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On the Hikami-Inoue conjecture

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Abstract

Given a braid presentation σ of a hyperbolic knot K, Hikami and Inoue consider a system of polynomial equations arising from a sequence of cluster mutations determined by σ. They show that any solution gives rise to shape variables and thus determines a boundary-parabolic $\mathrm{PSL}(2,\mathbb{C})$-representation of $\pi_1(S^3\setminus K)$. They conjecture the existence of a solution corresponding to the geometric representation. We show that a boundary-parabolic representation ρ arises from a solution if and only if the length of σ modulo 2 equals the obstruction to lifting ρ to a boundary-parabolic $\mathrm{SL}(2,\mathbb{C})$-representation (an element in $\mathbb{Z}_2$). In particular, the Hikami-Inoue conjecture holds if and only if the length of σ is odd. This can always be achieved by adding a kink to the braid if necessary. We also explicitly construct the solution corresponding to a boundary-parabolic representation given in the Wirtinger presentation of $\pi_1(S^3\setminus K)$.

Keywords: Hikami-Inoue conjecture, Ptolemy variety, braid presentation, hyperbolic knot, boundary-parabolic representation, cluster coordinates.
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1. Introduction

Let D be a braid of length n and width m. Hikami and Inoue [6] introduced $n+1$ cluster variables $\mathbf{x}^i=(x_1^i,\cdots,x_{3m+1}^i)$, $1\le i\le n+1$, where two consecutive variables $\mathbf{x}^i$ and $\mathbf{x}^{i+1}$ are related as follows. If D has a braid group presentation $\sigma_{k_1}^{\epsilon_1}\sigma_{k_2}^{\epsilon_2}\cdots\sigma_{k_n}^{\epsilon_n}$ (here σ_{k_i} is the standard generator of the m-braid group and $\epsilon_i=\pm1$), then we have

$$\mathbf{x}^2=R_{k_1}^{\epsilon_1}(\mathbf{x}^1),\ \mathbf{x}^3=R_{k_2}^{\epsilon_2}(\mathbf{x}^2),\cdots,\ \mathbf{x}^{n+1}=R_{k_n}^{\epsilon_n}(\mathbf{x}^n)$$

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where $R_k^{\pm}$ is the operator defined by

$$R_k^{\pm}(x_1, \cdots, x_{3m+1}) = (x_1, \cdots, x_{3k-3}, R^{\pm}(x_{3k-2}, \cdots, x_{3k+4}), x_{3k+5}, \cdots, x_{3m+1}).$$

We refer to the equations (10) and (11) for the definition of the operator $R^{\pm}$.

The initial cluster variable $\mathbf{x}^1 \in \mathbb{C}^{3m+1}$ is called a *solution* if $\mathbf{x}^1 = \mathbf{x}^{n+1}$. Hikami and Inoue showed that a solution $\mathbf{x}^1$ induces a boundary-parabolic representation $\rho_{\mathbf{x}^1} : \pi_1(S^3 \setminus K) \to \mathrm{PSL}(2, \mathbb{C})$, where K is the knot represented by the braid D. Assuming the following conjecture, they used these variables $\mathbf{x}^i$ to compute the volume and Chern-Simons invariant of $S^3 \setminus K$.

Conjecture 1.1. [6, Conjecture 3.2] Let D be a braid representing a hyperbolic knot K. Then there exists a solution $\mathbf{x}^1$ such that the induced representation $\rho_{\mathbf{x}^1} : \pi_1(S^3 \setminus K) \to \mathrm{PSL}(2, \mathbb{C})$ is geometric, i.e., discrete and faithful.

The main purpose of our paper is to analyze this conjecture. In particular we prove the following, which is a consequence of the more general results Theorems 1.4 and 1.5 below.

Theorem 1.2. Conjecture 1.1 holds if and only if the length of the braid is odd.

Note that one can always make the braid length odd by adding a kink if necessary. Let M be a compact 3-manifold with boundary.

Definition 1.3. A representation $\rho\colon \pi_1(M) \to G$, where G is either $\mathrm{PSL}(2, \mathbb{C})$ or $\mathrm{SL}(2, \mathbb{C})$, is *boundary-parabolic* if it maps peripheral subgroups to conjugates of the subgroup P of G consisting of upper triangular matrices with 1 on the diagonal. We shall sometimes call such a (G, P)-representation.

A representation $\pi_1(M) \to \mathrm{PSL}(2, \mathbb{C})$ may or may not lift to $\mathrm{SL}(2, \mathbb{C})$ and the obstruction to lifting is a class in $H^2(M; \{\pm 1\})$. Also, a boundary-parabolic $\mathrm{PSL}(2, \mathbb{C})$-representation may lift to an $\mathrm{SL}(2, \mathbb{C})$-representation which is not boundary-parabolic. The obstruction to lifting a boundary-parabolic $\mathrm{PSL}(2, \mathbb{C})$-representation ρ to a boundary-parabolic $\mathrm{SL}(2, \mathbb{C})$-representation is a class in $H^2(M, \partial M, \{\pm 1\})$ called the *obstruction class of* ρ [4, 3]. The image of this class in $H^2(M; \{\pm 1\})$ is the obstruction to lifting ρ to $\mathrm{SL}(2, \mathbb{C})$. If $M = S^3 \setminus \nu(K)$ is a knot exterior of K in S^3 (here $\nu(K)$ denotes a small open regular neighborhood of K), then $H^2(M, \partial M; \{\pm 1\}) \simeq \{\pm 1\}$ so the obstruction class of ρ can be viewed as an element of $\{\pm 1\}$.

Theorem 1.4. [Theorem 3.1] Let D be a braid of length n representing a knot K (not necessarily hyperbolic). Then the obstruction class of $\rho_{\mathbf{x}^1} : \pi_1(S^3 \setminus \nu(K)) \to \mathrm{PSL}(2, \mathbb{C})$ induced from a solution $\mathbf{x}^1$ is given by $(-1)^n$.

The obstruction class of the geometric representation of a hyperbolic knot is non-trivial. This follows from the fact that any lift of the geometric representation maps a longitude to an element with trace -2 (see e.g. [1], [9, §3.2] and also Proposition 2.2 below). Hence, Theorem 1.4 shows that having odd braid length is necessary for Conjecture 1.1 to hold. The fact that this is also sufficient follows from the result below, which is proved in Section 4.

Theorem 1.5. Let D be a braid of length n representing a knot K (not necessarily hyperbolic). Let $\rho : \pi_1(S^3 \setminus \nu(K)) \to \mathrm{PSL}(2, \mathbb{C})$ be a boundary-parabolic representation whose obstruction class is $(-1)^n$. Then there exists a solution $\mathbf{x}^1$ such that the induced representation $\rho_{\mathbf{x}^1}$ coincides with ρ up to conjugation.

We stress that the solution can be constructed explicitly when ρ is given using the Wirtinger presentation of π_1. This uses techniques developed in [2] and [13].

The paper is organized as follows. In Section 2, we give a review on Ptolemy varieties with obstruction classes. In Section 3, we give a short review on the Hikami-Inoue variables and prove Theorem 1.4. In Section 4, we present an explicit way to compute a solution described in Theorem 1.5 when a boundary-parabolic representation is given in the Wirtinger presentation of π_1.

1.1. Acknowledgements

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2. Ptolemy varieties with obstruction class

Let M be an oriented compact 3-manifold with non-empty boundary. We fix an ideal triangulation $\mathcal{T}$ of the interior of M. This endows M with a decomposition into truncated tetrahedra whose triangular faces triangulate ∂M (see Figure 1). We denote by M^i or ∂M^i the set of the oriented i-cells (unoriented when $i = 0$). For an oriented 1-cell e, we let $-e$ denote e with its opposite orientation.

2.1. Obstruction classes

For a group G the set $C^i(M; G)$ of all set maps from M^i to G forms a group with the operation naturally induced from G. We call $\sigma \in C^1(M; G)$ a *G-cocycle* if it satisfies

(i) $\sigma(e)\sigma(-e) = 1$ for all $e \in M^1$;
(ii) $\sigma(e_1)\sigma(e_2)\cdots\sigma(e_m) = 1$ for each face f of M where $e_1, \cdots, e_m$ are the boundary edges of the face in the cyclic order determined by a choice of orientation of f.

The set $Z^1(M; G)$ of all G-cocycles admits a $C^0(M; G)$-action defined as follows.

$$Z^1(M; G) \times C^0(M; G) \to Z^1(M; G), \quad (\sigma, \tau) \mapsto \sigma \cdot \tau$$

where $\sigma \cdot \tau : M^1 \to G$ is given by $(\sigma \cdot \tau)(e) = \tau(v)^{-1}\sigma(e)\tau(w)$ for $e \in M^1$, where v and w are the initial and terminal vertices of e, respectively. The following fact is well-known (see e.g. [14, 10]).

Proposition 2.1. The orbit space $H^1(M; G) := Z^1(M; G)/C^0(M; G)$ has a natural bijection with the set of all conjugacy classes of representations $\rho : \pi_1(M) \to G$.

Let G be either $\mathrm{SL}(2, \mathbb{C})$ or $\mathrm{PSL}(2, \mathbb{C})$ and P be the subgroup of G consisting of the upper triangular matrices with ones in the diagonal. We denote by $C^i(M, \partial M; G, P)$ the subset of $C^i(M; G)$ consisting of elements $\sigma \in C^i(M; G)$ satisfying $\sigma(x) \in P$ for all $x \in \partial M^i$. We let $Z^1(M, \partial M; G, P) = Z^1(M; G) \cap C^1(M, \partial M; G, P)$ and $H^1(M, \partial M; G, P) =$

$Z^1(M,\partial M;G,P)/C^0(M,\partial M;G,P)$. An element of $Z^1(M,\partial M;G,P)$ is called a (G,P)-*cocycle*. One easily checks (see e.g. [14]) that every (G,P)-representation can be represented by a (G,P)-cocycle. In fact, $H^1(M,\partial M;G,P)$ is in natural bijection with the set of (conjugacy classes of) so-called *decorated* (G,P)*-representations* (see e.g. [14, 4]), but we shall not need this here.

From the short exact sequence of groups $1 \to \{\pm 1\} \to \mathrm{SL}(2,\mathbb{C}) \to \mathrm{PSL}(2,\mathbb{C}) \to 1$, we obtain exact sequences (the standard proof of exactness still works in low degree even though the terms are only sets, not groups)

$$H^1(M;\mathrm{SL}(2,\mathbb{C})) \to H^1(M;\mathrm{PSL}(2,\mathbb{C})) \to H^2(M;\{\pm 1\}) \quad \text{and} \tag{1}$$

$$H^1(M,\partial M;\mathrm{SL}(2,\mathbb{C}),P) \to H^1(M,\partial M;\mathrm{PSL}(2,\mathbb{C}),P) \xrightarrow{\delta} H^2(M,\partial M;\{\pm 1\}) \tag{2}$$

with the connecting map δ. In particular, the sequence (2) tells us that a $(\mathrm{PSL}(2,\mathbb{C}),P)$-representation ρ admits a $(\mathrm{SL}(2,\mathbb{C}),P)$-lifting if and only if $\delta(\rho) \in H^2(M,\partial M;\{\pm 1\})$ vanishes, where ρ is viewed as a $(\mathrm{PSL}(2,\mathbb{C}),P)$-cocycle. The element $\delta(\rho)$ is called the *obstruction class* of ρ. It does not depend on the choice of a $(\mathrm{PSL}(2,\mathbb{C}),P)$-cocycle representing ρ. Recall that we have the long exact sequence

$$H^1(M;\{\pm 1\}) \to H^1(\partial M;\{\pm 1\}) \to H^2(M,\partial M;\{\pm 1\}) \to H^2(M;\{\pm 1\}).$$

It thus follows that if ρ lifts to $\mathrm{SL}(2,\mathbb{C})$ (e.g. if $H^2(M;\{\pm 1\}) = 0$), then the obstruction class of ρ in $H^2(M,\partial M;\{\pm 1\})$ can be viewed as an element of $\mathrm{Coker}(H^1(M;\{\pm 1\}) \to H^1(\partial M;\{\pm 1\}))$. In particular, if M is a knot exterior in the 3-sphere S^3, the obstruction class of ρ is determined by the lift of the longitude. More precisely, the following holds.

Proposition 2.2. Let M be a knot exterior in S^3 and let $\rho : \pi_1(M) \to \mathrm{PSL}(2,\mathbb{C})$ be a $(\mathrm{PSL}(2,\mathbb{C}),P)$-representation. The obstruction class of ρ, viewed as an element of $\{\pm 1\}$, coincides with half of $\mathrm{Tr}(\widetilde{\rho}(\lambda))$ where $\widetilde{\rho} : \pi_1(M) \to \mathrm{SL}(2,\mathbb{C})$ is a $\mathrm{SL}(2,\mathbb{C})$-lifting of ρ and λ is a canonical longitude of the knot.

Proof. Considering any Wirtinger presenation of $\pi_1(M)$, it is easy to check that ρ has only two $\mathrm{SL}(2,\mathbb{C})$-liftings $\widetilde{\rho}_+$ and $\widetilde{\rho}_- : \pi_1(M) \to \mathrm{SL}(2,\mathbb{C})$ such that $\mathrm{Tr}(\widetilde{\rho}_+(\mu)) = 2$ and $\mathrm{Tr}(\widetilde{\rho}_-(\mu)) = -2$, respectively. Here μ is a merdian of the knot. Since $\pi_1(\partial M)$ is an abelian group generated by μ and the longitude λ, ρ admits a $(\mathrm{SL}(2,\mathbb{C}),P)$-lifting if and only if $\mathrm{Tr}(\widetilde{\rho}_+(\lambda)) = 2$. Therefore, by definition, the obstruction class $\delta(\rho) \in \{\pm 1\}$ coincides with half of $\mathrm{Tr}(\widetilde{\rho}_+(\lambda))$. On the other hand, the canonical longitude λ is contained in the commutator subgroup of $\pi_1(M)$. Thus it should be expressed in Wirtinger generators of even length and we have $\widetilde{\rho}_+(\lambda) = \widetilde{\rho}_-(\lambda)$. □

2.2. Ptolemy varieties

Recall that M is a compact 3-manifold with an ideal triangulation $\mathcal{T}$ of its interior. The third author with Garoufalidis and Thurston [4] (see also [14]) gave an efficient parametrization of $(\mathrm{PSL}(2,\mathbb{C}),P)$-representations with a given obstruction class. Precisely, for $\sigma \in Z^2(M,\partial M;\{\pm 1\})$ the *Ptolemy variety* $P^\sigma(\mathcal{T})$ *with obstruction cocycle* σ is defined by the set of all set maps $c : \mathcal{T}^1 \to \mathbb{C} \setminus \{0\}$ satisfying $-c(e) = c(-e)$ for all $e \in \mathcal{T}^1$ and

$$\sigma_2\, c(l_{02})c(l_{13}) = \sigma_3\, c(l_{03})c(l_{12}) + \sigma_1\, c(l_{01})c(l_{23}) \tag{3}$$

for each ideal tetrahedron Δ (with vertices $\{0,1,2,3\}$) of $\mathcal{T}$, where l_{ij} is the oriented edge of Δ going from vertex i to vertex j, and σ_i is the σ-value on the hexagonal face opposite to the vertex i. See Figure 1. Here $\mathcal{T}^1$ denotes the set of oriented 1-cells of $\mathcal{T}$.

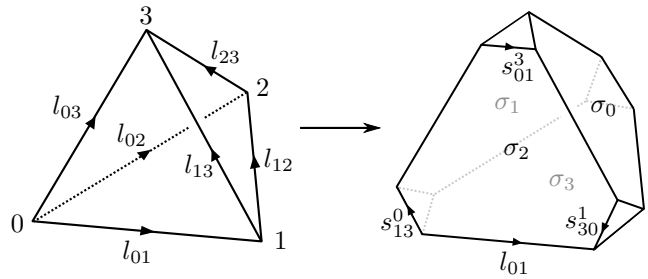


Figure 1: A truncated tetrahedron.

Each point c of $P^\sigma(\mathcal{T})$ corresponds to a $(\mathrm{PSL}(2,\mathbb{C}),P)$-cocycle Φ_c such that $\delta(\Phi_c) = [\sigma] \in H^2(M,\partial M;\{\pm 1\})$; hence it induces a $(\mathrm{PSL}(2,\mathbb{C}),P)$-representation $\rho_c : \pi_1(M) \to \mathrm{PSL}(2,\mathbb{C})$ whose obstruction class is $[\sigma]$. This cocycle is given explicitly from the Ptolemy coordinates $c(l_{ij})$ as follows:

$$\Phi_c(l_{ij}) = \pm \begin{pmatrix} 0 & -c(l_{ij})^{-1} \\ c(l_{ij}) & 0 \end{pmatrix}, \quad \Phi_c(s^k_{ij}) = \pm \begin{pmatrix} 1 & -\epsilon_l \frac{c(l_{ji})}{c(l_{ik})c(l_{kj})} \\ 0 & 1 \end{pmatrix}.$$

Here $s^k_{ij} \in \partial M^1$ is the edge contained in the face $[i,j,k]$ and parallel to l_{ij}; see Figure 1. The cocycle condition (ii) is then satisfied for the hexagonal faces, and the Ptolemy relation (3) ensures that it is also satisfied for the triangular faces. We refer [4, §9] for details.

Now let us consider the map

$$d : Z^1(\partial M;\{\pm 1\}) \to Z^2(M,\partial M;\{\pm 1\}),\ \epsilon \mapsto d(\epsilon) \tag{4}$$

defined by $d(\epsilon)$-value on a face of M by multiplying ϵ-values of all edges of ∂M that are contained in the face. Note that it induces the usual map $d_* : H^1(\partial M;\{\pm 1\}) \to H^2(M,\partial M;\{\pm 1\})$.

Proposition 2.3. Let $\epsilon \in Z^1(\partial M;\{\pm 1\})$. Then any $(\mathrm{PSL}(2,\mathbb{C}),P)$-representation ρ_c induced from $c \in P^{d(\epsilon)}(\mathcal{T})$ has a $\mathrm{SL}(2,\mathbb{C})$-lifting $\widetilde{\rho}_c : \pi_1(M) \to \mathrm{SL}(2,\mathbb{C})$ satisfying

$$\widetilde{\rho}_c(\gamma) = \begin{pmatrix} \overline{\epsilon}(\gamma) & * \\ 0 & \overline{\epsilon}(\gamma) \end{pmatrix} \tag{5}$$

for all $\gamma \in \pi_1(\partial M)$ up to conjugation, where $\overline{\epsilon} : \pi_1(\partial M) \to \{\pm 1\}$ is the homomorphism induced from the cocycle ϵ.

Proof. Let $\widetilde{\Phi}_c \in C^1(M;\mathrm{SL}(2,\mathbb{C}))$ be a lift of Φ_c satisfying that

$$\widetilde{\Phi}_c(l) = \begin{pmatrix} 0 & -c(l)^{-1} \\ c(l) & 0 \end{pmatrix}, \quad \widetilde{\Phi}_c(s) = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \tag{6}$$

for all $l \in M^1 \setminus \partial M^1$ and $s \in \partial M^1$. One can check that $\widetilde{\Phi}_c$ satisfies the $\mathrm{SL}(2,\mathbb{C})$-cocycle condition for the triangular faces (but may not for all faces of M). Let $\widetilde{\epsilon} \in C^1(M;\{\pm 1\})$ be the trivial extension of ϵ, i.e., $\widetilde{\epsilon}(e) := \epsilon(e)$ if $e \in \partial M^1$ and otherwise $\widetilde{\epsilon}(e) := 1$. Then by definition $\widetilde{\epsilon} \cdot \widetilde{\Phi}_c : M^1 \to \mathrm{SL}(2,\mathbb{C})$ is a $\mathrm{SL}(2,\mathbb{C})$-cocycle which satisfies

$$(\widetilde{\epsilon} \cdot \widetilde{\Phi}_c)(e) = \begin{pmatrix} \epsilon(e) & * \\ 0 & \epsilon(e) \end{pmatrix}$$

for all $e \in \partial M$. Letting $\widetilde{\rho}_c$ be a representation induced from the $\mathrm{SL}(2,\mathbb{C})$-cocycle $\widetilde{\epsilon} \cdot \widetilde{\Phi}_c$, the proposition follows. □

Combining Propositions 2.2 and 2.3, we obtain:

Theorem 2.4. Let M be a knot exterior in S^3 and let $\epsilon \in Z^1(\partial M;\{\pm 1\})$. Then any $(\mathrm{PSL}(2,\mathbb{C}), P)$-representation ρ_c induced from $c \in P^{d(\epsilon)}(\mathcal{T})$ has the obstruction class $\overline{\epsilon}(\lambda) \in \{\pm 1\}$, where $\overline{\epsilon} : \pi_1(\partial M) \to \{\pm 1\}$ is the induced homomorphism and λ is the canonical longitude of the knot.

3. The Hikami-Inoue cluster variables

3.1. The octahedral decomposition of a knot complement with two points removed

Let $K \subset S^3$ be a knot and let $\nu(K \cup \{p,q\})$ denote a tubular neigborhood of the union of K with two points $p \neq q \in S^3$ not in K. Whenever we choose a knot diagram representing K, we have a decomposition of the space $M = S^3 \setminus \nu(K \cup \{p,q\})$ into blocks each of which is a cube with two cylinders (whose core is the knot) removed. See Figure 2. Note that M is a 3-manifold with 3 boundary components (two spheres and a torus) whose interior is homeomorphic to $S^3 \setminus (K \cup \{p,q\})$. Now consider two quadrilaterals Q_1 and Q_2 in each block as in Figure 2 and collapse them horizontally so that their vertical edges are respectively identified. We call the resulting object a *pinched block*.

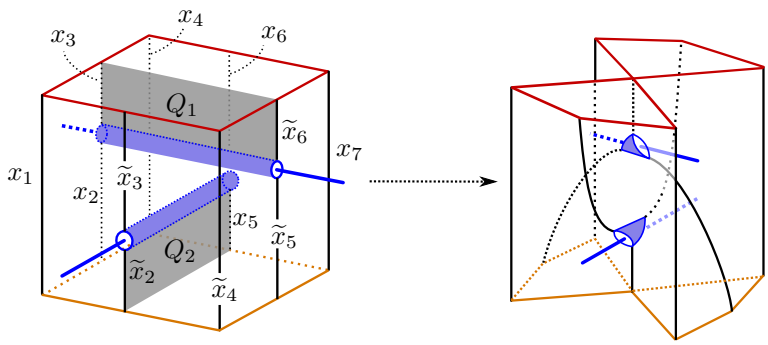


Figure 2: A pinched block

On the other hand, a pinched block can also be obtained from a truncated octahedron by identifying two pairs of edges as in Figure 3 (right). Therefore, one can obtain M by gluing truncated octahedra, and it thus follows that the interior of M can be decomposed into ideal octahedra (one per crossing). We denote this octahedral decomposition of $S^3 \setminus (K \cup \{p,q\})$ by $\mathcal{O}$. It is due to Dylan Thuston [11] (see also [12]).

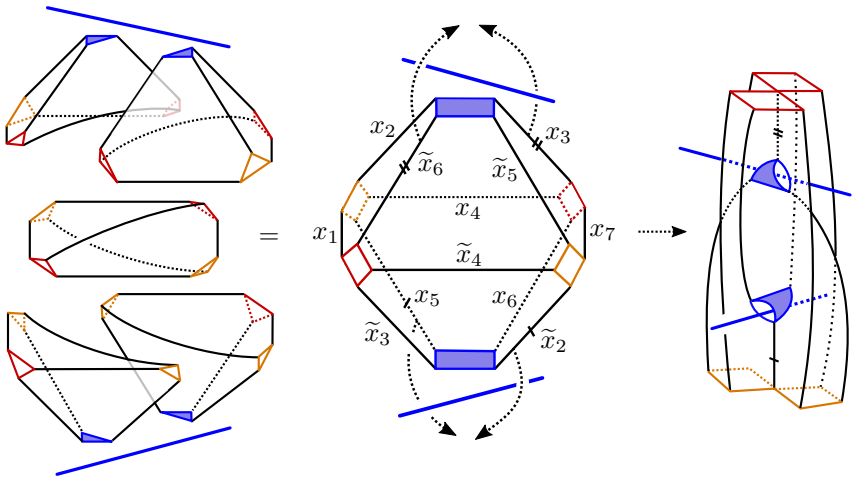


Figure 3: A truncated octahedron

3.2. The Hikami-Inoue cluster variables

An ideal octahedron as in Figure 3 has 12 edges each of which corresponds to a vertical edge of a cube in Figure 2. We may label those edges by $x_1, \cdots, x_7, \widetilde{x}_1, \cdots, \widetilde{x}_7$ as in Figure 4 with the obvious identifications $x_1 = \widetilde{x}_1$ and $x_7 = \widetilde{x}_7$. As indicated in Figure 4 (left) we shall regard the edges x_i as being above a crossing, and the edges $\widetilde{x}_i$ as below the crossing.

Assigning a complex-valued variable to each of the edges $x_1, \cdots, x_7$, $\widetilde{x}_1, \cdots, \widetilde{x}_7$ with the same label as the edge itself, Hikami and Inoue [5, §2.2] consider the equation $(\widetilde{x}_1, \cdots, \widetilde{x}_7) = R^{\pm}(x_1, \cdots, x_7)$ where $R^{\pm}$ is a certain operator defined by rational polynomial equations. As we shall see in Section 3.3, these equations are equivalent to Ptolemy relations for a particular obstruction cocycle.

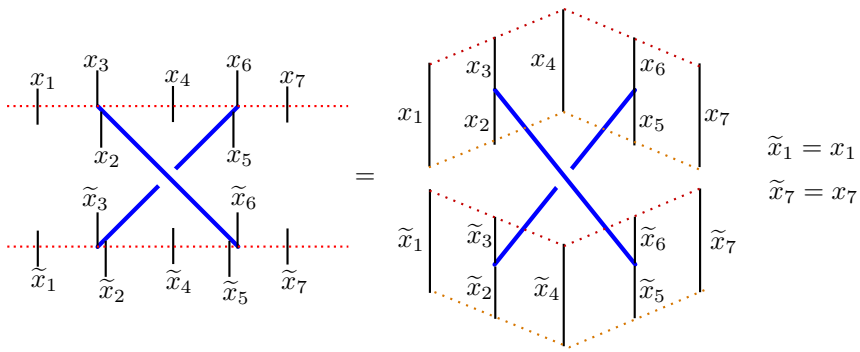


Figure 4: Edges of an octahedron at a crossing

Now suppose the knot diagram is given by a braid D with presentation $\sigma_{k_1}^{\epsilon_1} \cdots \sigma_{k_n}^{\epsilon_n}$. (Here σ_{k_i} is the standard generator of the m-braid group and $\epsilon_i = \pm 1$.) Similar to the

edge-labeling described in the previous paragraph, we label the oriented edges of the octahedral decomposition $\mathcal{O}$ as follows:

1. Draw $n+1$ imaginary horizontal lines on the braid D so that there is only a single crossing between two consecutive lines (see Figures 5 and 10).
2. As in Figure 4 (left), whenever a horizontal line meets the braid D there are two corresponding edges, and whenever a horizontal line meets a region of (the closure of) D, there is one corresponding edge. Since each of the horizontal lines meets the braid m times and the regions $m+1$ times, it corresponds to $3m+1$ edges of $\mathcal{O}$.
3. For the i-th horizontal line we orient the corresponding edges and denote them by $x_1^i, \cdots, x_{3m+1}^i$ as in Figure 5, and define $\mathbf{x}^i = (x_1^i, \cdots, x_{3m+1}^i)$.

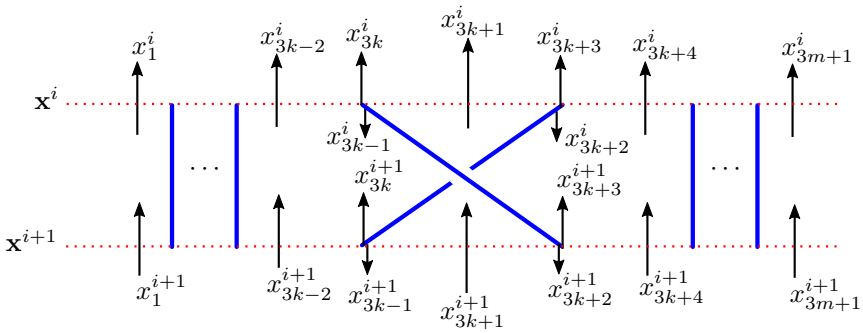


Figure 5: Edges of $\mathcal{O}$ around the i-th level of a braid

Note that there are many overlapped labelings; for instance, in Figure 5, we have $x_j^i = x_j^{i+1}$ for $j = 1, \cdots, 3k-2$ and $j = 3k+4, \cdots, 3m+1$.

We again assign a complex-valued variable to each oriented edge of $\mathcal{O}$ and denote the variable by the same as the edge itself. Hikami and Inoue [5] relate the variables $\mathbf{x}^i = (x_1^i, \cdots, x_{3m+1}^i)$ and $\mathbf{x}^{i+1} = (x_1^{i+1}, \cdots, x_{3m+1}^{i+1})$ by the equation

$$\mathbf{x}^{i+1} = R_{k_i}^{\epsilon_i}(\mathbf{x}^i)$$

for $1 \leq i \leq n$. Recall that the operator $R_k^{\pm}$ is defined by

$$R_k^{\pm}(x_1, \cdots, x_{3m+1}) = (x_1, \cdots, x_{3k-3}, R^{\pm}(x_{3k-2}, \cdots, x_{3k+4}), x_{3k+5}, \cdots, x_{3m+1}). \quad (7)$$

Note that $R_k^{\pm}$ only affects the variables above and below the k-th crossing.

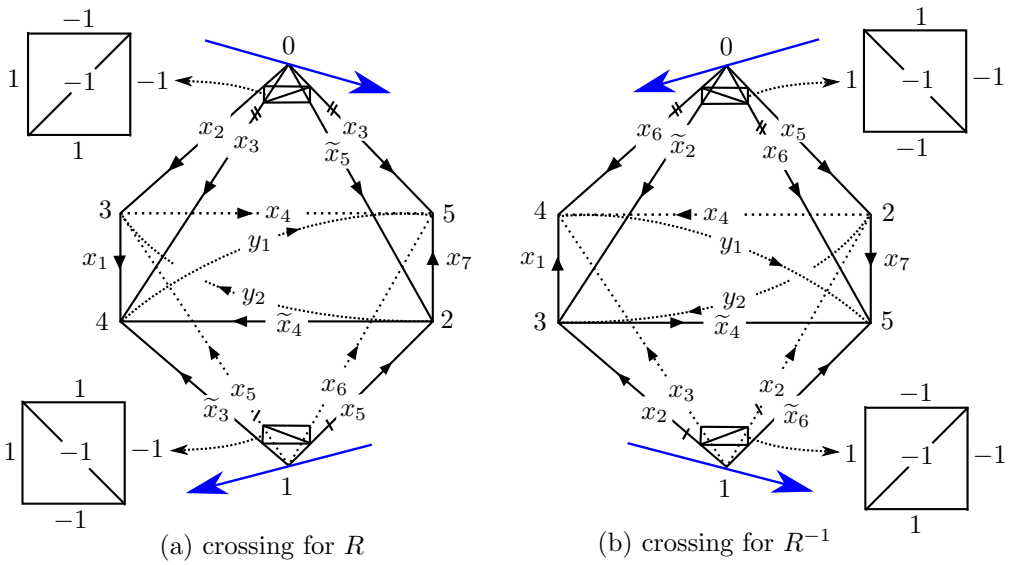
An initial variable $\mathbf{x}^1$ is called a solution if $\mathbf{x}^1 = \mathbf{x}^{n+1}$. Whenever we have a solution $\mathbf{x}^1 \in \mathbb{C}^{3m+1}$, we shall define the set map

$$c_{\mathbf{x}^1} : \mathcal{O}^1 \to \mathbb{C}$$

by assigning the variable x_j^i to the oriented edge of $\mathcal{O}$ labeled by the same name. The fact that this assignment respects the face identifications in $\mathcal{O}$ follows directly from the definitions of $R_k^{\pm}$ and $R^{\pm}$.

3.3. The obstruction cocycle

Let $\mathcal{T}$ be the ideal triangulation of $S^3 \setminus (K \cup \{p, q\})$ obtained by decomposing each octahedron of $\mathcal{O}$ into 5 ideal tetrahedra as in Figure 3 (left). As explained earlier this induces a triangulation of the boundary of M. We now define a cocycle $\epsilon \in Z^1(\partial M; \{\pm 1\})$ on ∂M by assigning signs to the short edges of the truncated tetrahedra. Note that each short edge either lies in the top/bottom of a truncated octahedron, or on one of the sides. We shall call the edges *top/bottom-edges* or *side-edges* accordingly. We assign signs to the top/bottom edges as indicated in Figure 6 and assign $+1$ do all of the side edges. This is clearly a cocycle, which respects the face pairings and thus gives rise to a cocycle in $\epsilon \in Z^1(\partial M; \{\pm 1\})$ as desired. We stress that ϵ depends on the decomposition of M, in particular the choice of a braid D representing K.



(a) crossing for R (b) crossing for R^{-1}

Figure 6: An ideal octahedron at a crossing

The cocycle ϵ is illustrated in 7, where μ and λ_{bf} denote the meridian and black-board framed longitude of the knot K, respectively. In particular, ϵ induces the homomorphism $\overline{\epsilon}$ that maps μ to -1 and λ_{bf} to 1.

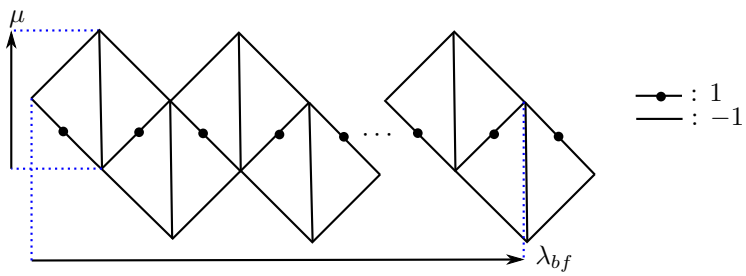


Figure 7: Configuration of ϵ on the boundary torus

3.4. Proof of Theorem 1.4

Let us consider an octahedron of $\mathcal{O}$. We index the vertices by $\{0, \cdots, 5\}$ and denote the oriented edges as in Figure 6. Let us compute the equation (3) for each of the ideal tetrahedra with obstruction cocycle $\sigma := d(\epsilon) \in Z^2(M, \partial M; \{\pm 1\})$, where d is the map in (4). For example, the tetrahedron with vertices $\{0, 3, 4, 5\}$ in Figure 6(a) gives

$$\sigma_4 c(l_{04})c(l_{35}) = \sigma_5\, c(l_{05})c(l_{34}) + \sigma_3 c(l_{03})c(l_{45}) \Leftrightarrow (-1)x_3x_4 = x_3x_1 + (-1)x_2y_1,$$

which is equivalent to $x_2y_1 = x_3x_4 + x_1x_3$. Similar computations give:

$$\begin{array}{lrcl}
\{0,3,4,5\}: & x_2y_1 & = & x_3x_4 + x_1x_3 \\
\{1,2,3,5\}: & x_6y_2 & = & x_5x_7 + x_4x_5 \\
\{2,3,4,5\}: & x_4\widetilde{x}_4 & = & x_1x_7 + y_1y_2 \\
\{0,2,4,5\}: & \widetilde{x}_5y_1 & = & x_3\widetilde{x}_4 + x_3x_7 \\
\{1,2,3,4\}: & \widetilde{x}_3y_2 & = & x_5\widetilde{x}_4 + x_1x_5
\end{array}$$

for Figure 6(a) and

$$\begin{array}{lrcl}
\{0,2,4,5\}: & y_1x_5 & = & x_4x_6 + x_6x_7 \\
\{1,2,3,4\}: & x_3y_2 & = & x_1x_2 + x_2x_4 \\
\{2,3,4,5\}: & x_4\widetilde{x}_4 & = & y_1y_2 + x_1x_7 \\
\{0,3,4,5\}: & \widetilde{x}_2y_1 & = & x_6\widetilde{x}_4 + x_1x_6 \\
\{1,2,3,5\}: & \widetilde{x}_6y_2 & = & x_2x_7 + x_2\widetilde{x}_4
\end{array}$$

for Figure 6(b). Considering $x_1, \cdots, x_7$ as given variables, we have

$$(y_1, y_2) = \left(\frac{x_3(x_1 + x_4)}{x_2}, \ \frac{x_5(x_4 + x_7)}{x_6} \right) \tag{8}$$

$$(\widetilde{x}_3, \widetilde{x}_4, \widetilde{x}_5) = \begin{pmatrix} \dfrac{x_1x_3x_5 + x_3x_4x_5 + x_1x_2x_6}{x_2x_4} \\ \dfrac{x_1x_3x_4x_5 + x_3x_4^2x_5 + x_1x_3x_5x_7 + x_3x_4x_5x_7 + x_1x_2x_6x_7}{x_2x_4x_6} \\ \dfrac{x_3x_4x_5 + x_3x_5x_7 + x_2x_6x_7}{x_4x_6} \end{pmatrix}^T$$

for Figure 6(a) and

$$(y_1, y_2) = \left(\frac{x_6(x_4 + x_7)}{x_5}, \ \frac{x_2(x_1 + x_4)}{x_3} \right) \tag{9}$$

$$(\widetilde{x}_2, \widetilde{x}_4, \widetilde{x}_6) = \begin{pmatrix} \dfrac{x_1x_3x_5 + x_1x_2x_6 + x_2x_4x_6}{x_3x_4} \\ \dfrac{x_1x_2x_4x_6 + x_2x_4^2x_6 + x_1x_3x_5x_7 + x_1x_2x_6x_7 + x_2x_4x_6x_7}{x_3x_4x_5} \\ \dfrac{x_2x_4x_6 + x_3x_5x_7 + x_2x_6x_7}{x_4x_5} \end{pmatrix}^T$$

for Figure 6(b).

Letting $\widetilde{x}_1 = x_1,\ \widetilde{x}_2 = x_5,\ \widetilde{x}_6 = x_3,\ \widetilde{x}_7 = x_7$ for Figure 6(a) and $\widetilde{x}_1 = x_1,\ \widetilde{x}_3 =$

$x_6,\ \widetilde{x}_5 = x_2,\ \widetilde{x}_7 = x_7$ for Figure 6(b), we obtain

$$\begin{pmatrix} \widetilde{x}_1 \\ \widetilde{x}_2 \\ \widetilde{x}_3 \\ \widetilde{x}_4 \\ \widetilde{x}_5 \\ \widetilde{x}_6 \\ \widetilde{x}_7 \end{pmatrix}^T = \begin{pmatrix} x_1 \\ x_5 \\ \dfrac{x_1x_3x_5 + x_3x_4x_5 + x_1x_2x_6}{x_2x_4} \\ \dfrac{x_1x_3x_4x_5 + x_3x_4^2x_5 + x_1x_3x_5x_7 + x_3x_4x_5x_7 + x_1x_2x_6x_7}{x_2x_4x_6} \\ \dfrac{x_3x_4x_5 + x_3x_5x_7 + x_2x_6x_7}{x_4x_6} \\ x_3 \\ x_7 \end{pmatrix}^T = R \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix}^T \tag{10}$$

for Figure 6(a) and

$$\begin{pmatrix} \widetilde{x}_1 \\ \widetilde{x}_2 \\ \widetilde{x}_3 \\ \widetilde{x}_4 \\ \widetilde{x}_5 \\ \widetilde{x}_6 \\ \widetilde{x}_7 \end{pmatrix}^T = \begin{pmatrix} x_1 \\ \dfrac{x_1x_3x_5 + x_1x_2x_6 + x_2x_4x_6}{x_3x_4} \\ x_6 \\ \dfrac{x_1x_2x_4x_6 + x_2x_4^2x_6 + x_1x_3x_5x_7 + x_1x_2x_6x_7 + x_2x_4x_6x_7}{x_3x_4x_5} \\ x_2 \\ \dfrac{x_3x_5x_7 + x_2x_4x_6 + x_2x_6x_7}{x_4x_5} \\ x_7 \end{pmatrix}^T = R^{-1} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix}^T \tag{11}$$

for Figure 6(b). The equations (10) and (11) exactly coincide with the definition of the operators $R^{\pm}$ in [6]. See [6, Equation (2.17)].

Now let D be a braid of length n and width m. Let $\mathbf{x}^1$ be a solution and $c_{\mathbf{x}^1} : \mathcal{O}^1 \to \mathbb{C}$ be the induced set map. Recall that $\mathcal{T}$ has two additional edges per crossing compared to $\mathcal{O}$. We extend the set map to $c_{\mathbf{x}^1} : \mathcal{T}^1 \to \mathbb{C}$ by defining the values on the added edges using the equations (8) and (9).

We say that a solution $\mathbf{x}^1 \in \mathbb{C}^{3m+1}$ is *non-trivial* if $c_{\mathbf{x}^1}(e) \neq 0$ for all $e \in \mathcal{T}^1$. The previous computation in this section tells us that the set map $c_{\mathbf{x}^1}$ induced from a non-trivial solution $\mathbf{x}^1$ is a point of the Ptolemy variety $P^\sigma(\mathcal{T})$ with obstruction cocycle $\sigma(= d(\epsilon)) \in Z^2(M, \partial M; \{\pm 1\})$. Therefore, a non-trivial solution $\mathbf{x}^1$ induces a $(\mathrm{PSL}(2,\mathbb{C}), P)$-representation $\rho_{\mathbf{x}^1}$ whose obstruction class is $\sigma \in H^2(M, \partial M; \{\pm 1\})$.

Theorem 3.1. Let D be a braid of length n and width m representing a knot. Let $\mathbf{x}^1 \in \mathbb{C}^{3m+1}$ be a non-trivial solution. Then the obstruction class of the induced representation $\rho_{\mathbf{x}^1}$ is $(-1)^n$.

Proof. By Theorem 2.4 it suffices to show that $\bar{\epsilon}(\lambda) = (-1)^n$ for the canonical longitude λ where $\bar{\epsilon}$ is the homomorphism on the boundary torus induced from the cocycle ϵ. Recall

Section 3.3 that we have $\overline{\epsilon}(\mu) = -1$ and $\overline{\epsilon}(\lambda_{bf}) = 1$ for the meridian μ and blackboard framed longitude λ_{bf}. We thus have

$$\begin{aligned}\overline{\epsilon}(\lambda) &= \overline{\epsilon}(\lambda_{bf})\ \overline{\epsilon}(\mu)^{-w(D)} \\ &= \overline{\epsilon}(\lambda_{bf})\ \overline{\epsilon}(\mu)^{-n} = (-1)^n.\end{aligned}$$

Here $w(D)$ denotes the writhe of the closure of D which is congruent to the length n in modulo 2. □

4. Explicit computation from a representation

Let D be a braid of length n representing a knot and M be the knot exterior. Let $\rho : \pi_1(M) \to \mathrm{PSL}(2,\mathbb{C})$ be a $(\mathrm{PSL}(2,\mathbb{C}), P)$-representation which has a $\mathrm{SL}(2,\mathbb{C})$-lifting $\widetilde{\rho} : \pi_1(M) \to \mathrm{SL}(2,\mathbb{C})$ such that

$$\widetilde{\rho}(\mu) = \begin{pmatrix} -1 & * \\ 0 & -1 \end{pmatrix} \neq -\mathrm{Id} \quad \text{and} \quad \widetilde{\rho}(\lambda) = \begin{pmatrix} (-1)^n & * \\ 0 & (-1)^n \end{pmatrix}$$

up to conjugation. As we stated in Theorem 1.5, there exists $c \in P^\sigma(\mathcal{T})$ such that the induced representation ρ_c coincides with ρ up to conjugation. In this section, we present an explicit way to compute such $c \in P^\sigma(\mathcal{T})$ using the notions of *arc-colorings* and *region-colorings* of a knot diagram (see e.g. [7, 8]).

Remark 4.1. The existence of such c follows from [4]. Namely, the Ptolemy variety parametrizes generically decorated representations, so it is enough to prove the existence of a generic decoration of ρ. Since the 5 simplex per crossing triangulation has the property that no edge joins a torus boundary component to itself, and since decorations of spherical boundary components can be chosen freely, a decoration with non-zero Ptolemy coordinates always exists. We refer to [4] for details on decorations. The above (non-constructive) existence argument was pointed out to the third author by Seonhwa Kim.

We index the regions of D by $1 \leq j \leq n+2$ and the arcs by $1 \leq i \leq n$. We then assign a 2-dimensional non-zero column vector, called a *region-coloring* V_j to the j-th region so that these vectors satisfy

$$V_{j_2} = \widetilde{\rho}(m_i)^{-1} V_{j_1}$$

for Figure 8 (left) where m_i is the Wirtinger generator corresponding to the i-th arc. The region-colorings are well-determined whenever an initial coloring is chosen arbitrarily.

We also assign a 2-dimensional non-zero column vector, called an *arc-coloring*, H_i to the i-th arc so that these vectors satisfy $\widetilde{\rho}(m_i)H_i = -H_i$ for $1 \leq i \leq m$ (recall that the eigenvalue of $\widetilde{\rho}(m_i)$ is -1) and

$$H_{i_3} = \widetilde{\rho}(m_{i_2})^{-1} H_{i_1} \tag{12}$$

for Figure 8 (right). We stress that the fact that the eigenvalue of $\widetilde{\rho}(\lambda_{bf})$ is 1 (equivalently, the eigenvalue of $\widetilde{\rho}(\lambda)$ is $(-1)^n$) is required to satisfy all the equations (12).

Recall that the octahedral decomposition $\mathcal{O}$ has $3n+2$ edges; (i) n of them, called *over-edges*, stand above the knot; (ii) other n of them, called *under-edges*, stand below

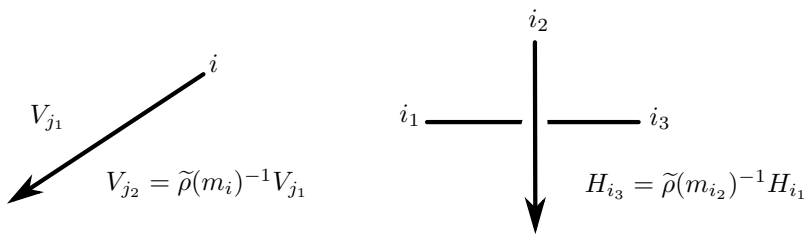


Figure 8: Rules for region- and arc-colorings.

the knot; (iii) last $n+2$ of them, called *regional edges*, stand on the regions. See Figure 9. We choose an additional non-zero column vector V_0 arbitrarily and define the set map $c : \mathcal{O}^1 \to \mathbb{C}$ as follows.

(i) $c(e) := \det(H_i, V_0)$ if e is the over-edge standing over the i-th arc;

(ii) $c(e) := \det(V_j, H_i)$ if e is the under-edge standing below the i-th arc whose left-side region is indexed by j;

(iii) $c(e) := \det(V_j, V_0)$ if e is the regional edge corresponding the j-th region.

Here we oriented the edge e as in Figure 9.

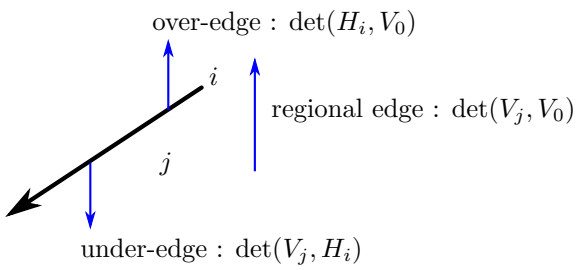


Figure 9: Edges of $\mathcal{O}$ with c-values.

We again extend the above set map to $c : \mathcal{T}^1 \to \mathbb{C}$ by using the equations (8) and (9). The argument of the proof of [2, Lemma 2.1] tells us that there exists V_0 so that $c(e) \neq 0$ for all $e \in \mathcal{T}^1$. For such V_0 we obtain $c \in P^\sigma(\mathcal{T})$ and the induced representation ρ_c coincides with ρ that we started with up to conjugation. See [2, 13] for further details.

Example 4.2 (The 4_1 knot with a kink)**.** Let us consider a braid of the 4_1 knot as in Figure 10. The geometric representation ρ lifts to a $\mathrm{SL}(2,\mathbb{C})$-representation $\widetilde{\rho}$ such that

$$\widetilde{\rho}(m_1) = \widetilde{\rho}(m_2) = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}, \quad \widetilde{\rho}(m_3) = \begin{pmatrix} -1 & 0 \\ -\lambda & -1 \end{pmatrix}$$
$$\widetilde{\rho}(m_4) = \begin{pmatrix} -1-\lambda & \lambda \\ -\lambda & -1+\lambda \end{pmatrix}, \quad \widetilde{\rho}(m_4) = \begin{pmatrix} -2 & \lambda \\ -1+\lambda & 0 \end{pmatrix}$$

where $\lambda^2 - \lambda + 1 = 0$. Note that $\widetilde{\rho}(\lambda)$ shall have trace -2.

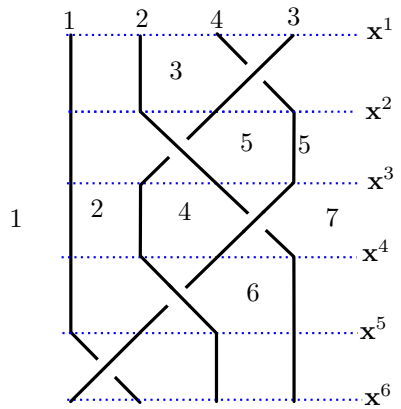


Figure 10: A braid presentation of the 4_1 knot.

Choosing the initial arc-coloring $H_1 = \binom{1}{0}$, we have

$$H_2 = \widetilde{\rho}(m_2)^{-1}H_1 = \binom{-1}{0}, \quad H_3 = \widetilde{\rho}(m_5)^{-1}H_2 = \binom{0}{-1+\lambda}$$

$$H_4 = \widetilde{\rho}(m_2)H_3 = \binom{1-\lambda}{1-\lambda}, \qquad H_5 = \widetilde{\rho}(m_3)^{-1}H_4 = \binom{-1+\lambda}{\lambda}.$$

Similarly, choosing the initial region-coloring $V_1 = \binom{\alpha}{\beta}$ for some $\alpha, \beta \in \mathbb{C}$, we have

$$V_2 = \widetilde{\rho}(m_1)^{-1}V_1 = \binom{-\alpha+\beta}{-\beta}, \qquad V_3 = \widetilde{\rho}(m_2)^{-1}V_2 = \binom{\alpha-2\beta}{\beta}$$

$$V_4 = \widetilde{\rho}(m_4)^{-1}V_2 = \binom{\alpha(1-\lambda)+\beta(-1+2\lambda)}{-\alpha\lambda+\beta(1+2\lambda)}, \quad V_5 = \widetilde{\rho}(m_3)^{-1}V_3 = \binom{-\alpha+2\beta}{\alpha\lambda-\beta(1+2\lambda)}$$

$$V_6 = \widetilde{\rho}(m_5)^{-1}V_4 = \binom{\alpha(-1+\lambda)+\beta(2-3\lambda)}{\alpha\lambda-\beta(1+3\lambda)}, \quad V_7 = \widetilde{\rho}(m_5)^{-1}V_5 = \binom{\alpha(1-\lambda)+\beta(-2+3\lambda)}{-\alpha(1+\lambda)+2\beta(2+\lambda)}.$$

Then finally, letting $V_0 = \binom{\gamma}{1}$ for some $\gamma \in \mathbb{C}$, we obtain : (here we abbreviate $\det(V, W)$ by $|V, W|$)

$$\mathbf{x}^1 = \begin{pmatrix} |V_1, V_0| \\ |V_2, H_1| \\ |H_1, V_0| \\ |V_2, V_0| \\ |V_3, H_2| \\ |H_2, V_0| \\ |V_3, V_0| \\ |V_6, H_4| \\ |H_4, V_0| \\ |V_6, V_0| \\ |V_7, H_3| \\ |H_3, V_0| \\ |V_7, V_0| \end{pmatrix}^T = \begin{pmatrix} \alpha - \beta\gamma \\ \beta \\ 1 \\ -\alpha + \beta\gamma + \beta \\ \beta \\ -1 \\ \alpha - \beta(\gamma + 2) \\ (\lambda - 1)(\alpha - 3\beta) \\ (\gamma - 1)(\lambda - 1) \\ \alpha(-\gamma\lambda + \lambda - 1) + \beta(3(\gamma - 1)\lambda + \gamma + 2) \\ \alpha\lambda - \beta(2\lambda + 1) \\ \gamma - \gamma\lambda \\ \alpha((\gamma - 1)\lambda + \gamma + 1) - \beta(2\gamma(\lambda + 2) - 3\lambda + 2) \end{pmatrix}^T$$

$$\mathbf{x}^2 = \begin{pmatrix} |V_1, V_0| \\ |V_2, H_1| \\ |H_1, V_0| \\ |V_2, V_0| \\ |V_3, H_2| \\ |H_2, V_0| \\ |V_3, V_0| \\ |V_5, H_3| \\ |H_3, V_0| \\ |V_5, V_0| \\ |V_7, H_5| \\ |H_5, V_0| \\ |V_7, V_0| \end{pmatrix}^T = \begin{pmatrix} \alpha - \beta\gamma \\ \beta \\ 1 \\ -\alpha + \beta\gamma + \beta \\ \beta \\ -1 \\ \alpha - \beta(\gamma + 2) \\ \lambda^2(-(\alpha - 2\beta)) \\ \gamma - \gamma\lambda \\ \beta(2\gamma\lambda + \gamma + 2) - \alpha(\gamma\lambda + 1) \\ (\lambda - 1)(\alpha - 3\beta) \\ -\gamma\lambda + \lambda - 1 \\ \alpha((\gamma - 1)\lambda + \gamma + 1) - \beta(2\gamma(\lambda + 2) - 3\lambda + 2) \end{pmatrix}^T$$

$$\mathbf{x}^3 = \begin{pmatrix} |V_1, V_0| \\ |V_2, H_1| \\ |H_1, V_0| \\ |V_2, V_0| \\ |V_4, H_4| \\ |H_4, V_0| \\ |V_4, V_0| \\ |V_5, H_2| \\ |H_2, V_0| \\ |V_5, V_0| \\ |V_7, H_5| \\ |H_5, V_0| \\ |V_7, V_0| \end{pmatrix}^T = \begin{pmatrix} \alpha - \beta\gamma \\ \beta \\ 1 \\ -\alpha + \beta\gamma + \beta \\ (\lambda - 1)(-(\alpha - 2\beta)) \\ (\gamma - 1)(\lambda - 1) \\ (\gamma - 1)\lambda(\alpha - 2\beta) + \alpha - \beta(\gamma + 1) \\ \alpha\lambda - \beta(2\lambda + 1) \\ -1 \\ \beta(2\gamma\lambda + \gamma + 2) - \alpha(\gamma\lambda + 1) \\ (\lambda - 1)(\alpha - 3\beta) \\ -\gamma\lambda + \lambda - 1 \\ \alpha((\gamma - 1)\lambda + \gamma + 1) - \beta(2\gamma(\lambda + 2) - 3\lambda + 2) \end{pmatrix}^T$$

$$\mathbf{x}^4 = \begin{pmatrix} |V_1, V_0| \\ |V_2, H_1| \\ |H_1, V_0| \\ |V_2, V_0| \\ |V_4, H_4| \\ |H_4, V_0| \\ |V_4, V_0| \\ |V_6, H_5| \\ |H_5, V_0| \\ |V_6, V_0| \\ |V_7, H_3| \\ |H_3, V_0| \\ |V_7, V_0| \end{pmatrix}^T = \begin{pmatrix} \alpha - \beta\gamma \\ \beta \\ 1 \\ -\alpha + \beta\gamma + \beta \\ (\lambda - 1)(-(\alpha - 2\beta)) \\ (\gamma - 1)(\lambda - 1) \\ (\gamma - 1)\lambda(\alpha - 2\beta) + \alpha - \beta(\gamma + 1) \\ -\beta \\ -\gamma\lambda + \lambda - 1 \\ \alpha(-\gamma\lambda + \lambda - 1) + \beta(3(\gamma - 1)\lambda + \gamma + 2) \\ \alpha\lambda - \beta(2\lambda + 1) \\ \gamma - \gamma\lambda \\ \alpha((\gamma - 1)\lambda + \gamma + 1) - \beta(2\gamma(\lambda + 2) - 3\lambda + 2) \end{pmatrix}^T$$

$$\mathbf{x}^5 = \begin{pmatrix} |V_1, V_0| \\ |V_2, H_1| \\ |H_1, V_0| \\ |V_2, V_0| \\ |V_3, H_1| \\ |H_1, V_0| \\ |V_3, V_0| \\ |V_6, H_4| \\ |H_4, V_0| \\ |V_6, V_0| \\ |V_7, H_3| \\ |H_3, V_0| \\ |V_7, V_0| \end{pmatrix}^T = \begin{pmatrix} \alpha - \beta\gamma \\ \beta \\ 1 \\ -\alpha + \beta\gamma + \beta \\ -\beta \\ 1 \\ \alpha - \beta(\gamma + 2) \\ (\lambda - 1)(\alpha - 3\beta) \\ (\gamma - 1)(\lambda - 1) \\ \alpha(-\gamma\lambda + \lambda - 1) + \beta(3(\gamma - 1)\lambda + \gamma + 2) \\ \alpha\lambda - \beta(2\lambda + 1) \\ \gamma - \gamma\lambda \\ \alpha((\gamma - 1)\lambda + \gamma + 1) - \beta(2\gamma(\lambda + 2) - 3\lambda + 2) \end{pmatrix}^T$$

$$\mathbf{x}^6 = \begin{pmatrix} |V_1, V_0| \\ |V_2, H_1| \\ |H_1, V_0| \\ |V_2, V_0| \\ |V_3, H_2| \\ |H_2, V_0| \\ |V_3, V_0| \\ |V_6, H_4| \\ |H_4, V_0| \\ |V_6, V_0| \\ |V_7, H_3| \\ |H_3, V_0| \\ |V_7, V_0| \end{pmatrix}^T = \begin{pmatrix} \alpha - \beta\gamma \\ \beta \\ 1 \\ -\alpha + \beta\gamma + \beta \\ \beta \\ -1 \\ \alpha - \beta(\gamma + 2) \\ (\lambda - 1)(\alpha - 3\beta) \\ (\gamma - 1)(\lambda - 1) \\ \alpha(-\gamma\lambda + \lambda - 1) + \beta(3(\gamma - 1)\lambda + \gamma + 2) \\ \alpha\lambda - \beta(2\lambda + 1) \\ \gamma - \gamma\lambda \\ \alpha((\gamma - 1)\lambda + \gamma + 1) - \beta(2\gamma(\lambda + 2) - 3\lambda + 2) \end{pmatrix}^T$$

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