilon^2}\}\widetilde{\mathbb{E}}_{W,\tilde{W}}^{x/\varepsilon,\tilde{x}/\varepsilon}\left(u_0(\varepsilon W_{\varepsilon^{-2}t}) - \overline{u}(t,x) - \mathscr{I}_t^\varepsilon[W]\right)\left(u_0(\varepsilon\tilde{W}_{\varepsilon^{-2}t}) - \overline{u}(t,\tilde{x}) - \mathscr{I}_t^\varepsilon[\tilde{W}]\right)\\
&\quad\times\left(\exp\left\{\beta^2\mathscr{R}_{\varepsilon^{-2}t}[W,\tilde{W}]\right\} - 1\right)\mathscr{G}(w_N)\mathscr{G}(\tilde{w}_N).
\end{aligned} \tag{7.6}$$

In the last equality of (7.6) we used Theorem 2.1.

In line with the framework of Section 2, we will proceed to approximate the times r_k by nearby regeneration times of the Markov chains. Thus, we define

$$\sigma^W(k) = (\varepsilon^{-2}t)\wedge\min\{r\geq r_k \mid \eta_r^W = 1\}.$$

Before we begin our argument in earnest, we recall bounds on the relevant error terms. Put

$$Y = \max_{0\leq k\leq K_t^\varepsilon}(\sigma^W(k) - r_k),\quad F(\tau) = \max_{r\in[0,\varepsilon^{-2}t-\tau]}|W_{r+\tau} - W_r|,$$

and

$$Z = \varepsilon^{\gamma/2}F(\varepsilon^{-\gamma} + Y).$$

Lemma 7.2. *We have constants C and c so that*

$$\widetilde{\mathbb{P}}_W(Y\geq C|\log\varepsilon| + \xi)\leq C\mathrm{e}^{-c\xi}, \tag{7.7}$$

$$\widetilde{\mathbb{P}}_W(F(Y)\geq C|\log\varepsilon| + \xi)\leq C\mathrm{e}^{-c\xi}, \tag{7.8}$$

and

$$\widetilde{\mathbb{P}}_W(Z\geq C|\log\varepsilon| + \xi)\leq C\mathrm{e}^{-c\xi}. \tag{7.9}$$

These bounds are simple consequences of the regeneration structure of the Markov chain described in Section 2 and of [13, Lemma A.1]. We begin our approximation procedure by replacing the deterministic times r_k in the definition (7.2) of $\mathscr{I}_{t,x}^\varepsilon$ by their regeneration approximations.

Lemma 7.3. *Let*

$$\tilde{\mathscr{I}}_t^\varepsilon[W] = \sum_{k=0}^{K_t^\varepsilon}(\varepsilon W_{\sigma^W(k+1)} - \varepsilon W_{\sigma^W(k)})\cdot\nabla\overline{u}(t-\varepsilon^2\sigma^W(k),\varepsilon W_{\sigma^W(k)}), \tag{7.10}$$

then for any $1\leq p<\infty$ and any $\zeta<\gamma-1$ there exists a constant $C = C(p,\zeta,\|u_0\|_{\mathcal{C}^2})\leq\infty$ so that

$$\left(\widetilde{\mathbb{E}}_W^x|\mathscr{I}_t^\varepsilon[W] - \tilde{\mathscr{I}}_t^\varepsilon[W]|^p\right)^{1/p}\leq C\varepsilon^\zeta. \tag{7.11}$$

Proof. We have

$$\begin{aligned}
&\mathscr{I}_t^\varepsilon[W]-\tilde{\mathscr{I}}_t^\varepsilon[W]\\
&=\sum_{k=0}^{K_t^\varepsilon}(\varepsilon W_{r_{k+1}}-\varepsilon W_{r_k})\cdot\nabla\overline{u}(t-\varepsilon^2 r_k,\varepsilon W_{r_k})-(\varepsilon W_{\sigma^W(k+1)}-\varepsilon W_{\sigma^W(k)})\cdot\nabla\overline{u}(t-\varepsilon^2\sigma^W(k),\varepsilon W_{\sigma^W(k)}),
\end{aligned}$$

hence

$$\begin{aligned}
|\mathscr{I}_t^\varepsilon[W]-\tilde{\mathscr{I}}_t^\varepsilon[W]| &\le \sum_{k=0}^{K_t^\varepsilon}|(\varepsilon W_{r_{k+1}}-\varepsilon W_{r_k})-(\varepsilon W_{\sigma^W(k+1)}-\varepsilon W_{\sigma^W(k)})|\cdot|\nabla\overline{u}(t-\varepsilon^2 r_k,\varepsilon W_{r_k})|\\
&+\sum_{k=0}^{K_t^\varepsilon}|\varepsilon W_{\sigma^W(k+1)}-\varepsilon W_{\sigma^W(k)}|\cdot|\nabla\overline{u}(t-\varepsilon^2 r_k,\varepsilon W_{r_k})-\nabla\overline{u}(t-\varepsilon^2\sigma^W(k),\varepsilon W_{\sigma^W(k)})|. \quad (7.12)
\end{aligned}$$

We estimate the first term in the right side by

$$|(\varepsilon W_{r_{k+1}}-\varepsilon W_{r_k})-(\varepsilon W_{\sigma^W(k+1)}-\varepsilon W_{\sigma^W(k)})|\cdot|\nabla\overline{u}(t-\varepsilon^2 r_k),\varepsilon W_{r_k})|\le 2F(Y)\varepsilon\|\overline{u}\|_{\mathcal{C}^1}, \quad (7.13)$$

and the second by

$$\begin{aligned}
&|\varepsilon W_{\sigma^W(k+1)}-\varepsilon W_{\sigma^W(k)}|\cdot|\nabla\overline{u}(t-\varepsilon^2 r_k,\varepsilon W_{r_k})-\nabla\overline{u}(t-\varepsilon^2\sigma^W(k),\varepsilon W_{\sigma^W(k)})|\\
&\qquad\le \varepsilon F(Y+\varepsilon^{-\gamma})\|\overline{u}\|_{\mathcal{C}^2}(\varepsilon^2 Y+\varepsilon F(Y))=\varepsilon^{1-\gamma/2}Z\|\overline{u}\|_{\mathcal{C}^2}(\varepsilon^2 Y+\varepsilon F(Y)). \quad (7.14)
\end{aligned}$$

Combining (7.12), (7.13) and (7.14), and recalling (7.3), gives us

$$|\mathscr{I}_t^\varepsilon[W]-\tilde{\mathscr{I}}_t^\varepsilon[W]|\le\varepsilon^{\gamma-2}t\big[2F(Y)\varepsilon\|\overline{u}\|_{\mathcal{C}^1}+\varepsilon^{1-\gamma/2}Z\|\overline{u}\|_{\mathcal{C}^2}(\varepsilon^2 Y+\varepsilon F(Y))\big],$$

which in light of Lemma 7.2 implies (7.11). □

Lemma 7.4. *For any power $1\le p<\infty$, there exists a $C=C(p,t,\|u_0\|_{\mathcal{C}^2})<\infty$ so that*

$$\left(\widetilde{\mathbb{E}}_W^{x/\varepsilon}\left|u_0(\varepsilon W_{\varepsilon^{-2}t})-\overline{u}(t,x)-\tilde{\mathscr{I}}_t^\varepsilon[W]\right|^p\right)^{1/p}\le C\varepsilon^\zeta, \quad (7.15)$$

for any $\zeta<1-\gamma/2$.

Proof. To ease the notation, in this proof we will abbreviate $\sigma=\sigma^W$. We write the Taylor expansion

$$\begin{aligned}
&\overline{u}(t-\varepsilon^2\sigma(k+1),\varepsilon W_{\sigma(k+1)})-\overline{u}(t-\varepsilon^2\sigma(k),\varepsilon W_{\sigma(k)})\\
&=-\varepsilon^2\left(\sigma(k+1)-\sigma(k)\right)\partial_t\overline{u}(t-\varepsilon^2\sigma(k),\varepsilon W_{\sigma(k)})+\varepsilon(W_{\sigma(k+1)}-W_{\sigma(k)})\cdot\nabla\overline{u}(t-\varepsilon^2\sigma(k),\varepsilon W_{\sigma(k)})\\
&\quad+\frac{1}{2}\varepsilon^2\,\mathrm{Q}\,\overline{u}(t-\varepsilon^2\sigma(k),\varepsilon W_{\sigma(k)})(W_{\sigma(k+1)}-W_{\sigma(k)})+\mathscr{Y}_k[W], \quad (7.16)
\end{aligned}$$

where $\mathrm{Q}\,\overline{u}$ is the quadratic form associated to the Hessian of $\overline{u}$ (so $\mathrm{Q}\,\overline{u}(V)=\mathrm{Hess}\,\overline{u}(V,V)$) and $\mathscr{Y}_k[W]$ is the remainder term. By Taylor's theorem, we have

$$|\mathscr{Y}_k[W]|\le C\|\overline{u}\|_{\mathcal{C}^3}\left(\varepsilon^4|\sigma(k+1)-\sigma(k)|^2+\varepsilon^3|W_{\sigma(k+1)}-W_{\sigma(k)}|^3\right). \quad (7.17)$$

Note that the second term in the second line of (7.16) appears in the definition (7.10) of $\tilde{\mathscr{I}}_t^\varepsilon$. Thus, we can telescope the left side of (7.16) to obtain

$$\tilde{\mathscr{I}}_t^\varepsilon[W]=u_0(\varepsilon W_{\varepsilon^{-2}t})-\overline{u}(t-\varepsilon^2\sigma(0),\varepsilon W_{\sigma(0)})+\sum_{k=0}^{K_t^\varepsilon}\left(\varepsilon^2\mathscr{X}_k[W]+\mathscr{Y}_k[W]\right), \quad (7.18)$$

where

$$\mathscr{X}_k[W] = (\sigma(k+1)-\sigma(k))\,\partial_t\overline{u}(t-\varepsilon^2\sigma(k),\varepsilon W_{\sigma(k)}) - \frac{1}{2}\,\mathrm{Q}\,\overline{u}(t-\varepsilon^2\sigma(k),\varepsilon W_{\sigma(k)})(W_{\sigma(k+1)}-W_{\sigma(k)})$$
$$= (\sigma(k+1)-\sigma(k))\,\frac{1}{2}a\Delta\overline{u}(t-\varepsilon^2\sigma(k),\varepsilon W_{\sigma(k)}) - \frac{1}{2}\,\mathrm{Q}\,\overline{u}(t-\varepsilon^2\sigma(k),\varepsilon W_{\sigma(k)})(W_{\sigma(k+1)}-W_{\sigma(k)}).$$

We now deal with each piece of this expression in turn.

The drift terms. We first define

$$\widetilde{\mathscr{X}_k} = (\widetilde{\sigma}(k+1)-\widetilde{\sigma}(k))\,\frac{1}{2}a\Delta\overline{u}(t-\varepsilon^2\sigma(k),\varepsilon W_{\sigma(k)}) - \frac{1}{2}\,\mathrm{Q}\,\overline{u}(t-\varepsilon^2\sigma(k),\varepsilon W_{\sigma(k)})(W_{\widetilde{\sigma}(k+1)}-W_{\widetilde{\sigma}(k)}),$$

where $\widetilde{\sigma}(k)=\min\{r\ge r_k \mid \eta_r^W=1\}$. Using the relation (2.8) between the effective diffusivity a and the variance of the increments $W_{\sigma_{n+1}^W}-W_{\sigma_n^W}$, as well as the isotropy of W, we see that

$$\widetilde{\mathbb{E}}_W\,\widetilde{\mathscr{X}_k}[W]=0 \tag{7.19}$$

for each k. We also note the simple bound

$$|\widetilde{\mathscr{X}_k}[W]| \le a\|\overline{u}\|_{\mathcal{C}^2}(\varepsilon^{-\gamma}+Y)+\|\overline{u}\|_{\mathcal{C}^2}\left(F(\varepsilon^{-\gamma}+Y)\right)^2 \le a\|\overline{u}\|_{\mathcal{C}^2}(\varepsilon^{-\gamma}+Y)+\|\overline{u}\|_{\mathcal{C}^2}\varepsilon^{-\gamma}Z^2. \tag{7.20}$$

Therefore, by Lemma 7.2, we have

$$\widetilde{\mathbb{E}}_W|\widetilde{\mathscr{X}_k}[W]|^p \le C\varepsilon^{-p\xi}, \tag{7.21}$$

for any $\xi>\gamma$. We further define

$$M_\ell=\sum_{k=0}^{\ell}\widetilde{\mathscr{X}_k}[W]. \tag{7.22}$$

For each $\ell\ge 0$, define $\mathcal{G}_\ell$ to be the σ-algebra generated by $\{W_t \mid t\le\sigma(\ell)\}\cup\{\eta_t^W \mid t\le\sigma(\ell)\}$. Then, according to (7.19), $\{M_\ell\}$ is a martingale with respect to the filtration $\{\mathcal{G}_\ell\}$. An L^p-version of the Burkholder-Gundy inequality [7] (see also [3, Theorem 9]) implies that

$$\left(\widetilde{\mathbb{E}}_W|\varepsilon^2 M_{K_t^\varepsilon}|^p\right)^{1/p} \le C\varepsilon^2\left[(K_t^\varepsilon+1)^{p/2-1}\sum_{k=0}^{K_t^\varepsilon}\widetilde{\mathbb{E}}_W|\widetilde{\mathscr{X}_k}[W]|^p\right]^{1/p} \le C\varepsilon^\zeta \tag{7.23}$$

for any $\zeta<1-\gamma/2$. We used (7.3) and (7.21) in the second inequality above.

On the other hand, we note that $\mathscr{X}_k[W]-\widetilde{\mathscr{X}_k}[W]$ can be nonzero for at most one k, so we have

$$\left|\sum_{k=0}^{K_t^\varepsilon}(\mathscr{X}_k[W]-\widetilde{\mathscr{X}_k}[W])\right| = \max_{k=0}^{K_t^\varepsilon}\left|\mathscr{X}_k[W]-\widetilde{\mathscr{X}_k}[W]\right| \le C\|\overline{u}\|_{\mathcal{C}^2}\varepsilon^{-\gamma/2}Z, \tag{7.24}$$

so

$$\left(\widetilde{\mathbb{E}}_W\left|\varepsilon^2\sum_{k=0}^{K_t^\varepsilon}(\mathscr{X}_k[W]-\widetilde{\mathscr{X}_k}[W])\right|^p\right)^{1/p} \le C\varepsilon^\zeta \tag{7.25}$$

for any $\zeta<2-\gamma/2$.

The error term. By (7.17), we have

$$\Big|\sum_{k=0}^{K_t^\varepsilon}\mathscr{Y}_j^\varepsilon[W]\Big| \le C\|\overline{u}\|_{\mathcal{C}^3}\sum_{j\ge 0}^{K_t^\varepsilon}\left(\varepsilon^4|\sigma(k+1)-\sigma(j)|^2+|\varepsilon W_{\sigma(k+1)}-\varepsilon W_{\sigma(k)}|^3\right)$$
$$\le C\|\overline{u}\|_{\mathcal{C}^3}K_t^\varepsilon\left(\varepsilon^4\left(\varepsilon^{-\gamma}+Y\right)^2+|\varepsilon F(\varepsilon^{-\gamma}+Y)|^3\right) \le C\|\overline{u}\|_{\mathcal{C}^3}\left(\varepsilon^{2-\gamma}\left(1+\varepsilon^\gamma Y\right)^2+\varepsilon^{1-\gamma/2}Z^3\right),$$

so by Lemma 7.2 we have

$$\Big(\widetilde{\mathbb{E}}_W\Big|\sum_{k=0}^{K_t^\varepsilon}\mathscr{Y}_j^\varepsilon[W]\Big|^p\Big)^{1/p}\le C\varepsilon^\zeta \tag{7.26}$$

for any $\zeta<1-\gamma/2$.

The initial term. Finally, we observe that

$$\left(\widetilde{\mathbb{E}}_W\left|\overline{u}(t-\varepsilon^2\sigma(0),\varepsilon W_{\sigma(0)})-\overline{u}(t,x)\right|^p\right)^{1/p}\le\|\overline{u}\|_{\mathcal{C}^1}\left(\widetilde{\mathbb{E}}_W(\varepsilon^2\sigma(0)+\varepsilon|W_{\sigma(0)}|)^p\right)^{1/p}\le C\varepsilon^\zeta \tag{7.27}$$

for any $\zeta<1-\gamma/2$.

Applying the bounds (7.23), (7.25), (7.26), and (7.27) to (7.18) gives us (7.15). □

Corollary 7.5. *For any $1\le p<\infty$ and $\zeta<(\gamma-1)\wedge(1-\gamma/2)$ there exists $C=C(p,\zeta,t,\|u_0\|_{\mathcal{C}^2})$ so that*

$$\left(\widetilde{\mathbb{E}}_W^{x/\varepsilon}|u_0(\varepsilon W_{\varepsilon^{-2}t})-\overline{u}(t,x)-\mathscr{I}_t^\varepsilon[W]|^p\right)^{1/p}\le C\varepsilon^\zeta. \tag{7.28}$$

Proof. This is a simple consequence of the triangle inequality applied to Lemma 7.3 and Lemma 7.4. □

We will need the following auxiliary lemma.

Lemma 7.6. *There is a $\beta_0>0$ so that if $\chi\beta^2<\beta_0^2$, then there is a constant $C=C(\alpha,\beta)<\infty$ so that for any $\varepsilon>0$ and $x,\tilde{x}\in\mathbf{R}^2$ we have*

$$\widetilde{\mathbb{E}}_{W,\tilde{W}}^{x/\varepsilon,\tilde{x}/\varepsilon}\left(\exp\left\{\beta^2\mathscr{R}_{\varepsilon^{-2}t}[W,\tilde{W}]\right\}-1\right)^\chi<C\left(\frac{\varepsilon}{|x-\tilde{x}|}\wedge 1\right)^{d-2}. \tag{7.29}$$

Proof. As $\mathscr{R}_t[W,\tilde{W}]\ge 0$, we have

$$\begin{aligned}\widetilde{\mathbb{E}}_{W,\tilde{W}}^{x/\varepsilon,\tilde{x}/\varepsilon}\left(\exp\left\{\beta^2\mathscr{R}_{\varepsilon^{-2}t}[W,\tilde{W}]\right\}-1\right)^\chi&\le\widetilde{\mathbb{P}}_{W,\tilde{W}}^{x/\varepsilon,\tilde{x}/\varepsilon}\left(\mathscr{R}_{\varepsilon^{-2}t}[W,\tilde{W}]>0\right)\\&\times\sup_{r>0,W\restriction_{[0,r]},\tilde{W}\restriction_{[0,r]}}\widetilde{\mathbb{E}}_{W,\tilde{W}}^{x/\varepsilon,\tilde{x}/\varepsilon}\exp\left\{\chi\beta^2\mathscr{R}_{[r,\varepsilon^{-2}t]}[W,\tilde{W}]\right\}\le C\left(\frac{\varepsilon}{|x-\tilde{x}|}\wedge 1\right)^{d-2},\end{aligned} \tag{7.30}$$

by Proposition 2.11 and Proposition 2.5, as long as $\chi\beta^2<\beta_0^2$ is sufficiently small. □

Proposition 7.7. *For all $\chi>1$ and for any $\zeta<(1-\gamma/2)\wedge(\gamma-1)$, there exists C so that*

$$|\mathbb{E}q^\varepsilon(t,x)q^\varepsilon(t,\tilde{x})-\mathbb{E}q^\varepsilon(t,x)\mathbb{E}q^\varepsilon(t,\tilde{x})|\le C\left(|x-\tilde{x}|^{-\frac{d-2}{\chi}}\varepsilon^{2\zeta+\frac{d-2}{\chi}}\wedge\varepsilon^{2\zeta}\right).$$

Proof. Take p so that $\frac{1}{\chi}+\frac{2}{p}=1$. We go back to (7.6) and apply Hölder's inequality, as well as (7.28) and (7.29) to get the bound

$$\begin{aligned}&|\mathbb{E}q^\varepsilon(t,x)q^\varepsilon(t,\tilde{x})-\mathbb{E}q^\varepsilon(t,x)\mathbb{E}q^\varepsilon(t,\tilde{x})|\le\|\mathscr{G}\|_\infty^2\left(\sup_x\widetilde{\mathbb{E}}_W^{x/\varepsilon}|u_0(\varepsilon W_{\varepsilon^{-2}t})-\overline{u}(t,x)-\mathscr{I}_t^\varepsilon[W]|^p\right)^{2/p}\\&\times\left(\widetilde{\mathbb{E}}_{W,\tilde{W}}^{x/\varepsilon,\tilde{x}/\varepsilon}\left(\exp\left\{\beta^2\mathscr{R}_{\varepsilon^{-2}t}[W,\tilde{W}]\right\}-1\right)^\chi\right)^{1/\chi}\le C\varepsilon^{2\zeta}\left(\frac{\varepsilon}{|x-\tilde{x}|}\wedge 1\right)^{\frac{d-2}{\chi}}.\end{aligned}$$

□

We are finally ready to prove Theorem 1.4.

Proof of Theorem 1.4. By Proposition 7.7, we have, for any $\zeta < (1-\gamma/2) \wedge (\gamma - 1)$ and any $\chi > 1$,

$$
\begin{aligned}
&\varepsilon^{-(d-2)}\mathbb{E}\left(\int g(x)q^\varepsilon(t,x)\,\mathrm{d}x - \mathbb{E}\int g(x)q^\varepsilon(t,x)\,\mathrm{d}x\right)^2 \\
&\quad = \varepsilon^{-(d-2)}\int\int g(x)g(\tilde{x})\left[\mathbb{E}q^\varepsilon(t,x)q^\varepsilon(t,\tilde{x}) - \mathbb{E}q^\varepsilon(t,x)\mathbb{E}q^\varepsilon(t,\tilde{x})\right]\,\mathrm{d}x\,\mathrm{d}\tilde{x} \\
&\quad \le C\varepsilon^{(d-2)(1/\chi-1)+2\zeta}\int\int g(x)g(\tilde{x})|x-\tilde{x}|^{-\frac{d-2}{\chi}}\,\mathrm{d}x\,\mathrm{d}\tilde{x}
\end{aligned}
$$

by Proposition 7.7. The integral in the last line is finite because g is smooth and compactly supported. Now by taking χ sufficiently close to 1, and reducing ζ slightly, we achieve (1.23). □

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