

POLYLOGARITHMS, BLOCH COMPLEXES, AND QUIVER MUTATIONS

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ABSTRACT. For an integer $n \geq 2$ we define a polylogarithm $\widehat{\mathcal{L}}_n$, which is a holomorphic function on the universal abelian cover of $\mathbb{C} \setminus \{0, 1\}$ defined modulo $(2\pi i)^n / (n-1)!$. We analyze its functional relations and give a method for producing relations from quivers. We define higher weight analogues $\widehat{\mathcal{B}}_n(\mathbb{C})$ of Neumann's extended Bloch group $\widehat{\mathcal{B}}(\mathbb{C})$, and show that the imaginary part (when n is even) or real part (when n is odd) of $\widehat{\mathcal{L}}_n$ agrees with Goncharov's real valued polylogarithm \mathcal{L}_n on $\widehat{\mathcal{B}}_n(\mathbb{C})$. For a field F we also define an extended version $\widehat{\Gamma}(F, n)$ of Goncharov's Bloch complex $\Gamma(F, n)$. Goncharov's complex conjecturally computes the rational motivic cohomology of F , and one may speculate whether the extended complex computes the integral motivic cohomology. Finally, we use $\widehat{\mathcal{L}}_3$ to construct a lift of Goncharov's regulator $H_5(\mathrm{SL}(3, \mathbb{C})) \rightarrow \mathbb{R}$ to a complex regulator whose real part agrees with that of Goncharov.

1. INTRODUCTION

For a natural number n , the polylogarithm of weight n is defined by the power series

$$(1.1) \quad \mathrm{Li}_n(z) = \sum_{k=1}^{\infty} z^k / k^n, \quad |z| \leq 1.$$

It extends holomorphically to $\mathbb{C} \setminus (1, \infty)$, but is multivalued on \mathbb{C} with branch points at 0 and 1. There are several real single valued analogues of the polylogarithm (see [Zag91] for definitions and basic properties). We shall only consider

$$(1.2) \quad \mathcal{L}_n(z) = \Re_n \left(\sum_{r=0}^{n-1} \frac{2^r B_r}{r!} \mathrm{Li}_{n-r}(z) (\log |z|)^r \right),$$

where $\Re_n(x)$ denotes the real part of x when n is odd and the imaginary part when n is even, and $B_1 = -1/2$, $B_2 = 1$, $B_3 = 0$, $B_4 = -1/30$, etc., are the Bernoulli numbers. The functions $\mathcal{L}_n(z)$ are continuous on $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$, and

$$(1.3) \quad \mathcal{L}_2(z) = \mathrm{Im}(\mathrm{Li}_2(z)) + \mathrm{Im}(\log(1-z)) \log(|z|)$$

is the Bloch-Wigner dilogarithm. Letting $\mathbb{Z}[X]$ denote the free abelian group on a set X , we may regard \mathcal{L}_n as functions on $\mathbb{Z}[\mathbb{C}P^1]$ by linear extension.

1.1. Regulators and algebraic K -theory. The study of \mathcal{L}_n and its functional relations is related to algebraic K -theory. Goncharov has defined subgroups R_n of $\mathbb{Z}[\mathbb{C}P^1]$ consisting of functional relations of \mathcal{L}_n , i.e. $\mathcal{L}_n(\alpha) = 0$ for $\alpha \in R_n$. We refer to [Gon95] and Section 3 below for their definition. Let

$$(1.4) \quad \mathcal{P}_n(\mathbb{C}) = \mathbb{Z}[\mathbb{C}P^1] / R_n, \quad \mathcal{B}_n(\mathbb{C}) = \mathrm{Ker}(\nu_n),$$

C. Z. was supported by DMS-1711405

2010 *Mathematics Classification*. Primary 11G55. Secondary 13F60, 58J28, 19E15.

Key words and phrases: Polylogarithms, Bloch complexes, cluster coordinates, regulators, motivic cohomology, Cheeger-Chern-Simons classes.

where

$$(1.5) \quad \begin{aligned} \nu_n: \mathcal{P}_n(\mathbb{C}) &\rightarrow \mathcal{P}_{n-1}(\mathbb{C}) \otimes \mathbb{C}^*, & [z] &\mapsto [z] \otimes z \text{ for } n > 2, \\ \nu_2: \mathcal{P}_2(\mathbb{C}) &\rightarrow \wedge^2(\mathbb{C}^*), & [z] &\mapsto z \wedge (1-z) \text{ for } n = 2. \end{aligned}$$

Remark 1.1. Goncharov uses the symbol \mathcal{B}_n to denote the groups \mathcal{P}_n . We use \mathcal{B}_n to denote the kernel of ν_n . Our use is consistent with common notation when $n = 2$.

For an abelian group A , let $A_{\mathbb{Q}} = A \otimes_{\mathbb{Z}} \mathbb{Q}$. Goncharov [Gon94, Gon05b] has conjectured the existence of a commutative diagram

$$(1.6) \quad \begin{array}{ccc} K_{2n-1}^{(n)}(\mathbb{C})_{\mathbb{Q}} & \xrightarrow{\cong} & \mathcal{B}_n(\mathbb{C})_{\mathbb{Q}} \\ & \searrow \text{reg}_n & \swarrow \mathcal{L}_n \\ & \mathbb{R} & \end{array},$$

where $K_q^{(p)}(\mathbb{C}) = \text{gr}_{\gamma}^p K_q(\mathbb{C})$ denotes the associated graded groups for the γ -filtration on $K_q(\mathbb{C})$, and reg_n is a regulator map (see e.g. [Gon05b]). If this conjecture holds, it follows that a proper understanding of the groups R_n would give concrete descriptions of $K_{2n-1}^{(n)}(\mathbb{C})_{\mathbb{Q}}$.

When $n = 2$ Goncharov conjectures that R_2 is generated by the elements

$$(1.7) \quad [x] - [y] + \left[\frac{y}{x}\right] - \left[\frac{1-x^{-1}}{1-y^{-1}}\right] + \left[\frac{1-x}{1-y}\right], \quad x, y \in \mathbb{C}P^1,$$

corresponding to the well known functional relation

$$(1.8) \quad \mathcal{L}_2(x) - \mathcal{L}_2(y) + \mathcal{L}_2\left(\frac{y}{x}\right) - \mathcal{L}_2\left(\frac{1-x^{-1}}{1-y^{-1}}\right) + \mathcal{L}_2\left(\frac{1-x}{1-y}\right) = 0$$

for the Bloch-Wigner dilogarithm. If so, it follows that $\mathcal{B}_2(\mathbb{C})$ is the classical Bloch group $\mathcal{B}(\mathbb{C})$ and the isomorphism in (1.6) follows from Suslin's result [Sus90] that $K_3^{\text{ind}}(F)_{\mathbb{Q}} \cong \mathcal{B}(F)_{\mathbb{Q}}$ for any infinite field. Goncharov also conjectures that R_3 is generated by elements

$$(1.9) \quad [x] - [x^{-1}], \quad [x] + [1-x] + [1-x^{-1}] - ([y] + [1-y] + [1-y]^{-1}), \quad R_3(x, y, z),$$

where $R_3(x, y, z)$ is an explicit 3-variable relation (see Remark 3.17). For $n > 3$ few explicit elements in R_n are known other than the elements $[x] + (-1)^n[x^{-1}]$ which are in R_n for any n . Gangl [Gan16] has constructed a 931 term relation in R_4 , and an alternative conjectural description of R_4 has been given in [GR18].

One can define $\mathcal{P}_n(F)$ for any field F and Goncharov has shown that the maps ν_n in (1.5) are the first maps in a chain complex $\Gamma(F, n)$. When $n = 2$ this complex is simply $\nu_2: \mathcal{P}_2(F) \rightarrow \wedge^2(F^*)$ and when $n > 2$ the complex $\Gamma(F, n)$ is given by

$$(1.10) \quad \mathcal{P}_n(F) \xrightarrow{\delta_1} \dots \xrightarrow{\delta_k} \mathcal{P}_{n-k}(F) \otimes \wedge^k(F^*) \xrightarrow{\delta_{k+1}} \dots \xrightarrow{\delta_{n-2}} \mathcal{P}_2(F) \otimes \wedge^{n-2}(F^*) \xrightarrow{\delta_{n-1}} \wedge^n(F^*),$$

with $\mathcal{P}_n(F)$ in degree 1 and $\wedge^n(F^*)$ in degree n and

$$(1.11) \quad \delta_1([z]) = [z] \otimes z, \quad \delta_k([z] \otimes a) = [z] \otimes (z \wedge a) \text{ for } 1 < k < n-1, \quad \delta_{n-1}([z] \otimes a) = z \wedge (1-z) \wedge a.$$

We stress that this is only a chain complex modulo two-torsion ($2z \wedge z$ is always zero, but $z \wedge z$ may not be).

Generalizing the isomorphism in (1.6) Goncharov conjectures [Gon94, Conj. 2.1] that there are isomorphisms

$$(1.12) \quad H^i(\Gamma(F, n)_{\mathbb{Q}}) \cong K_{2n-i}^{(n)}(F)_{\mathbb{Q}}.$$

It is interesting to note that the chain complex $\Gamma(n, F)$ is concentrated exactly in the degrees i , where $K_{2n-i}^{(n)}(F)$ may be non-vanishing (it is known that $K_{2n-i}^{(n)}(F)_{\mathbb{Q}} = 0$ for $i > n$ and conjectured that $K_{2n-i}^{(n)}(F)_{\mathbb{Q}} = 0$ for $i \leq 0$).

Remark 1.2. For any smooth scheme X over a field, Voevodsky has defined motivic cohomology groups $H_{\mathcal{M}}^i(X, \mathbb{Z}(n))$ which are rationally isomorphic to $K_{2n-i}^{(n)}(X)$. Hence, the complex $\Gamma(n, F)_{\mathbb{Q}}$ conjecturally computes the rational motivic cohomology of F . Voevodsky also showed that $H_{\mathcal{M}}^i(X, \mathbb{Z}(n))$ is isomorphic to Bloch's higher Chow groups $CH^n(X, 2n - i)$ [MVW06, Voe02]. For any field F one has $CH^p(F, q)[1/(q - 1)!] \cong K_q^{(p)}(F)[1/(q - 1)!]$ [Lev97].

1.2. Complex valued regulators and Neumann's extended Bloch group. Bloch [Blo86] defined regulator maps $CH^n(X, 2n - i) \rightarrow H_{\mathcal{D}}^i(X, (2\pi i)^n \mathbb{Z})$, where X is a complex variety and $H_{\mathcal{D}}^*$ is Deligne cohomology [EV88]. One has $H_{\mathcal{D}}^1(\mathbb{C}, (2\pi i)^n \mathbb{Z}) = \mathbb{C}/(2\pi i)^n \mathbb{Z}$ and under the isomorphism $K_{2n-1}^{(n)}(\mathbb{C})_{\mathbb{Q}} \cong CH^n(\mathbb{C}, 2n - 1)_{\mathbb{Q}}$ the regulator map reg_n from (1.6) is the composition of Bloch's regulator with $\mathfrak{R}_n: \mathbb{C}/(2\pi i)^n \mathbb{Q} \rightarrow \mathbb{R}$. By Remark 1.2 reg_n thus lifts to a complex valued regulator. We shall also denote this regulator by reg_n .

1.2.1. Neumann's extended Bloch group. Neumann [Neu04] (modified slightly in [GZ07]) defined a holomorphic function $\mathcal{R}: \widehat{\mathbb{C}} \rightarrow \mathbb{C}/4\pi^2 \mathbb{Z}$ where $\widehat{\mathbb{C}}$ is the universal abelian cover of $\mathbb{C} \setminus \{0, 1\}$ and \mathcal{R} is a variant of Roger's dilogarithm. He defined a group $\widehat{\mathcal{P}}(\mathbb{C})$ to be the quotient of $\mathbb{Z}[\widehat{\mathbb{C}}]$ by a certain lift of the five term relation (1.7) and defined the *extended Bloch group* $\widehat{\mathcal{B}}(\mathbb{C})$ to be the kernel of a map $\widehat{\mathcal{P}}(\mathbb{C}) \rightarrow \wedge^2(\mathbb{C})$ lifting the map ν_2 in (1.11). There is a commutative diagram (see [Neu04, GZ07, Zic15])

$$(1.13) \quad \begin{array}{ccccc} K_3^{\text{ind}}(\mathbb{C}) & \xrightarrow{\cong} & H_3(\text{SL}(2, \mathbb{C})) & \xrightarrow{\cong} & \widehat{\mathcal{B}}(\mathbb{C}) \\ & \searrow \text{reg}_2 & \downarrow \widehat{c}_2 & \swarrow \mathcal{R} & \\ & & \mathbb{C}/4\pi^2 \mathbb{Z} & & \end{array}$$

where \widehat{c}_2 is the second Cheeger-Chern-Simons class [CS85]. Although Neumann defined $\widehat{\mathcal{B}}(\mathbb{C})$ using analytic continuation, Zickert [Zic15] proved that one can define an extended Bloch group $\widehat{\mathcal{B}}(F)$ for any field F given a choice of a \mathbb{Z} -extension $0 \rightarrow \mathbb{Z} \rightarrow E \rightarrow F^* \rightarrow 0$ of the unit group F^* . When $F = \mathbb{C}$ the extension is simply the one given by the exponential map. He also showed that Suslin's isomorphism $K_3^{\text{ind}}(F)_{\mathbb{Q}} \cong \mathcal{B}(F)_{\mathbb{Q}}$ lifts to an isomorphism $K_3^{\text{ind}}(F) \cong \widehat{\mathcal{B}}(F)$ at least when F is a number field and E is a generator of the group $\text{Ext}(F^*, \mathbb{Z})$.

Remark 1.3. For any field F , one has $CH^2(F, 3) \cong K_3^{\text{ind}}(F)$ [Sus87], and since $K_3(\mathbb{C})$ is divisible (see e.g. [Sah89]) we thus have $K_3^{(2)}(\mathbb{C}) \cong K_3^{\text{ind}}(\mathbb{C})$.

1.3. Goals of the paper. We briefly describe the main goals of the paper. Explicit results are listed in Section 2 and more details can be found in the remainder of the paper.

- Define complex valued polylogarithms $\widehat{\mathcal{L}}_n: \widehat{\mathbb{C}} \rightarrow \mathbb{C}/\frac{(2\pi i)^n}{(n-1)!} \mathbb{Z}$ generalizing Neumann's map \mathcal{R} when $n = 2$. Analyze its functional relations.
- Define higher weight analogues $\widehat{\mathcal{P}}_n(\mathbb{C})$ and $\widehat{\mathcal{B}}_n(\mathbb{C})$ of Neumann's groups $\widehat{\mathcal{P}}(\mathbb{C})$ and $\widehat{\mathcal{B}}(\mathbb{C})$. The group $\widehat{\mathcal{P}}_n(\mathbb{C})$ is the quotient of $\mathbb{Z}[\widehat{\mathbb{C}}]$ by a subgroup \widehat{R}_n of functional relations for $\widehat{\mathcal{L}}_n$, and $\widehat{\mathcal{B}}_n(\mathbb{C})$ is the kernel of a map $\widehat{\mathcal{P}}_n(\mathbb{C}) \rightarrow \widehat{\mathcal{P}}_{n-1}(\mathbb{C}) \otimes \mathbb{C}$ when $n > 2$, or $\widehat{\mathcal{P}}_2(\mathbb{C}) \rightarrow \wedge^2(\mathbb{C})$ when $n = 2$.

- Give a method for constructing elements in \widetilde{R}_n from a quiver. When n is 2 or 3, this method is very effective, but we have found no new elements in \widetilde{R}_4 using this method.
- Show that $\mathfrak{R}_n \circ \widehat{\mathcal{L}}_n$ equals $\mathcal{L}_n \circ r$ on $\widehat{\mathcal{B}}_n$, where $r: \widehat{\mathbb{C}} \rightarrow \mathbb{C} \setminus \{0, 1\}$ is the covering map. We stress that they are not equal on $\widehat{\mathbb{C}}$. Note that $\widehat{\mathcal{L}}_n$ is holomorphic while \mathcal{L}_n is not the real/imaginary part of a holomorphic function. This suggests that $\widehat{\mathcal{L}}_n$ is more natural.
- Show that Goncharov's complex (1.10) lifts to a complex $\widehat{\Gamma}(\mathbb{C}, n)$:

$$(1.14) \quad \widehat{\mathcal{P}}_n(\mathbb{C}) \xrightarrow{\delta_1} \dots \xrightarrow{\delta_k} \widehat{\mathcal{P}}_{n-k}(\mathbb{C}) \otimes \wedge^k(\mathbb{C}) \xrightarrow{\delta_{k+1}} \dots \xrightarrow{\delta_{n-2}} \widehat{\mathcal{P}}_2(\mathbb{C}) \otimes \wedge^{n-2}(\mathbb{C}) \xrightarrow{\delta_{n-1}} \wedge^n(\mathbb{C}).$$

- Goncharov [Gon95] used \mathcal{L}_3 to define a map $H_5(\mathrm{SL}(3, \mathbb{C})) \rightarrow \mathbb{R}$, which he proved to be a non-zero rational multiple of Borel's regulator. We use $\widehat{\mathcal{L}}_3$ to construct a map

$$(1.15) \quad H_5(\mathrm{SL}(3, \mathbb{C})) \rightarrow \mathbb{C} / \frac{(\pi i)^3}{2} \mathbb{Z}$$

whose real part agrees with that of Goncharov.

Remark 1.4. We mainly work over the field \mathbb{C} , but the groups $\widehat{\mathcal{P}}_n(\mathbb{C})$, $\widehat{\mathcal{B}}_n(\mathbb{C})$ and the complex (1.14) can be defined over any field F together with a choice of \mathbb{Z} -extension of F^* .

1.3.1. *Some speculation.* The following questions may inspire further investigations.

- Is the complex $\widehat{\Gamma}(F, n)$ rationally isomorphic to Goncharov's complex $\Gamma(F, n)$?
- Does Goncharov's conjectural rational isomorphism (1.12) lift to an integral isomorphism

$$(1.16) \quad H^i(\widehat{\Gamma}(F, n)) \cong H^i_{\mathcal{M}}(F, \mathbb{Z}(n))?$$

- One has characteristic classes $\widehat{c}_n: H_{2n-1}(\mathrm{SL}(n, \mathbb{C})) \rightarrow \mathbb{C}/(2\pi i)^n \mathbb{Z}$ [CS85]. Does (1.13) generalize to a diagram

$$(1.17) \quad \begin{array}{ccccc} H^1_{\mathcal{M}}(\mathbb{C}, \mathbb{Z}(n)) & \xrightarrow{\cong} & H_{2n-1}(\mathrm{SL}(n, \mathbb{C})) & \xrightarrow{\cong} & \widehat{\mathcal{B}}_n(\mathbb{C}) \\ & \searrow \mathrm{reg}_n & \downarrow \widehat{c}_n & \swarrow (n-1)! \widehat{\mathcal{L}}_n & \\ & & \mathbb{C}/(2\pi i)^n \mathbb{Z} & & \end{array}$$

- Is the map (1.15) equal to \widehat{c}_3 modulo $\frac{(\pi i)^3}{2}$?

Remark 1.5. Neumann defined two variants of his extended Bloch group, one using $\widehat{\mathbb{C}}$ and another using a disconnected cover $\widehat{\mathbb{C}}_{\mathrm{signs}}$. We shall likewise consider different variants, the main difference being the ambiguity of definition of $\widehat{\mathcal{L}}_n$ (modulo $\frac{(2\pi i)^n}{(n-1)!}$ or $\frac{(\pi i)^n}{(n-1)!}$). In this section we have not distinguished the different variants.

Acknowledgment. We thank Soren Galatius and Matthias Goerner for helpful discussions.

2. STATEMENT OF RESULTS

Consider the space

$$(2.1) \quad \widehat{\mathbb{C}}_{\mathrm{signs}} = \left\{ (u, v) \in \mathbb{C}^2 \mid \epsilon_1 e^u + \epsilon_2 e^v = 1 \text{ for some } \epsilon_1, \epsilon_2 \in \{-1, 1\} \right\},$$

introduced by Neumann [Neu04] (see also Zagier [Zag07]). It has four components $\widehat{\mathbb{C}}_{++}$, $\widehat{\mathbb{C}}_{-+}$, $\widehat{\mathbb{C}}_{+-}$ and $\widehat{\mathbb{C}}_{--}$ corresponding to the signs of ϵ_1 and ϵ_2 . There is a holomorphic map

$$(2.2) \quad r: \widehat{\mathbb{C}}_{\mathrm{signs}} \rightarrow \mathbb{C} \setminus \{0, 1\}, \quad (u, v) \mapsto \epsilon_1 e^u \quad \text{if } (u, v) \in \widehat{\mathbb{C}}_{\epsilon_1, \epsilon_2},$$

which restricts to a $\mathbb{Z} \times \mathbb{Z}$ cover on each component. On $\widehat{\mathbb{C}}_{\text{signs}}$ we shall introduce the holomorphic one form

$$(2.3) \quad \omega_n = (-1)^n \frac{n-1}{n!} u^{n-2} (udv - vdu) \in \Omega^1(\widehat{\mathbb{C}}_{\text{signs}}),$$

which is closed since $\widehat{\mathbb{C}}_{\text{signs}}$ is complex 1-dimensional. Let ν_2 denote the 2-adic valuation and let

$$(2.4) \quad \kappa_n = \begin{cases} 2^{2-n} & \text{if } n \text{ is even,} \\ 2^{3+\nu_2(n-1)-n} & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 2.1 (Proof in Section 5.1). *The form w_n has periods in $\frac{(2\pi i)^n}{(n-1)!} \mathbb{Z}$ on $\widehat{\mathbb{C}}_{++}$ and $\widehat{\mathbb{C}}_{+-}$ and periods in $\kappa_n \frac{(2\pi i)^n}{(n-1)!} \mathbb{Z}$ on $\widehat{\mathbb{C}}_{-+}$ and $\widehat{\mathbb{C}}_{--}$.*

2.0.1. *Primitives for ω_n .* Letting Log denote the main branch of logarithm (argument in $(-\pi, \pi]$) we may uniquely write each element (u, v) in $\widehat{\mathbb{C}}_{\text{signs}}$ in the form

$$(2.5) \quad \langle z; p, q \rangle_{\epsilon_1, \epsilon_2} := (\text{Log}(\epsilon_1 z) + 2p\pi, \text{Log}(\epsilon_2(1-z)) + 2q\pi i).$$

For $z \in \mathbb{C} \setminus \{0, 1\}$ and an integer q let

$$(2.6) \quad \mathcal{L}i_k(z; q) = \text{Li}_k(z) - \frac{2q\pi i}{(k-1)!} \text{Log}(z)^{k-1}.$$

Theorem 2.2 (Proof in Section 5.1). *The function*

$$(2.7) \quad \widehat{\mathcal{L}}_n(u, v) = \sum_{r=0}^{n-1} \frac{(-1)^r}{r!} \mathcal{L}i_{n-r}(z; q) u^r - \frac{(-1)^n}{n!} u^{n-1} v$$

is holomorphic and well defined modulo $\frac{(2\pi i)^n}{(n-1)!}$ for $(u, v) \in \widehat{\mathbb{C}}_{++}$ or $\widehat{\mathbb{C}}_{+-}$ and modulo $\kappa_n \frac{(2\pi i)^n}{(n-1)!}$ for $(u, v) \in \widehat{\mathbb{C}}_{-+}$, where ϵ_1, z and q are defined by (2.5). It is a primitive for ω_n , i.e. $d\widehat{\mathcal{L}}_n = \omega_n$.

Remark 2.3. On $\widehat{\mathbb{C}}_{--}$ the function (2.7) is only defined modulo $\frac{(\pi i)^n}{(n-1)!}$, and in order to obtain a primitive defined modulo $\kappa_n \frac{(2\pi i)^n}{(n-1)!}$, one must modify it by multiples of $\frac{(\pi i)^n}{(n-1)!}$. We refer to Section 5 for details.

Remark 2.4. The map $\widehat{\mathcal{L}}_2$ equals $\mathcal{R} + \frac{\pi^2}{6}$ modulo π^2 , where \mathcal{R} is Neumann's polylogarithm [Neu04] (reviewed in Section 4).

2.0.2. *Inversions and order 3 symmetries in low degree.* It is well known that the polylogarithm \mathcal{L}_n in (1.2) satisfies the functional equations

$$(2.8) \quad \begin{aligned} \mathcal{L}_n(z) + (-1)^n \mathcal{L}_n(z^{-1}) &= 0, \\ \mathcal{L}_3(z) + \mathcal{L}_3\left(\frac{1}{1-z}\right) + \mathcal{L}_3(1-z^{-1}) &= \zeta(3), \quad \mathcal{L}_2(z) - \mathcal{L}_2\left(\frac{1}{1-z}\right) = 0. \end{aligned}$$

Consider the holomorphic maps

$$(2.9) \quad \widehat{\tau}: \widehat{\mathbb{C}}_{\text{signs}} \rightarrow \widehat{\mathbb{C}}_{\text{signs}}, \quad (u, v) \mapsto (-u, v-u), \quad \widehat{\sigma}: \widehat{\mathbb{C}}_{\text{signs}} \rightarrow \widehat{\mathbb{C}}_{\text{signs}}, \quad (u, v) \mapsto (-v, u-v).$$

One easily checks that τ and σ have order 2 and 3, respectively, and that they are lifts of the maps $\mathbb{C} \setminus \{0, 1\} \rightarrow \mathbb{C} \setminus \{0, 1\}$ given by $z \mapsto z^{-1}$ and $z \mapsto \frac{1}{1-z}$, respectively. An elementary calculation shows that

$$(2.10) \quad \tau^*(\omega_n) = -(-1)^n \omega_n, \quad \omega_3 + \sigma^* \omega_3 + (\sigma^*)^2 \omega_3 = 0, \quad \sigma^*(\omega_2) = \omega_2.$$

This implies that the functions

$$(2.11) \quad \widehat{\mathcal{L}}_n(u, v) + (-1)^n \widehat{\mathcal{L}}_n(-u, v - u), \\ \widehat{\mathcal{L}}_3(u, v) + \widehat{\mathcal{L}}_3(-v, u - v) + \widehat{\mathcal{L}}_3(v - u, -u), \quad \widehat{\mathcal{L}}_2(u, v) + \widehat{\mathcal{L}}_2(-v, u - v),$$

are locally constant.

Lemma 2.5. For $(u, v) \in \widehat{\mathbb{C}}_{-+}$ we have

$$(2.12) \quad \widehat{\mathcal{L}}_n(u, v) + (-1)^n \widehat{\mathcal{L}}_n(-u, v - u) = (2^n - 2)(\pi i)^n \frac{B_n}{n!} \in \mathbb{C}/\kappa_n \frac{(2\pi i)^n}{(n-1)!} \mathbb{Z}$$

if n is even and 0 if n is odd.

Proof. Since the function is constant on $\widehat{\mathbb{C}}_{-+}$ it is enough to consider $(u, v) = (0, \text{Log}(2)) \in \widehat{\mathbb{C}}_{-+}$. For this point $\tau(u, v) = (u, v)$, which proves the result for odd n . Since $\text{Li}_n(1) = \zeta(n)$ (this follows from (1.1)) it follows from the formula $\text{Li}_n(z) + \text{Li}_n(-z) = 2^{1-n} \text{Li}_n(z^2)$ [Lew91, p. 29] that $\text{Li}_n(-1) = -(1 - 2^{1-n})\zeta(n)$. When n is even, $\zeta(n) = (-1)^{\frac{n}{2}+1} \frac{B_n(2\pi)^n}{2n!}$, so we have

$$(2.13) \quad 2\widehat{\mathcal{L}}_n(0, \log(2)) = 2\text{Li}_n(-1) = -2(1 - 2^{1-n})\zeta(n) = (2^n - 2)(\pi i)^n \frac{B_n}{n!}.$$

This concludes the proof. \square

Remark 2.6. By the Staudt-Clausen formula for the denominator of even Bernoulli numbers, we see that the order of (2.12), when n is even, is equal to the denominator of $\frac{B_n}{2n}$. This is equal to the order of $K_n(\mathbb{Q})$ when n is divisible by 4 and half this, when n is 2 modulo 4 (see e.g. [Wei05]). It is equal to the order of the Harris-Segal summand of $K_n(\mathbb{Q})$ for any even n .

Lemma 2.7. For (u, v) in $\widehat{\mathbb{C}}_{++}$, $\widehat{\mathbb{C}}_{+-}$ or $\widehat{\mathbb{C}}_{-+}$ we have

$$(2.14) \quad \widehat{\mathcal{L}}_3(u, v) + \widehat{\mathcal{L}}_3(-v, u - v) + \widehat{\mathcal{L}}_3(v - u, -u) = \zeta(3) \pmod{4\pi^3 i}.$$

Proof. It is enough to check this for $(u, v) \in \widehat{\mathbb{C}}_{++}$. If (u, v) is a lift of -1 , then $(-v, u - v)$ and $(v - u, -u)$ are lifts of $\frac{1}{2}$ and 2, respectively. The result now follows from the formulas

$$(2.15) \quad \text{Li}_3(-1) = -\frac{3}{4}\zeta(3), \quad \text{Li}_3\left(\frac{1}{2}\right) = \frac{7}{8}\zeta(3) - \frac{\pi^2}{12}\log(2) + \frac{1}{6}\log(2)^3, \\ \text{Li}_3(2) = \frac{7}{8}\zeta(3) + \frac{\pi^2}{4}\log(2) - \frac{\pi}{2}i\log(2).$$

which can be found in [Lew81, A.2.6]. We leave the details of the computation to the reader. \square

Remark 2.8. One has $\widehat{\mathcal{L}}_3(u, v) + \widehat{\mathcal{L}}_3(-v, u - v) + \widehat{\mathcal{L}}_3(v - u, -u) = \zeta(3) - \frac{3}{2}\pi^3 i$ modulo $8\pi^3 i$ if $(u, v) \in \widehat{\mathbb{C}}_{--}$.

Lemma 2.9. We have

$$(2.16) \quad \widehat{\mathcal{L}}_2(u, v) + \widehat{\mathcal{L}}_2(-v, u - v) = -\frac{\pi^2}{6} \pmod{\frac{\pi^2}{2}}.$$

Proof. This is proved using elementary properties of the dilogarithm. We omit the details. \square

2.1. Relationship between $\widehat{\mathcal{L}}_n$ and \mathcal{L}_n . For $(u, v) \in \mathbb{C}^2$, let

$$(2.17) \quad \det(u \wedge v) = \operatorname{Re}(u) \operatorname{Im}(v) - \operatorname{Im}(u) \operatorname{Re}(v).$$

The result below relates $\widehat{\mathcal{L}}_n$ to \mathcal{L}_n . It is the key to proving that $\mathfrak{R}_n \circ \widehat{\mathcal{L}}_n$ agrees with $\mathcal{L}_n \circ r$ on $\widehat{\mathcal{B}}_n(\mathbb{C})$. It is a generalization of [DZ06, Prop. 4.6] for $n = 2$.

Theorem 2.10 (Proof in Section 10). *There exist rational numbers $c_{i,j}$ and $d_{i,j}$ such that*

$$(2.18) \quad \begin{aligned} \mathfrak{R}_n(\widehat{\mathcal{L}}_n(u, v)) - \mathcal{L}_n(r(u, v)) &= \sum_{s=1}^{n-2} \left(\mathfrak{R}_{n-s}(\widehat{\mathcal{L}}_{n-s}(u, v)) \sum_{i=0}^s c_{i,s-i} \operatorname{Re}(u)^i \operatorname{Im}(u)^{s-i} \right) + \\ &\quad \det(u \wedge v) \sum_{i=0}^{n-2} d_{i,n-2-i} \operatorname{Re}(u)^i \operatorname{Im}(u)^{n-2-i}. \end{aligned}$$

Explicit formulas for $c_{i,j}$ and $d_{i,j}$ are given in Section 10. For example, we have

$$(2.19) \quad \begin{aligned} \operatorname{Im}(\widehat{\mathcal{L}}_2(u, v)) - \mathcal{L}_2(r(u, v)) &= -\frac{1}{2} \det(u \wedge v) \\ \operatorname{Re}(\widehat{\mathcal{L}}_3(u, v)) - \mathcal{L}_3(r(u, v)) &= \operatorname{Im}(\widehat{\mathcal{L}}_2(u, v)) \operatorname{Im}(u) + \frac{1}{6} \det(u \wedge v) \\ \operatorname{Im}(\widehat{\mathcal{L}}_4(u, v)) - \mathcal{L}_4(r(u, v)) &= -\operatorname{Re}(\widehat{\mathcal{L}}_3(u, v)) \operatorname{Im}(u) + \frac{1}{6} (\operatorname{Re}(u)^2 + 3 \operatorname{Im}(u)^2) + \\ &\quad \frac{1}{24} \det(u \wedge v) (\operatorname{Re}(u)^2 + \operatorname{Im}(u)^2). \end{aligned}$$

2.2. Functional relations. We now define the notion of a *differential $\widehat{\mathcal{L}}_n$ relation*. Such gives rise to a genuine $\widehat{\mathcal{L}}_n$ relation if a certain *realization variety* is positive dimensional (see Proposition 2.18). To motivate our definitions we start with two trivial observations.

Observation 1. Let $\epsilon_1, \epsilon_2 \in \{\pm 1\}$ and let $k_j, l_j \in \mathbb{Z}$ and $a_j \in \mathbb{C}^*$. We then have

$$(2.20) \quad \epsilon_1 \prod_{j=1}^N a_j^{k_j} + \epsilon_2 \prod_{j=1}^N a_j^{l_j} = 1 \implies \left(\sum_{j=1}^N k_j \tilde{a}_j, \sum_{j=1}^N l_j \tilde{a}_j \right) \in \widehat{\mathbb{C}}_{\epsilon_1, \epsilon_2},$$

where \tilde{a}_j denotes any logarithm of a_j , i.e. $\exp(\tilde{a}_j) = a_j$.

The 1-form ω_n from (2.3) is also a 1-form on \mathbb{C}^2 . We shall also denote by ω_n the 1-form on $(\mathbb{C}^2)^K$ restricting to ω_n on each coordinate.

Observation 2. Suppose that $\Gamma^* \omega_n = 0$ for some linear map $\Gamma : \mathbb{C}^N \rightarrow (\mathbb{C}^2)^K$. If Y is a connected smooth manifold and $\tilde{p}_Y : Y \rightarrow \mathbb{C}^N$ is a smooth map with $\Gamma(\tilde{p}_Y(Y)) \subset \widehat{\mathbb{C}}^K$, then the function $\widehat{\mathcal{L}}_n \circ \Gamma \circ \tilde{p}_Y$ is constant on Y .

For $i \in \mathbb{N}$, let \tilde{a}_i and a_i be free variables and consider the polynomial rings

$$(2.21) \quad S = \mathbb{Z}[a_1^{\pm 1}, a_2^{\pm 1}, \dots], \quad \tilde{S} = \mathbb{Z}[\tilde{a}_1, \tilde{a}_2, \dots].$$

We shall think of \tilde{a}_i as a logarithm of a_i . Let $\tilde{S}_1 \subset \tilde{S}$ denote the subgroup generated by the \tilde{a}_i , and let $U(S)$ denote the free multiplicative group on the a_i . We have a canonical group homomorphism

$$(2.22) \quad \pi : \tilde{S}_1 \rightarrow U(S), \quad \tilde{a}_i \mapsto a_i.$$

Definition 2.11. A *symbolic log-pair* is a pair (u, v) of elements in \tilde{S}_1 . The set of symbolic log-pairs is denoted by $\operatorname{SymbLogs}$.

In the following we let

$$(2.23) \quad \alpha = \sum_{i=1}^K r_i [(u_i, v_i)] \in \mathbb{Z}[\text{SymbLogs}], \quad u_i = \sum_{j=1}^N k_{ji} \tilde{a}_j, \quad v_i = \sum_{j=1}^N l_{ji} \tilde{a}_j.$$

Definition 2.12. A *sign determination* for α is a vector $((\epsilon_1^1, \epsilon_2^1), \dots, (\epsilon_1^N, \epsilon_2^N))$ of sign pairs, i.e. $\epsilon_1^i, \epsilon_2^i \in \{\pm 1\}$.

Definition 2.13. Let \mathcal{V}_ϵ be a sign determination for α . The *realization variety* of α with respect to \mathcal{V}_ϵ is the scheme $X_\alpha^{\mathcal{V}_\epsilon}$ over \mathbb{Z} defined by the *realization equations*

$$(2.24) \quad \epsilon_1^i \pi(u_i) + \epsilon_2^i \pi(v_i) = 1 \in S, \quad i = 1, \dots, N.$$

Note that a point in $X_\alpha^{\mathcal{V}_\epsilon}(\mathbb{C})$ is a ring homomorphism $p: S \rightarrow \mathbb{C}$ and thus restricts to a group homomorphism $p_U: U(S) \rightarrow \mathbb{C}^*$.

Definition 2.14. Let p be a point in $X_\alpha^{\mathcal{V}_\epsilon}(\mathbb{C})$. A *lift* of p is a homomorphism $\tilde{p}: \tilde{S}_1 \rightarrow \mathbb{C}$ lifting p_U in the sense that $\exp \circ \tilde{p} = p_U \circ \pi$.

We may regard \tilde{p} as a choice of logarithm of each of the coordinates of p .

Definition 2.15. A *realization* of α is a triple $\tilde{p} = (\mathcal{V}_\epsilon, p, \tilde{p})$ where \mathcal{V}_ϵ is a sign determination for α , p is a smooth point in $X_\alpha^{\mathcal{V}_\epsilon}(\mathbb{C})$ and \tilde{p} is a lift of p .

Remark 2.16. We stress that a realization of α only depends on the terms (u_i, v_i) , not on the coefficients. In particular, a realization of α canonically determines a realization of any multiple of α and of all lower level projections (Definition 2.26).

Given a realization \tilde{p} it follows from Observation 1 that we have elements

$$(2.25) \quad \tilde{p}(\alpha) = \sum_{i=1}^K r_i [(\tilde{p}(u_i), \tilde{p}(v_i))] \in \mathbb{Z}[\widehat{\mathbb{C}}_{\text{signs}}], \quad p(\alpha) = \sum_{i=1}^K r_i [\epsilon_1^i p(\pi(u_i))] \in \mathbb{Z}[\mathbb{C} \setminus \{0, 1\}]$$

satisfying that $r(\tilde{p}(\alpha)) = p(\alpha)$, where r is the covering (2.2). If $Y \subset X_\alpha^{\mathcal{V}_\epsilon}(\mathbb{C})$ is a smooth submanifold and \tilde{p}_Y is a family of lifts of the points in Y , (2.25) gives rise to maps

$$(2.26) \quad \tilde{p}_Y(\alpha): Y \rightarrow \mathbb{Z}[\widehat{\mathbb{C}}_{\text{signs}}], \quad p_Y(\alpha): Y \rightarrow \mathbb{Z}[\mathbb{C} \setminus \{0, 1\}].$$

In particular, $\widehat{\mathcal{L}}_n \circ \tilde{p}_Y(\alpha)$ and $\mathcal{L}_n \circ p_Y(\alpha)$ are functions on Y . Note that if Y is simply connected, one may choose \tilde{p}_Y to be *smooth over Y* meaning that the logarithms of the coordinates are smooth on Y . If so, $\widehat{\mathcal{L}}_n \circ \tilde{p}_Y(\alpha)$ is smooth.

2.2.1. Differential $\widehat{\mathcal{L}}_n$ relations. For an integer $k > 0$ let $\Omega_k^1(\tilde{S})$ denote the group of 1-forms on S of degree k (finite formal sums of terms $f_I d\tilde{a}_i$ where f_I is a degree k monomial). To a symbolic log-pair (u, v) we can assign the one-form (compare with (2.3))

$$(2.27) \quad w_n(u, v) = u^{n-2}(udv - vdu) \in \Omega_{n-1}^1(\tilde{S}).$$

Definition 2.17. We say that α is a *differential $\widehat{\mathcal{L}}_n$ relation* if

$$(2.28) \quad w_n(\alpha) := \sum_{i=1}^K r_i w_n(u_i, v_i) = 0 \in \Omega_{n-1}^1(\tilde{S}).$$

The following is a trivial consequence of Observation 2.

Proposition 2.18. Suppose α is a differential $\widehat{\mathcal{L}}_n$ relation with a realization \tilde{p} . For any simply connected submanifold Y of $X_\alpha^{\mathcal{V}_\epsilon}(\mathbb{C})$, and any smooth lift \tilde{p}_Y over Y the function $\widehat{\mathcal{L}}_n \circ \tilde{p}_Y(\alpha)$ is constant on Y .

2.2.2. *Examples.*

Example 2.19 (The lifted five term relation). Consider the element

$$(2.29) \quad \alpha = [(u_0, v_0)] - [(u_1, v_1)] + [(u_2, v_2)] - [(u_3, v_3)] + [(u_4, v_4)] \in \mathbb{Z}[\text{SymbLogs}]$$

where (u_i, v_i) are the symbolic log-pairs defined by

$$(2.30) \quad \begin{aligned} (u_0, v_0) &= (\tilde{a}_1, \tilde{a}_3), & (u_1, v_1) &= (\tilde{a}_2, \tilde{a}_4), & (u_2, v_2) &= (\tilde{a}_2 - \tilde{a}_1, \tilde{a}_5 - \tilde{a}_1) \\ (u_3, v_3) &= (\tilde{a}_2 + \tilde{a}_3 - \tilde{a}_1 - \tilde{a}_4, \tilde{a}_5 - \tilde{a}_1 - \tilde{a}_4), & (u_4, v_4) &= (\tilde{a}_3 - \tilde{a}_4, \tilde{a}_5 - \tilde{a}_4). \end{aligned}$$

One easily checks that

$$(2.31) \quad w_2(\alpha) = \sum_{i=0}^4 (-1)^i (u_i dv_i - v_i du_i) = 0 \in \Omega_1^1(\tilde{S})$$

so α is a differential $\widehat{\mathcal{L}}_2$ relation. For any realization of α , $\tilde{p}(\alpha) \in \mathbb{Z}[\widehat{\mathbb{C}}_{\text{signs}}]$ is an instance of Neumann's lifted five term relation (see Definition 4.2).

Example 2.20. $[(\tilde{a}_1, \tilde{a}_2)] + (-1)^n [(-\tilde{a}_1, \tilde{a}_2 - \tilde{a}_1)]$ is a differential $\widehat{\mathcal{L}}_n$ relation. The corresponding relation is an instance of (2.12) if $\mathcal{V}_\epsilon = ((-1, 1), (-1, 1))$.

Example 2.21. $[(\tilde{a}_1, \tilde{a}_2)] + [(-\tilde{a}_2, \tilde{a}_1 - \tilde{a}_2)] + [(\tilde{a}_2 - \tilde{a}_1, -\tilde{a}_1)]$ is a differential $\widehat{\mathcal{L}}_3$ relation. The corresponding $\widehat{\mathcal{L}}_3$ relation for $\mathcal{V}_\epsilon = ((1, 1), (1, -1), (-1, 1))$ or any cyclic permutation of \mathcal{V}_ϵ is an instance of (2.14).

Example 2.22. One checks that we have a differential $\widehat{\mathcal{L}}_2$ relation

$$(2.32) \quad [(\tilde{a}_1, \tilde{a}_2 + \tilde{a}_5)] + [(\tilde{a}_2, \tilde{a}_3 + \tilde{a}_1)] + [(\tilde{a}_3, \tilde{a}_4 + \tilde{a}_2)] + [(\tilde{a}_4, \tilde{a}_5 + \tilde{a}_3)] + [(\tilde{a}_5, \tilde{a}_1 + \tilde{a}_4)].$$

For the realization vector with $\epsilon_1^i = -1$ and $\epsilon_2^i = 1$, the corresponding $\widehat{\mathcal{L}}_2$ relation is constant and equal to $-\frac{\pi^2}{2}$ modulo $4\pi^2$. It is a lift of the non-alternating five term relation (3.6) below.

Remark 2.23. The examples above are all lifts of \mathcal{L}_n relations. One can similarly obtain lifts of Goncharov's 22 term relation for \mathcal{L}_3 (see Section 3.5), and the 40 term relation for \mathcal{L}_3 found in [GG⁺13] (see Section 8.3). One can also lift Gangl's 931 term relation for \mathcal{L}_4 to a 931 term relation for $\widehat{\mathcal{L}}_4$. We shall not do this here.

Example 2.24. One has a differential $\widehat{\mathcal{L}}_4$ relation

$$(2.33) \quad [(2\tilde{a}_1 + 2\tilde{a}_3, -\tilde{a}_1 + \tilde{a}_2)] - 8[(\tilde{a}_1 + \tilde{a}_3, -2\tilde{a}_1 + \tilde{a}_2 - \tilde{a}_3)].$$

The corresponding $\widehat{\mathcal{L}}_4$ relations depend not only on the choice of sign determination, but also on the choice of lift. It does not give rise to elements in $\widetilde{R}_4(\mathbb{C})$ (defined below).

2.2.3. *Characterization of differential $\widehat{\mathcal{L}}_n$ relations.* Consider the homomorphism

$$(2.34) \quad \nu: \mathbb{Z}[\text{SymbLogs}] \rightarrow \wedge^2(\tilde{S}_1), \quad (u, v) \mapsto u \wedge v.$$

The following is elementary.

Lemma 2.25. α is a differential $\widehat{\mathcal{L}}_2$ relation if and only if $2\nu(\alpha) = 0$.

Definition 2.26. Let $l \in \{2, \dots, n-1\}$. For integers $j_1, \dots, j_{n-l} \in \{1, \dots, N\}$ the elements

$$(2.35) \quad \pi_{j_1, \dots, j_{n-l}}(\alpha) = \sum_{i=1}^K r_i k_{j_1 i} \cdots k_{j_{n-l} i}(u_i, v_i) \in \mathbb{Z}[\text{SymbLogs}]$$

are called *level l projections* of α . We shall occasionally allow l to be n and define $\pi_\emptyset(\alpha) = \alpha$.

Theorem 2.27 (Proof in Section 6). *For $n > 2$, α is a differential $\widehat{\mathcal{L}}_n$ relation if and only if all level l projections of α (with $l \in \{2, \dots, n\}$) are differential $\widehat{\mathcal{L}}_1$ relations.*

Example 2.28. For α as in Example 2.21 we have

$$(2.36) \quad \pi_1(\alpha) = [(\tilde{a}_1, \tilde{a}_2)] - [(\tilde{a}_2 - \tilde{a}_1, -\tilde{a}_1)], \quad \pi_2(\alpha) = -[(-\tilde{a}_2, \tilde{a}_1 - \tilde{a}_2)] + [(\tilde{a}_2 - \tilde{a}_1, -\tilde{a}_1)].$$

For α as in Example 2.24, one has $\pi_{1,3}(\alpha) = 4[(2\tilde{a}_1 + 2\tilde{a}_3, -\tilde{a}_1 + \tilde{a}_2)] - 8[(\tilde{a}_1 + \tilde{a}_3, -2\tilde{a}_1 + \tilde{a}_2 - \tilde{a}_3)]$. One easily verifies that these are indeed differential $\widehat{\mathcal{L}}_2$ relations.

2.3. The higher extended Bloch groups. We now construct subsets $\widetilde{R}_n(\mathbb{C}) \subset \mathbb{Z}[\widehat{\mathcal{C}}_{\text{signs}}]$ of functional relations for $\widehat{\mathcal{L}}_n$ that map to Goncharov's functional relations $R_n(\mathbb{C})$ for \mathcal{L}_n under the covering map $r: \widehat{\mathcal{C}}_{\text{signs}} \rightarrow \mathbb{C} \setminus \{0, 1\}$. We stress that our definitions, although inspired by complex analysis, are defined purely algebraically and work for arbitrary fields as well as \mathbb{C} (see Section 2.6).

2.3.1. Definition of $\widetilde{R}_n(\mathbb{C})$. The reader is encouraged to compare with the definition of $R_n(\mathbb{C})$ in Definition 3.11 below.

Let α denote an element as in (2.23). Given a realization of α we let $\overline{X_\alpha^{\mathcal{V}_\epsilon}(\mathbb{C})}$ denote the Zariski closure of $X_\alpha^{\mathcal{V}_\epsilon}(\mathbb{C})$ in \mathbb{C}^N . For each point $q \in \overline{X_\alpha^{\mathcal{V}_\epsilon}(\mathbb{C})}$ can define $q(\alpha) \in \mathbb{Z}[\mathbb{C} \cup \{\infty\}]$ as in (2.25). A lift of $q \in \overline{X_\alpha^{\mathcal{V}_\epsilon}(\mathbb{C})}$ is defined as in Definition 2.14 (although it takes values in $\mathbb{C} \oplus \log(0)\mathbb{Z}$, where $\log(0)$ is a symbol).

Definition 2.29. A point $q \in \overline{X_\alpha^{\mathcal{V}_\epsilon}(\mathbb{C})}$ is *zero-degenerate* with respect to α if $q(\alpha) \in \mathbb{Z}\{0\}$. A lift \tilde{q} of a zero-degenerate point q is *permissible* if $\tilde{q}(v_i) = 0$ whenever $q(\pi(u_i)) = 0$.

To define $\widetilde{R}_n(\mathbb{C})$ we need the following concepts whose definition is given in Section 7: *Proper realization*, *equivalence* of realizations, and *proper ambiguity* of a differential $\widehat{\mathcal{L}}_n$ relation.

Definition 2.30. The set $\widetilde{R}_n(\mathbb{C})$ is the subset of $\mathbb{Z}[\widehat{\mathcal{C}}_{\text{signs}}]$ generated by the following two types of relations, where α is a differential $\widehat{\mathcal{L}}_n$ relation with proper ambiguity.

- (1) $\tilde{p}(\alpha) - \tilde{q}(\alpha)$, where \tilde{p} is a proper realization of α and \tilde{q} is any realization equivalent to \tilde{p} .
- (2) $\tilde{p}(\alpha)$, where \tilde{p} is a proper realization such that the component of p in $\overline{X_\alpha^{\mathcal{V}_\epsilon}(\mathbb{C})}$ contains a point q , which is zero-degenerate with respect to α and has a permissible lift.

Example 2.31. Consider the differential $\widehat{\mathcal{L}}_2$ relation

$$(2.37) \quad \alpha = -[(\tilde{a}_3, \tilde{a}_1)] + [(\tilde{a}_4, \tilde{a}_2)] - [(\tilde{a}_5 - \tilde{a}_1, \tilde{a}_2 - \tilde{a}_1)] + [(\tilde{a}_5 - \tilde{a}_1 - \tilde{a}_4, \tilde{a}_2 + \tilde{a}_3 - \tilde{a}_1 - \tilde{a}_4)] - [(\tilde{a}_5 - \tilde{a}_4, \tilde{a}_3 - \tilde{a}_4)]$$

obtained from the $\widehat{\mathcal{L}}_2$ relation in Example 2.19 by applying $[(u, v)] \mapsto -[(v, u)]$. It has proper ambiguity. Its realization variety for the realization vector $\mathcal{V}_\epsilon = ((1, 1), \dots, (1, 1))$ consists of the points $(a_1, \dots, a_5) \in \mathbb{C}^5$ where $a_i \neq 0$ and $a_3 = 1 - a_2$, $a_4 = 1 - a_2$, $a_5 = a_1 - a_2$. A point with $a_1 = a_2$, $a_3 = a_4$, $a_5 = 0$ is zero-degenerate and has a permissible lift given by $(\text{Log}(a_1), \text{Log}(a_2), \text{Log}(a_3), \text{Log}(a_4), \text{Log}(0))$. It thus follows that $\tilde{p}(\alpha) \in \widetilde{R}_2(\mathbb{C})$ for any realization with sign determination \mathcal{V}_ϵ and any realization equivalent to such. These are exactly the *inverted lifted five term relations* (see Section 4.1).

We can now define

$$(2.38) \quad \widehat{\mathcal{P}}_n(\mathbb{C}) = \mathbb{Z}[\widehat{\mathcal{C}}_{\text{signs}}] / \widetilde{R}_n(\mathbb{C}).$$

2.3.2. The main theorems.

Theorem 2.32 (Proof in Section 7). *The projection $\mathbb{Z}[\widehat{\mathbb{C}}_{\text{signs}}] \rightarrow \mathbb{Z}[\mathbb{C} \setminus \{0, 1\}]$ induced by the covering map $r: \widehat{\mathbb{C}}_{\text{signs}} \rightarrow \mathbb{C} \setminus \{0, 1\}$ takes $2\widetilde{R}_n(\mathbb{C})$ to $R_n(\mathbb{C})$.*

Theorem 2.33 (Proof in Section 7). *If β is in $\widetilde{R}_n(\mathbb{C})$ then $\widehat{\mathcal{L}}_n(\beta) = 0$ in $\mathbb{C}/\frac{(\pi i)^n}{(n-1)!}\mathbb{Z}$.*

We have the following homomorphisms lifting (1.5)

$$(2.39) \quad \begin{aligned} \widehat{\nu}_n: \mathbb{Z}[\widehat{\mathbb{C}}_{\text{signs}}] &\rightarrow \widehat{\mathcal{P}}_{n-1}(\mathbb{C}) \otimes \mathbb{C}, & [(u, v)] &\mapsto [(u, v)] \otimes u, & n > 2 \\ \widehat{\nu}_2: \mathbb{Z}[\widehat{\mathbb{C}}_{\text{signs}}] &\rightarrow \wedge^2(\mathbb{C}), & [(u, v)] &\mapsto u \wedge v. \end{aligned}$$

Theorem 2.34 (Proof in Section 7). *$\widehat{\nu}_n$ takes $\widetilde{R}_n(\mathbb{C})$ to 0. We thus have homomorphisms*

$$(2.40) \quad \widehat{\nu}_n: \widehat{\mathcal{P}}_n(\mathbb{C}) \rightarrow \widehat{\mathcal{P}}_{n-1}(\mathbb{C}) \otimes \mathbb{C}, \quad \widehat{\nu}_2: \widehat{\mathcal{P}}_2(\mathbb{C}) \rightarrow \wedge^2(\mathbb{C})$$

for $n > 2$ and $n = 2$, respectively.

Definition 2.35. The *weight n extended Bloch group* $\widehat{\mathcal{B}}_n(\mathbb{C}) \subset \widehat{\mathcal{P}}_n(\mathbb{C})$ is the kernel of $\widehat{\nu}_n$.

Theorem 2.36 (Proof in Section 10). *If $\beta \in \widehat{\mathcal{B}}_n(\mathbb{C})$ we then have*

$$(2.41) \quad \mathfrak{R}_n(\widehat{\mathcal{L}}_n(\beta)) = \mathcal{L}_n(r(\beta)).$$

Proposition 2.37 (Proof in Section 4.1). Assuming the conjecture that $R_2(\mathbb{C})$ is generated by (inverted) five term relations (Conjecture 3.13), one has an isomorphism modulo 2-torsion

$$(2.42) \quad \widehat{\mathcal{B}}_2(\mathbb{C}) \cong \widehat{\mathcal{B}}(\mathbb{C})$$

induced by $[(u, v)] \mapsto -[(v, u)]$.

Remark 2.38. The reason that (2.42) only holds modulo 2-torsion is that Neumann's *transfer relation* may not be zero in $\widehat{\mathcal{B}}_2(\mathbb{C})$ (see Section 4).

2.3.3. *The lifted Bloch complex.* By Theorem 2.34 we have a chain complex $\widehat{\Gamma}(\mathbb{C}, n)$:

$$(2.43) \quad \widehat{\mathcal{P}}_n(\mathbb{C}) \xrightarrow{\delta_1} \dots \xrightarrow{\delta_k} \widehat{\mathcal{P}}_{n-k}(\mathbb{C}) \otimes \wedge^k(\mathbb{C}) \xrightarrow{\delta_{k+1}} \dots \xrightarrow{\delta_{n-2}} \widehat{\mathcal{P}}_2(\mathbb{C}) \otimes \wedge^{n-2}(\mathbb{C}) \xrightarrow{\delta_{n-1}} \wedge^n(\mathbb{C}),$$

with maps given by

$$(2.44) \quad \begin{aligned} \delta_1([(u, v)]) &= [(u, v)] \otimes u, & \delta_{n-1}([(u, v)] \otimes a) &= u \wedge v \wedge a, \\ \delta_k([(u, v)] \otimes a) &= [(u, v)] \otimes u \wedge a & \text{for } 1 < k < n-1. \end{aligned}$$

Moreover, the map

$$(2.45) \quad \widehat{\mathcal{P}}_{n-k}(\mathbb{C}) \otimes \wedge^k(\mathbb{C}) \rightarrow \mathcal{P}_{n-k}(\mathbb{C}) \otimes \wedge^k(\mathbb{C}^*), \quad [(u, v)] \otimes a_1 \cdots \wedge a_k \rightarrow [r(u)] \otimes e^{a_1} \wedge \cdots \wedge e^{a_k}$$

gives rise to a chain map $r: \widehat{\Gamma}(\mathbb{C}, n) \rightarrow \Gamma(\mathbb{C}, n)$.

2.3.4. *A refinement for $\widehat{\mathbb{C}}_{++}$.* One also has subsets $\widetilde{R}_n(\mathbb{C})_{++}$ defined as above, but using only sign determinations where $\epsilon_1^i = \epsilon_2^i = 1$ for all i (and replacing $\widetilde{R}_l(\mathbb{C})$ by $\widetilde{R}_l(\mathbb{C})_{++}$ in (7.1)). One can then define $\widehat{\mathcal{P}}(\mathbb{C})_{++}$, $\widehat{\mathcal{B}}_n(\mathbb{C})_{++}$ and $\widehat{\Gamma}(\mathbb{C}, n)_{++}$. The main advantage of this variant is that Theorem 2.33 holds modulo $\frac{(2\pi i)^n}{(n-1)!}$ instead of $\frac{(\pi i)^n}{(n-1)!}$ (see Remark 7.21).

2.4. Relations from quivers. We refer to Section 8 for details and terminology regarding quivers and mutation. For a quiver Q_0 with m vertices, let $\text{Mut}(Q_0)$ denote the set of triples (Q, a, k) where k is a mutable vertex and (Q, a) is a seed which is mutation equivalent to the initial seed of Q_0 . The set of \mathcal{A} -coordinates (see Definition 8.8) of Q_0 is the set of all coordinates in the seeds appearing in $\text{Mut}(Q_0)$. By enumerating the \mathcal{A} -coordinates we can associate to each \mathcal{A} -coordinate one of the generators a_i of the polynomial ring S in (2.21) and its lift $\tilde{a}_i \in \tilde{S}$.

The key observation is that the *exchange relation* (see Definition 8.2)

$$(2.46) \quad a'_k = \frac{1}{a_k} \left(\prod_{j|\varepsilon_{kj}>0} a_j^{\varepsilon_{kj}} + \prod_{j|\varepsilon_{kj}<0} a_j^{-\varepsilon_{kj}} \right)$$

for the seed mutation $(Q, a) \mapsto (\mu_k(Q), \mu_k(a))$ is equivalent to the realization equation (2.24) for the symbolic log-pair

$$(2.47) \quad \hat{\mathcal{X}}(Q, a, k) = \left(\sum_j \varepsilon_{kj} \tilde{a}_j, \tilde{a}_k + \tilde{a}'_k + \sum_{j|\varepsilon_{kj}<0} \varepsilon_{kj} \tilde{a}_j \right)$$

with respect to the sign determination $(-1, 1)$. Note that if $\mathcal{X}(Q, a, k) = (u, v)$ then

$$(2.48) \quad \hat{\mathcal{X}}(\mu_k(Q), \mu_k(a)) = (-u, v - u),$$

and that $\pi(u) = \prod_j a_j^{\varepsilon_{kj}}$ is the \mathcal{X} -coordinate (see Definition 8.9) of (Q, a, k) . Define

$$(2.49) \quad \hat{\mathcal{X}}(Q_0, n) = \frac{\mathbb{Z}[\hat{\mathcal{X}}(\text{Mut}(Q_0))]}{\langle (u, v) + (-1)^n (-u, v - u) \rangle},$$

where $\hat{\mathcal{X}}(\text{Mut}(Q_0)) \subset \text{SymbLogs}$ is the image of (2.47). We may think of it as the free abelian group generated by the \mathcal{X} -coordinates of Q_0 up to inversion. The map

$$(2.50) \quad w_n: \mathbb{Z}[\text{SymbLogs}] \rightarrow \Omega_{n-1}^1(\tilde{S}), \quad (u, v) \mapsto u^{n-2}(udv - vdu)$$

induces a map

$$(2.51) \quad w_n: \hat{\mathcal{X}}(Q_0, n) \rightarrow \Omega_{n-1}^1(\tilde{S}),$$

and we have the following result (see Section 8.3), which is an immediate consequence of our definitions.

Proposition 2.39. Every element in the kernel of w_n is a differential $\hat{\mathcal{L}}_n$ relation.

2.4.1. Examples. There is a quiver $Q_{p,q}$ associated to a Grassmannian $\text{Gr}(p, q)$ (see Section 8.1). If Q_0 is the quiver of $\text{Gr}(2, 5)$, $\mathcal{X}(Q_0, 2)$ is five dimensional, and the kernel of w_2 is one dimensional generated by a five term relation, which is an instance of the relation in Example 2.22. If Q_0 is the quiver of $\text{Gr}(3, 6)$, $\mathcal{X}(Q_0, 3)$ is 52 dimensional, and the kernel of w_3 is one-dimensional generated by a 40 term relation (see Section 8.3). It is a lift of the 40 term relation for \mathcal{L}_3 found in [GG⁺13].

Remark 2.40. One may speculate whether the quivers for $\text{Gr}(n, 2n)$ or $\text{Gr}(n, 2n + 1)$ give rise to $\hat{\mathcal{L}}_n$ relations in a similar fashion, but since the mutation class is infinite for $n > 3$, this can not be verified by exhaustive search.

2.5. A lift of Goncharov's regulator. Let $\widetilde{\text{Gr}}(p, q)$ denote the affine cone over the Grassmannian of p -planes in q -space. An element can be represented by a $p \times q$ matrix defined up to the action by $\text{SL}(p)$. For each $i = 1, \dots, q$ one has a (simplicial) map $\delta_i: \widetilde{\text{Gr}}(p, q) \rightarrow \widetilde{\text{Gr}}(p, q-1)$ given by omitting the i th column of a representing matrix, and one has a chain complex $G_*(p)$ given by

$$(2.52) \quad G_q(p) = \mathbb{Z}[\widetilde{\text{Gr}}(p, q+1)(\mathbb{C})], \quad \partial: G_q(p) \rightarrow G_{q-1}(p), \quad \partial = \sum_{i=1}^q (-1)^{i-1} \delta_i.$$

Goncharov [Gon94] constructed a commutative diagram

$$(2.53) \quad \begin{array}{ccccccc} G_6(3) & \longrightarrow & G_5(3) & \longrightarrow & G_4(3) & \longrightarrow & G_3(3) \\ \downarrow & & \downarrow g_5 & & \downarrow g_4 & & \downarrow g_3 \\ 0 & \longrightarrow & \mathcal{P}_3(\mathbb{C})_{\mathbb{Q}} & \longrightarrow & (\mathcal{P}_2(\mathbb{C}) \otimes \mathbb{C}^*)_{\mathbb{Q}} & \longrightarrow & \wedge^3(\mathbb{C}^*)_{\mathbb{Q}}. \end{array}$$

The definition of the g_i are reviewed in Section 9. For each p , there is a canonical map $\Gamma: H_*(\text{SL}(p, \mathbb{C})) \rightarrow H_*(G_*(p))$, and Goncharov showed that the composition

$$(2.54) \quad H_5(\text{SL}(3, \mathbb{C})) \xrightarrow{\Gamma} H_5(G_*(3)) \xrightarrow{g_5} \mathcal{B}_3(\mathbb{C})_{\mathbb{Q}} \xrightarrow{\mathcal{L}_3} \mathbb{R}$$

is a rational multiple of the Borel regulator. Defining $\Gamma_i(\mathbb{C}, n) = \Gamma^{2n-i}(\mathbb{C}, n)$ one may view (2.53) as a chain map $G_*(3) \rightarrow \Gamma_*(\mathbb{C}, 3)_{\mathbb{Q}}$. We shall define a map $f_*: G_*(3) \rightarrow \widehat{\Gamma}_*(\mathbb{C}, 3)$ such that the composition with $\widehat{\Gamma}_*(\mathbb{C}, 3) \rightarrow \Gamma_*(\mathbb{C}, 3)_{\mathbb{Q}}$ agrees with Goncharov's on H_5 . We suspect that the composition

$$(2.55) \quad H_5(\text{SL}(3, \mathbb{C})) \xrightarrow{\Gamma} H_5(G_*(3)) \xrightarrow{f_5} \widehat{\mathcal{B}}_3(\mathbb{C}) \xrightarrow{\widehat{\mathcal{L}}_3} \mathbb{C}/\frac{(\pi i)^3}{2} \mathbb{Z}$$

agrees with the universal Cheeger-Simons class \widehat{c}_3 [CS85].

2.6. Other fields. In Zickert [Zic15], an extended Bloch group $\widehat{\mathcal{B}}(F)$ was defined for an arbitrary field F together with a \mathbb{Z} -extension

$$(2.56) \quad 0 \rightarrow \mathbb{Z} \rightarrow E \xrightarrow{\pi} F^* \rightarrow 1$$

of the unit group F^* of F . Given such one can define

$$(2.57) \quad \widehat{F}_{\text{signs}} = \{(u, v) \in E \times E \mid \epsilon_1 \pi(u) + \epsilon_2 \pi(v) = 1, \epsilon_1, \epsilon_2 \in \{-1, 1\} \subset F\}.$$

A *realization* of a differential $\widehat{\mathcal{L}}_n$ relation α can now be defined as in Definition 2.15 but with p being a smooth point in $X_{\alpha}^{\mathcal{V}_{\epsilon}}(F)$ and lifts \widetilde{p} of p taking values in E . All definitions carry over to this setup, e.g. we can define $\widehat{R}_n(F)$, $\widehat{\mathcal{P}}_n(F)$, $\widehat{\mathcal{B}}_n(F)$ and $\widehat{\Gamma}(F, n)$ (which is now only a chain complex modulo 2-torsion).

Our main exposition focuses on $F = \mathbb{C}$ for simplicity, and because this is our main interest.

Remark 2.41. We have omitted the dependence on E from the notation. Zickert [Zic15] showed that if F^* is free modulo torsion, and E is primitive, then $\widehat{\mathcal{B}}(F)$ is independent of E up to canonical isomorphism. We suspect this to hold for the higher analogues as well.

3. THE BLOCH GROUP AND GONCHAROV'S HIGHER BLOCH COMPLEXES

3.1. The classical Bloch group. We use the conventions of Suslin [Sus90]. Let F be a field. Let $\mathcal{A}(F)$ be the kernel of the homomorphism

$$(3.1) \quad \nu: \mathbb{Z}[F \setminus \{0, 1\}] \rightarrow \wedge^2(F^*), \quad [z] \mapsto z \wedge (1 - z),$$

and let $FT(F)$ be the subgroup of $\mathbb{Z}[F \setminus \{0, 1\}]$ generated by *five term relations*, i.e. elements of the form

$$(3.2) \quad FT[x, y] = [x] - [y] + \left[\frac{y}{x}\right] - \left[\frac{1-x^{-1}}{1-y^{-1}}\right] + \left[\frac{1-x}{1-y}\right], \quad x \neq y \in F \setminus \{0, 1\}.$$

One can check that $\nu(FT[x, y]) = 0$, so $FT(F)$ is a subgroup of $\mathcal{A}(F)$.

Definition 3.1. The *Bloch group* of F is the group $\mathcal{B}(F) = \mathcal{A}(F)/FT(F)$. The group $\mathcal{P}(F) = \mathbb{Z}[F \setminus \{0, 1\}]/FT(F)$ is called the *pre-Bloch group* of F .

If $F = \mathbb{C}$, one can show that $\mathcal{L}_2(FT[x, y]) = 0$, so we may regard \mathcal{L}_2 as a map $\mathcal{P}(\mathbb{C}) \rightarrow \mathbb{R}$.

Theorem 3.2 ([Sus90]). *If F is infinite there is an exact sequence*

$$(3.3) \quad 0 \rightarrow \tilde{\mu}_F \rightarrow K_3^{\text{ind}}(F) \rightarrow \mathcal{B}(F) \rightarrow 0.$$

In particular, $K_3^{\text{ind}}(F)_{\mathbb{Q}} \cong \mathcal{B}(F)_{\mathbb{Q}}$. Here $\tilde{\mu}_F$ denotes the non-trivial \mathbb{Z}_2 -extension of the group μ_F of roots of unity in F (except when F has characteristic 2, where $\tilde{\mu}_F$ denotes μ_F).

One can show (see [Sus90, DS82]) that $c = [x] + [1-x] \in \mathcal{B}(F)$ is independent of x and that

$$(3.4) \quad 2([x] + [x^{-1}]) = [x^2] + [x^{-2}] \in \mathcal{B}(F), \quad 6c = 0 \in \mathcal{B}(F).$$

It follows that if we ignore 2-torsion we have

$$(3.5) \quad [x] = \left[\frac{1}{1-x}\right] = \left[1 - \frac{1}{x}\right] = -[x^{-1}] = -[1-x] = -\left[\frac{x}{x-1}\right] \in \mathcal{P}(F)/\langle c \rangle.$$

Remark 3.3. There is also a *non-alternating five term relation* (see e.g. [Lew91, p. 245], [GGs+13, eq. (4.12)])

$$(3.6) \quad FT^+[x, y] = [-x] + [-y] + \left[-\frac{1+y}{x}\right] + \left[-\frac{1+x+y}{xy}\right] + \left[-\frac{1+x}{y}\right].$$

Note that $FT^+[x, y] = FT[-x, y+1] + 2c$, so $FT^+[x, y] = 0 \in \mathcal{B}(F)/\langle c \rangle$.

3.2. An alternative variant. Let $\mathcal{P}'(F)$ and $\mathcal{B}'(F)$ be the quotients of $\mathbb{Z}[F \setminus \{0, 1\}]$, respectively, $\mathcal{A}(F)$ by the subgroup generated by elements $\phi(FT[x, y])$, where $\phi([x]) = -[1-x]$. The element $c = [x] + [1-x] \in \mathcal{B}'(F)$ is still independent of x , and one easily shows that $FT[x, y] = c \in \mathcal{B}'(F)$ and that ϕ induces isomorphisms $\mathcal{P}(F) \cong \mathcal{P}'(F)$ and $\mathcal{B}(F) \cong \mathcal{B}'(F)$. One easily checks that the following holds in $\mathcal{B}'(F)$ modulo 2-torsion

$$(3.7) \quad [x] + [x^{-1}] = 2c, \quad [x] - \left[\frac{1}{1-x}\right] = -c, \quad FT^+[x, y] = 0.$$

3.3. The higher Bloch complexes. We stress that our definition is slightly different from that of Goncharov since we (like Suslin) don't allow the elements $[0]$, $[1]$ and $[\infty]$. All of our relations are also relations in the sense of Goncharov, so Goncharov's $\mathcal{P}_n(F)$ are quotients of ours. The main difference is that the element $2([x] + (-1)^n[x^{-1}])$ may be non-zero in $\mathcal{P}_n(F)$ (see Remark 3.15) and that our $\mathcal{B}_2(F)$ is isomorphic to $\mathcal{B}(F)$ on the nose (pending Conjecture 3.13 that $R_2(F)$ is generated by (inverted) 5-term relations).

3.3.1. *Goncharov's original definition* [Gon95, p. 221]. Let F be a field, and let $P_F^1 = F \cup \{\infty\}$. The complex involves groups $\mathcal{P}_n(F)$, which are defined using subgroups $\mathcal{A}_n(F)$ and $R_n(F)$ of $\mathbb{Z}[P_F^1]$, that are defined inductively starting with $n = 2$. Note that if X is a smooth curve over F with function field $F(X)$ and p is a point in X then p defines an evaluation map

$$(3.8) \quad p: \mathbb{Z}[F(X)] \rightarrow \mathbb{Z}[P_F^1].$$

Assume that $\mathcal{A}_n(K)$ has been defined for every field K . Define $R_n(F)$ to be the subgroup generated by $[0]$, $[\infty]$, and all elements of the form $p_1(\alpha) - p_2(\alpha)$ where p_1 and p_2 are smooth points in a geometrically irreducible curve X over F , and α is an element in $\mathcal{A}_2(F(X))$. One can then define $\mathcal{P}_n(F) = \mathbb{Z}[P_F^1]/R_n(F)$. It remains to give the inductive definition of $\mathcal{A}_n(F)$. If, for some $n > 2$, $\mathcal{A}_{n-1}(F)$ has been defined, we can define a homomorphism

$$(3.9) \quad \nu_{n,F}: \mathbb{Z}[P_F^1] \rightarrow \mathcal{P}_{n-1}(F) \otimes_{\mathbb{Z}} F^*, \quad [x] \mapsto [x] \otimes x, \quad [0], [1], [\infty] \mapsto 0,$$

and define $\mathcal{A}_n(F) = \text{Ker}(\nu_{n,F})$. Finally, define $\mathcal{A}_2 = \text{Ker}(\nu_{2,F})$, where $\nu_{2,F}$ is the map in (3.1) extended to $\mathbb{Z}[P_F^1]$ by taking $[0]$, $[1]$ and $[\infty]$ to 0.

Example 3.4. A simple induction argument shows that $2([t] + (-1)^n[t^{-1}]) \in \mathcal{A}_n(F(t))$ for all n . Hence, for any $x \in P_F^1$, $2([x] + (-1)^n[x^{-1}]) = 2([0] + (-1)^n[\infty]) = 0 \in \mathcal{P}_n(F)$. As we shall see below (Remark 3.15), this element may be non-zero in our version of $\mathcal{P}_n(F)$.

Example 3.5. One easily checks that $[t] + [1-t] \in \mathcal{A}_2(F(t))$. Hence, the element $c = [x] + [1-x] \in \mathcal{B}_2(F)$ is independent of $x \in P_F^1$. In fact, it is 0 modulo 2-torsion.

Goncharov shows that $\nu_{n,F}$ induces maps $\mathcal{P}_n(F) \rightarrow \mathcal{P}_{n-1}(F) \otimes F^*$, and we thus have a chain complex $\Gamma(F, n)$ with $\mathcal{P}_n(F)$ in degree 1 and $\wedge^n(F^*)$ in degree n

$$(3.10) \quad \mathcal{P}_n(F) \xrightarrow{\delta_1} \dots \xrightarrow{\delta_k} \mathcal{P}_{n-k}(F) \otimes \wedge^k(F^*) \xrightarrow{\delta_{k+1}} \dots \xrightarrow{\delta_{n-2}} \mathcal{P}_2(F) \otimes \wedge^{n-2}(F^*) \xrightarrow{\delta_{n-1}} \wedge^n(F^*)$$

modulo 2-torsion. The maps δ_i are defined in (1.11).

Goncharov also shows that all elements in $R_n(\mathbb{C})$ are functional relations for \mathcal{L}_n , so there is a map $\mathcal{L}_n: \mathcal{P}_n(\mathbb{C}) \rightarrow \mathbb{R}$.

Conjecture 3.6 ([Gon94, Conj. 2.1]). For $i = 1, \dots, n$ there are isomorphisms

$$(3.11) \quad H^i(\Gamma(F, n)_{\mathbb{Q}}) \cong K_{2n-i}^{(n)}(F)_{\mathbb{Q}},$$

and when $F = \mathbb{C}$ and $i = 1$, the map \mathcal{L}_n agrees with the regulator reg_n .

Definition 3.7. The n th Bloch group $\mathcal{B}_n(F)$ is the group $\text{Ker}(\delta_1) = H^1(\Gamma(F, n))$.

Conjecture 3.8 ([Gon95, Conj 1.20]). The group $R_2(F)$ is the subgroup generated by elements $FT[x, y]$, where $x, y \in P_F^1$ (with the conventions that $\frac{0}{0} = 1, \frac{0}{\infty} = 0$, etc.).

Note that in $\mathcal{B}_2(F)$ we have $FT[x, y] = FT[x, 0] = [x] + [1-x] = c$, whereas in $\mathcal{B}(F)$ the element $FT[x, y]$ is zero and c is an element of order dividing 6.

Proposition 3.9. If Conjecture 3.8 holds, the inclusion $F \setminus \{0, 1\} \rightarrow P_F^1$ induces an isomorphism

$$(3.12) \quad \mathcal{B}(F)/\langle c \rangle \cong \mathcal{B}_2(F)$$

modulo 2-torsion, where $c = [x] + [1-x]$.

Proof. For any fixed y , $FT[x, y] \in \mathcal{A}(F(x))$, so all five term relations are equal in $\mathcal{B}_2(F)$. In particular, $FT[x, y] = FT[x, x] = [1]$. Since $[x^2] + [x^{-2}] \in \mathcal{A}_2(F(x))$, $2[1] = [0] + [\infty] = 0 \in \mathcal{P}_2(F)$, so $[1] = 0$ (modulo 2-torsion). This shows that the map $\mathcal{B}(F) \rightarrow \mathcal{B}_2(F)$ is well defined and surjective. Assuming Conjecture 3.8, the kernel is generated by elements $FT[x, y]$ where $\{x, y\} \cap \{0, 1, \infty\} \neq \emptyset$. One now checks that all such elements are in the group generated by c . For example, $FT[x, \infty] = [x] - [1 - x^{-1}] = [x] + [x^{-1}] - ([x^{-1}] + [1 - x^{-1}]) = -c$. \square

3.3.2. Our definition.

Definition 3.10. Let X be a curve over F and let $\alpha \in \mathcal{A}_n(F(X))$. A point p in X is called *non-degenerate* with respect to α if $p(\alpha)$ is in $\mathbb{Z}[F \setminus \{0, 1\}]$, and *zero-degenerate* if $p(\alpha)$ is in $\mathbb{Z}[\{0\}]$.

Definition 3.11. Define $\mathcal{A}_n(F)$ and $\mathcal{P}_n(F)$ as above, but using $F \setminus \{0, 1\}$ instead of F_F^1 . Define $R_n(F)$ as the subgroup of $\mathbb{Z}[F \setminus \{0, 1\}]$ generated by the following two types of elements:

- (1) $p(\alpha) - q(\alpha)$ where p and q are smooth points in a geometrically irreducible smooth curve X over F that are non-degenerate with respect to some $\alpha \in \mathcal{A}_n(F(X))$.
- (2) $p(\alpha)$ with p and α as above if X contains a point which is zero-degenerate with respect to α .

The groups $\mathcal{P}_k(F)$ fit into a chain complex as in (3.10).

Example 3.12. Let $\phi([x]) = -[1 - x]$. For any fixed $y \in F \setminus \{0, 1\}$, one easily checks that

$$(3.13) \quad \phi(FT[x, y]) = -[1 - x] + [1 - y] - \left[\frac{x - y}{x}\right] + \left[\frac{x - y}{x(1 - y)}\right] - \left[\frac{x - y}{1 - y}\right]$$

is in $\mathcal{A}_2(F(x))$, and that $x = y$ is a zero-degenerate point. Hence, $\phi(FT[x, y]) \in R_2(F)$ for all $x, y \in F \setminus \{0, 1\}$. We call such a relation an *inverted five term relation*.

The analogue of Conjecture 3.8 is the following.

Conjecture 3.13. The group $R_2(F)$ is the subgroup generated by inverted five term relations.

Proposition 3.14. Assuming Conjecture 3.13, we have an isomorphism $\mathcal{B}(F) \cong \mathcal{B}_2(F)$ induced by $\phi: [x] \mapsto -[1 - x]$.

Proof. Let $\mathcal{B}'(F)$ be the group defined in Section 3.2. Conjecture 3.13 implies that $\mathcal{B}'(F)$ is equal to $\mathcal{B}_2(F)$. Since $\mathcal{B}'(F) \cong \mathcal{B}(F)$ induced by ϕ , the result follows. \square

Remark 3.15. If F is totally real c has order 6 [Sus90]. Hence, the element $2([x] + (-1)^n[x^{-1}])$ may be non-zero in our variant of $\mathcal{P}_n(F)$. We suspect that its order is finite and divides the order of $K_{2n-1}(\mathbb{Q})$; see Remark 2.6.

3.4. Goncharov's 22 term relation. For $\alpha_1, \alpha_2, \alpha_3$ in F and $\beta_i = 1 - \alpha_i(1 - \alpha_{i-1})$, we may write Goncharov's 22 term relation [Gon95] as follows (see [Zag91, p. 428])

$$(3.14) \quad \mathcal{R}_3(\alpha_1, \alpha_2, \alpha_3) = A_1 + A_2 + A_3 + [-\alpha_1\alpha_2\alpha_3],$$

where (indices modulo 3)

$$(3.15) \quad A_i = [\alpha_i] + [\beta_i] - \left[\frac{\alpha_{i-1}}{\beta_i}\right] + \left[\frac{\beta_i}{\alpha_{i-1}\alpha_i}\right] + \left[\frac{\alpha_i\beta_{i-1}}{\beta_{i+1}}\right] + \left[-\frac{\beta_i}{\alpha_i\beta_{i-1}}\right] - \left[\frac{\alpha_{i-1}\alpha_i\beta_{i+1}}{\beta_i}\right].$$

We have replaced β_i/α_{i-1} in [Zag91] by α_{i-1}/β_i in order to make Lemma 3.16 below hold modulo 2 torsion instead of 6 torsion. If $F = \mathbb{C}$, one has

$$(3.16) \quad \mathcal{L}_3(\mathcal{R}_3(\alpha_1, \alpha_2, \alpha_3)) = 3\zeta(3).$$

We may also regard $\mathcal{R}_3(\alpha_1, \alpha_2, \alpha_3)$ as an element of $\mathbb{Z}[F(\alpha_1, \alpha_2, \alpha_3) \setminus \{0, 1\}]$. The result below was stated without proof (modulo 6 torsion) in [Gon95, Thm 1.3].

Lemma 3.16. Modulo 2-torsion we have

$$(3.17) \quad \nu_{3,F(\alpha_1, \alpha_2, \alpha_3)}(\mathcal{R}_3(\alpha_1, \alpha_2, \alpha_3)) = 0 \in \mathcal{P}'(F(\alpha_1, \alpha_2, \alpha_3)) \otimes F(\alpha_1, \alpha_2, \alpha_3)^*.$$

Proof. Modulo 2-torsion one has

$$(3.18) \quad \nu_{3,F(\alpha_1, \alpha_2, \alpha_3)}(\mathcal{R}_3(\alpha_1, \alpha_2, \alpha_3)) = \sum_{i=1}^3 R_{1,i} \otimes \alpha_i + \sum_{i=1}^3 R_{2,i} \otimes \beta_i,$$

where

$$(3.19) \quad \begin{aligned} R_{1,i} &= [\alpha_i] - \left[\frac{\beta_i}{\alpha_{i-1}\alpha_i} \right] + \left[\frac{\alpha_i\beta_{i-1}}{\beta_{i+1}} \right] - \left[-\frac{\beta_i}{\alpha_i\beta_{i-1}} \right] - \left[\frac{\alpha_{i-1}\alpha_i\beta_{i+1}}{\beta_i} \right] + \\ &\quad - \left[\frac{\alpha_i}{\beta_{i+1}} \right] - \left[\frac{\beta_{i+1}}{\alpha_i\alpha_{i+1}} \right] - \left[\frac{\alpha_i\alpha_{i+1}\beta_{i-1}}{\beta_{i+1}} \right] + [-\alpha_1\alpha_2\alpha_3] \\ R_{2,i} &= [\beta_i] + \left[\frac{\alpha_{i-1}}{\beta_i} \right] + \left[\frac{\beta_i}{\alpha_{i-1}\alpha_i} \right] + \left[-\frac{\beta_i}{\alpha_i\beta_{i-1}} \right] + \left[\frac{\alpha_{i-1}\alpha_i\beta_{i+1}}{\beta_i} \right] + \\ &\quad \left[\frac{\alpha_{i+1}\beta_i}{\beta_{i-1}} \right] - \left[-\frac{\beta_{i+1}}{\alpha_{i+1}\beta_i} \right] - \left[\frac{\alpha_{i-1}\beta_{i+1}}{\beta_i} \right] - \left[\frac{\alpha_{i-1}\alpha_{i+1}\beta_i}{\beta_{i-1}} \right]. \end{aligned}$$

Letting $\gamma_i = 1 - \alpha_i$ and $\delta = 1 + \alpha_1\alpha_2\alpha_3$ and using (3.7) one checks that

$$(3.20) \quad \begin{aligned} R_{1,i} &= -FT\left[\gamma_{i+1}, \frac{\beta_{i+1}}{\delta}\right] - FT\left[\frac{\beta_{i+1}}{\delta}, \beta_{i+1}\right] + FT[\gamma_{i+1}, \beta_{i+1}] + c = 0 \in \mathcal{P}'(F(\alpha_1, \alpha_2, \alpha_3)), \\ R_{2,i} &= FT\left[\alpha_i^{-1}, \frac{\beta_{i-1}}{\delta}\right] + FT\left[\frac{\beta_{i-1}}{\delta}, \gamma_{i-1}\right] - FT[\alpha_i^{-1}, \gamma_{i-1}] - c = 0 \in \mathcal{P}'(F(\alpha_1, \alpha_2, \alpha_3)). \end{aligned}$$

This proves the result. \square

Remark 3.17. Goncharov shows that $R(\alpha_1, \alpha_2, \alpha_3) = \mathcal{R}(\alpha_1, \alpha_2, \alpha_3) - 3[1] \in R_3(F)$. Since we don't allow [1], we can replace [1] by $[x] + [\frac{1}{1-x}] + [1 - \frac{1}{x}]$. We shall not need this.

3.5. Lifting the 22 term relation. We now lift Goncharov's relation to a relation for $\widehat{\mathcal{L}}_3$. This involves a few concepts that are not introduced until later in the paper.

With $\alpha_i, \beta_i, \gamma_i$ and δ as above, one easily checks that

$$(3.21) \quad \begin{aligned} 1 - \alpha_i &= \gamma_i, \quad 1 - \beta_i = \alpha_i\gamma_i, \quad 1 - \frac{\alpha_{i-1}}{\beta_i} = \frac{\gamma_{i-1}\gamma_i}{\beta_i}, \quad 1 - \frac{\beta_i}{\alpha_{i-1}\alpha_i} = -\frac{\gamma_i}{\alpha_{i-1}\alpha_i} \\ 1 - \frac{\alpha_i\beta_{i-1}}{\beta_{i+1}} &= \frac{\gamma_{i+1}\beta_i}{\beta_{i+1}}, \quad 1 + \frac{\beta_i}{\alpha_i\beta_{i-1}} = \frac{\delta}{\alpha_i\beta_{i-1}}, \quad 1 - \frac{\alpha_i\alpha_{i-1}\beta_{i+1}}{\beta_i} = \frac{\delta\gamma_i}{\beta_i}, \quad 1 + \alpha_1\alpha_2\alpha_3 = \delta. \end{aligned}$$

By introducing *free* variables $\alpha_i, \beta_i, \gamma_i$ and δ and their lifts $\tilde{\alpha}_i, \tilde{\beta}_i, \tilde{\gamma}_i$ and $\tilde{\delta}$, the relations (3.21) lead us to associate a symbolic log-pair to each of the terms in $\mathcal{R}_3(\alpha_1, \alpha_2, \alpha_3)$. In particular, we have an element

$$(3.22) \quad \alpha = \tilde{A}_1 + \tilde{A}_2 + \tilde{A}_3 + [(\tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3, \tilde{\delta})] \in \mathbb{Z}[\text{SymbLogs}],$$

where

$$(3.23) \quad \begin{aligned} \tilde{A}_i &= [(\tilde{\alpha}_i, \tilde{\gamma}_i)] + [(\tilde{\beta}_i, \tilde{\alpha}_i + \tilde{\gamma}_i)] - [(\tilde{\alpha}_{i-1} - \tilde{\beta}_i, \tilde{\gamma}_{i-1} + \tilde{\gamma}_i - \tilde{\beta}_i)] + \\ &\quad [(\tilde{\beta}_i - \tilde{\alpha}_{i-1} - \tilde{\alpha}_i, \tilde{\gamma}_i - \tilde{\alpha}_{i-1} - \tilde{\alpha}_i)] + [(\tilde{\alpha}_i + \tilde{\beta}_{i-1} - \tilde{\beta}_{i+1}, \tilde{\gamma}_{i+1} + \tilde{\beta}_i - \tilde{\beta}_{i+1})] + \\ &\quad [(\tilde{\beta}_i - \tilde{\alpha}_i - \tilde{\beta}_{i-1}, \tilde{\delta} - \tilde{\alpha}_i - \tilde{\beta}_{i-1})] - [(\tilde{\alpha}_{i-1} + \tilde{\alpha}_i + \tilde{\beta}_{i+1} - \tilde{\beta}_i, \tilde{\delta} + \tilde{\gamma}_i - \tilde{\beta}_i)]. \end{aligned}$$

If we index the first 21 terms of α by (i, j) with $i \in \{1, 2, 3\}$ and $j \in \{1, \dots, 7\}$ and the last by $i = 22$, it follows from (3.21) that $p(\alpha)$ is an instance of $\mathcal{R}(\alpha_1, \alpha_2, \alpha_3)$ for any realization \tilde{p} of α with sign determination \mathcal{V}_ϵ given by

$$(3.24) \quad \begin{aligned} (\epsilon_1^{(i,j)}, \epsilon_2^{(i,j)}) &= (1, 1) \text{ if } j \in \{1, 2, 3, 5, 7\}, \\ (\epsilon_1^{(4,i)}, \epsilon_2^{(4,i)}) &= (1, -1), \quad (\epsilon_1^{(6,i)}, \epsilon_2^{(6,i)}) = (-1, 1), \quad (\epsilon_1^{22}, \epsilon_2^{22}) = (-1, 1). \end{aligned}$$

One similarly has lifts $\tilde{R}_{1,i}$ and $\tilde{R}_{2,i}$ of the elements $R_{1,i}$ and $R_{2,i}$. Note that these are the (non-zero) level 2 projections (Definition 2.26) of α .

Terminology 3.18. In this section a *good* realization means a realization with sign determination as in (3.24).

Proposition 3.19. The element α is a differential $\widehat{\mathcal{L}}_3$ relation with proper ambiguity (Definition 7.4). Moreover, any good realization is a proper realization (Definition 7.7) of 2α .

Proof. By inspecting the terms one checks that each monomial one form appears with non-trivial coefficient in either 0, 2, 3, or 4 of the expressions $w_3(u_i, v_i)$ and that these coefficients always cancel out. For example, the one form $\tilde{\alpha}_1^2 d\tilde{\gamma}_1$ appears in $w_3(\tilde{\alpha}_1, \tilde{\gamma}_1)$, $w_3(\tilde{\alpha}_1 - \tilde{\beta}_2, \tilde{\gamma}_1 + \tilde{\gamma}_2 - \tilde{\beta}_2)$, $w_3(\tilde{\beta}_1 - \tilde{\alpha}_3 - \tilde{\alpha}_1, \tilde{\gamma}_1 - \tilde{\alpha}_3 - \tilde{\alpha}_1)$, and $w_3(\tilde{\alpha}_3 + \tilde{\alpha}_1 + \tilde{\beta}_2 - \tilde{\beta}_1, \tilde{\delta} + \tilde{\gamma}_1 - \tilde{\beta}_1)$ with coefficients in $w_3(\alpha)$ being 1, -1 , 1, and -1 , respectively. This proves the first statement. When writing α in the form (2.23) with $K = 22$, $N = 10$, a simple inspection shows that each of the 10 integers $\sum_i r_i k_{ji}^2 l_{ji}$ is either 0 or 3, proving the second statement. For the third statement, note that (3.20) shows that $2R_{1,i}$ and $2R_{2,i}$ are linear combinations of inverted five term relations. A natural lift of (3.20) similarly shows that good realizations of $2\tilde{R}_{1,i}$ and $2\tilde{R}_{2,i}$ are linear combinations of inverted lifted five term relations (see Section 4.1). This concludes the proof. \square

Corollary 3.20. For a good realization \tilde{p} (or any realization sign equivalent to such) one has

$$(3.25) \quad \widehat{\mathcal{L}}_3(\tilde{p}(\alpha)) = 3\zeta(3) \in \mathbb{C}/\frac{(\pi i)^3}{2}\mathbb{Z}.$$

Proof. If \tilde{p} is a good realization it follows from Proposition 3.19 (and the fact that \mathbb{C} is 2-divisible) that $\tilde{p}(\alpha) \in \widehat{\mathcal{B}}_3(\mathbb{C})$. Since $p(\alpha)$ is an instance of $\mathcal{R}_3(\alpha_1, \alpha_2, \alpha_3)$ it follows from (3.16) and Theorem 2.36 that the real part of $\widehat{\mathcal{L}}_3(\tilde{p}(\alpha))$ is $3\zeta(3)$. If one chooses a (good) realization where all α_i , β_i , γ_i and δ are positive real numbers and all logarithms are main branches, one easily checks that the imaginary part of $\widehat{\mathcal{L}}_3(\tilde{p}(\alpha))$ is zero. This concludes the proof. The parenthetical remark follows from Remark 7.22. \square

4. THE EXTENDED BLOCH GROUP

We review Neumann's extended Bloch group [Neu04]. We use the definition in [Zic15] (which works for arbitrary fields) and the modified variant of dilogarithm from [GZ07]. We stress that there are two variants of the extended Bloch group: $\widehat{\mathcal{B}}(\mathbb{C})$ using the space $\widehat{\mathbb{C}}_{\text{signs}}$ from (2.1), and $\widehat{\mathcal{B}}(\mathbb{C})_{++}$ using only the component $\widehat{\mathbb{C}} = \widehat{\mathbb{C}}_{++}$.

Definition 4.1. An element

$$(4.1) \quad [(u_0, v_0)] - [(u_1, v_1)] + [(u_2, v_2)] - [(u_3, v_3)] + [(u_4, v_4)] \in \mathbb{Z}[\widehat{\mathbb{C}}_{\text{signs}}]$$

is a *lifted five term relation* if

$$(4.2) \quad \begin{aligned} u_2 &= u_1 - u_0, & u_3 &= u_1 - u_0 - v_1 + v_0, \\ v_3 &= v_2 - v_1, & u_4 &= v_0 - v_1, & v_4 &= v_2 - v_1 + u_0. \end{aligned}$$

One easily checks that the covering map $r: \widehat{\mathbb{C}}_{\text{signs}} \rightarrow \mathbb{C} \setminus \{0, 1\}$ takes a lifted five term relation to a standard five term relation (3.2) and that the map

$$(4.3) \quad \widehat{\nu}: \mathbb{Z}[\widehat{\mathbb{C}}_{\text{signs}}] \rightarrow \wedge^2(\mathbb{C}), \quad (u, v) \mapsto u \wedge v$$

takes lifted five term relations to 0.

Definition 4.2. Let $\widehat{\mathcal{P}}(\mathbb{C})$ be the quotient of $\mathbb{Z}[\widehat{\mathbb{C}}_{\text{signs}}]$ by the lifted five term relations as well as the *transfer relation*

$$(4.4) \quad [(u + \pi i, v + \pi i)] + [(u, v)] - [(u + \pi i, v)] - [(u, v + \pi i)],$$

and let $\widehat{\mathcal{B}}(\mathbb{C})$ be the kernel of $\widehat{\nu}: \widehat{\mathcal{P}}(\mathbb{C}) \rightarrow \wedge^2(\mathbb{C})$.

Definition 4.3. Let $\widehat{\mathcal{P}}(\mathbb{C})_{++}$ be the quotient of $\mathbb{Z}[\widehat{\mathbb{C}}_{++}]$ by the lifted five term relations in $\mathbb{Z}[\widehat{\mathbb{C}}_{++}]$, and let $\widehat{\mathcal{B}}(\mathbb{C})_{++} \subset \widehat{\mathcal{P}}(\mathbb{C})_{++}$ be the kernel of $\widehat{\nu}$.

Remark 4.4. The transfer relation is not needed in the definition of $\widehat{\mathcal{B}}(\mathbb{C})_{++}$ and not including it changes $\widehat{\mathcal{B}}(\mathbb{C})$ only by a trivial $\mathbb{Z}/2\mathbb{Z}$ -extension [Neu04, Prop. 7.2].

Proposition 4.5. Both $\widehat{\mathcal{B}}(\mathbb{C})$ and $\widehat{\mathcal{B}}(\mathbb{C})_{++}$ are extensions of $\mathcal{B}(\mathbb{C})$ by \mathbb{Q}/\mathbb{Z} and the canonical map $\widehat{\mathcal{B}}(\mathbb{C})_{++} \rightarrow \widehat{\mathcal{B}}(\mathbb{C})$ is surjective with cyclic kernel of order 4.

One can write an element in $\widehat{\mathbb{C}}_{\text{signs}}$ uniquely as $(\text{Log}(z) + p\pi i, \text{Log}(1 - z) + q\pi i)$ for integers p and q . We may thus denote elements by $[z; p, q]$. Neumann shows that the map

$$(4.5) \quad \mathcal{R}([z; p, q]) = \text{Li}_2(z) + \frac{1}{2}(\text{Log}(z) + p\pi i)(\text{Log}(1 - z) - q\pi i) - \frac{\pi^2}{6}$$

is defined on $\widehat{\mathcal{P}}(\mathbb{C})$ and $\widehat{\mathcal{P}}(\mathbb{C})_{++}$ modulo π^2 and $4\pi^2$, respectively (see [GZ07] for the latter).

Proposition 4.6. The map \mathcal{R} is injective on the kernel $\widehat{\mathcal{B}}(\mathbb{C}) \rightarrow \mathcal{B}(\mathbb{C})$, and similarly for $\widehat{\mathcal{B}}(\mathbb{C})_{++}$.

Let $\tilde{c} = (u, v) + (v, u) \in \widehat{\mathcal{B}}(\mathbb{C})$. Neumann [Neu04] shows \tilde{c} is independent of $(u, v) \in \widehat{\mathbb{C}}_{\text{signs}}$ and that it has order 6.

Lemma 4.7. The following holds in $\widehat{\mathcal{B}}(\mathbb{C})$ modulo 2-torsion.

$$(4.6) \quad (u, v) + (-u, v - u) = 0, \quad (u, v) - (-v, u - v) = \tilde{c}.$$

Proof. This is a restatement of [Neu04, Prop. 13.1]. □

Remark 4.8. There is also a *non-alternating lifted five term relation*

$$(4.7) \quad [(u_0, v_0)] + [(v_1, u_1)] + [(u_2, v_2)] + [(v_3, u_3)] + [(u_4, v_4)]$$

for (u_i, v_i) as in (4.2), which is a lift of a non-alternating five term relation.

4.1. An alternative variant. One can also define $\widehat{\mathcal{B}}'(\mathbb{C})$ as above but without the transfer relation and with the lifted five term relations replaced by the *inverted lifted five term relations*

$$(4.8) \quad \sum_{i=0}^4 (-1)^i [(v_i, u_i)] = 0,$$

when the (u_i, v_i) satisfy (4.2). It follows that the map $\widehat{\phi}: \mathbb{Z}[\widehat{\mathbb{C}}_{\text{signs}}] \rightarrow \mathbb{Z}[\widehat{\mathbb{C}}_{\text{signs}}]$ taking $[(u, v)]$ to $-[(v, u)]$ defines a map $\widehat{\mathcal{B}}'(\mathbb{C}) \rightarrow \widehat{\mathcal{B}}(\mathbb{C})$, which is an isomorphism modulo 2-torsion. Compare with the definition of $\mathcal{B}'(\mathbb{C})$ in Section 3.2. A lifted five term relation equals \tilde{c} in $\widehat{\mathcal{B}}'(\mathbb{C})$, and a

non-alternating lifted five term relation is 0 modulo 2-torsion. Also, it follows from Lemma 4.7 that in $\widehat{\mathcal{B}}'(\mathbb{C})$ modulo 2-torsion one has (compare with (3.7))

$$(4.9) \quad (u, v) + (-u, v - u) = 2\tilde{c}, \quad (u, v) - (-v, u - v) = -\tilde{c}.$$

One easily checks that $\widehat{\mathcal{L}}_2(\tilde{c}) = \pi^2/6$ modulo π^2 , so there is a commutative diagram

$$(4.10) \quad \begin{array}{ccc} \widehat{\mathcal{B}}'(\mathbb{C}) & \xrightarrow{\widehat{\phi}} & \widehat{\mathcal{B}}(\mathbb{C}) \\ & \searrow \widehat{\mathcal{L}}_2 & \swarrow \mathcal{R} \\ & \mathbb{C}/\pi^2\mathbb{Z} & \end{array}$$

Proposition 4.9. Assuming Conjecture 3.13 one has $\widehat{\mathcal{B}}_2(\mathbb{C}) = \widehat{\mathcal{B}}'(\mathbb{C})$ modulo 2-torsion.

Proof. By Example 2.31 every inverted lifted five term relation is in $\widetilde{R}_2(\mathbb{C})$, so there is a canonical projection $\widehat{\mathcal{B}}'(\mathbb{C}) \rightarrow \widehat{\mathcal{B}}_2(\mathbb{C})$. If $\beta \neq 0 \in \mathcal{B}'(\mathbb{C})$ goes to zero, it follows from the fact that $\mathcal{B}'(\mathbb{C}) = \mathcal{B}_2(\mathbb{C})$ (Proposition 3.14) and the fact that \mathcal{R} is injective on the kernel of $\widehat{\mathcal{B}}(\mathbb{C}) \rightarrow \mathcal{B}(\mathbb{C})$ (Proposition 4.6) that β is a transfer relation. This proves the result. \square

Remark 4.10. By a similar argument we have $\widehat{\mathcal{B}}_2(\mathbb{C})_{++} = \widehat{\mathcal{B}}'(\mathbb{C})_{++}$, which holds on the nose.

4.2. Other fields. Let F be a field and E a fixed \mathbb{Z} -extension of F^* . We can then define $\widehat{F}_{\text{signs}}$ as in (2.57). It is a union of subsets $\widehat{F}_{++}, \widehat{F}_{-+}, \widehat{F}_{+-}, \widehat{F}_{--}$, which are disjoint unless F has characteristic 2 in which case they are equal. One can define $\widehat{\mathcal{B}}(F)_{++}$ as above and $\widehat{\mathcal{B}}(F)$ as above, but without including the transfer relation (which only affects 2-torsion).

5. BASIC PROPERTIES OF THE LIFTED POLYLOGARITHMS

We now give concrete models for the four components of $\widehat{\mathcal{C}}_{\text{signs}}$ following [Neu04, GZ07]. For signs ϵ_1 and ϵ_2 (regarded when convenient as elements of $\{-1, 1\}$), let

$$(5.1) \quad \mathbb{C}_{\epsilon_1\epsilon_2}^{\text{cut}} = \mathbb{C} \setminus \{z \in \mathbb{R} \mid \epsilon_1 z \leq 0, \epsilon_2(1-z) \leq 0\}.$$

Note that $\mathbb{C}_{\epsilon_1\epsilon_2}^{\text{cut}}$ is disconnected. Let

$$(5.2) \quad \overline{\mathbb{C}}_{\epsilon_1\epsilon_2}^{\text{cut}} = \mathbb{C}_{\epsilon_1\epsilon_2}^{\text{cut}} \cup \{z \pm 0i \mid z \in \mathbb{R}, \epsilon_1 z < 0, \epsilon_2(1-z) < 0\}.$$

The functions Li_k and Log extend continuously to $\overline{\mathbb{C}}_{\epsilon_1\epsilon_2}^{\text{cut}}$. Define $\widehat{\mathcal{C}}_{\epsilon_1\epsilon_2}$ to be the Riemann surface obtained from $\overline{\mathbb{C}}_{\epsilon_1\epsilon_2}^{\text{cut}} \times \mathbb{Z}^2$ as the quotient by the relations

$$(5.3) \quad \begin{aligned} (z + 0i, p, q) &\sim (z - 0i, p + \epsilon_1, q) && \text{if } \epsilon_1 z < 0, \quad \epsilon_2(1-z) > 0 \\ (z + 0i, p, q) &\sim (z - 0i, p, q - \epsilon_2) && \text{if } \epsilon_1 z > 0, \quad \epsilon_2(1-z) < 0 \\ (z + 0i, p, q) &\sim (z - 0i, p + \epsilon_1, q - \epsilon_2) && \text{if } \epsilon_1 z < 0, \quad \epsilon_2(1-z) < 0. \end{aligned}$$

An equivalence class is denoted by $\langle z; p, q \rangle_{\epsilon_1\epsilon_2}$. The map

$$(5.4) \quad \langle z; p, q \rangle_{\epsilon_1\epsilon_2} \mapsto (\text{Log}(\epsilon_1 z) + 2p\pi i, \text{Log}(\epsilon_2(1-z)) + 2q\pi i)$$

identifies $\widehat{\mathcal{C}}_{\epsilon_1\epsilon_2}$ with the appropriate component of $\widehat{\mathcal{C}}_{\text{signs}}$.

5.1. Definition of $\widehat{\mathcal{L}}_n$. We begin with the definition of a map \widehat{L}_n , which agrees with $\widehat{\mathcal{L}}_n$ modulo $\frac{(\pi i)^n}{(n-1)!}$. Consider the map

$$(5.5) \quad \widehat{L}_n: \overline{\mathbb{C}}_{\epsilon_1 \epsilon_2}^{\text{cut}} \times \mathbb{Z}^2 \rightarrow \mathbb{C}$$

taking $\langle z; p, q \rangle_{\epsilon_1 \epsilon_2}$ to

$$(5.6) \quad \sum_{r=0}^{n-1} \frac{(-1)^r}{r!} \mathcal{L}i_{n-r}(z; q) \text{Log}(\epsilon_1 z; p)^r - \frac{(-1)^n}{n!} \text{Log}(\epsilon_1 z; p)^{n-1} \text{Log}(\epsilon_2(1-z); q),$$

where

$$(5.7) \quad \mathcal{L}i_k(z; q) = \text{Li}_k(z) - \frac{2q\pi i}{(k-1)!} \text{Log}(z)^{k-1}, \quad \text{Log}(z; p) = \text{Log}(z) + 2p\pi i.$$

We wish to show that \widehat{L}_n descends to a holomorphic function on $\widehat{\mathbb{C}}_{\epsilon_1 \epsilon_2}$. For $z \in \mathbb{R} \setminus \{0, 1\}$, let $z_{\pm} = z \pm 0i$, and let

$$(5.8) \quad \Delta_{\epsilon_1 \epsilon_2}(z, p, q) = \frac{(n-1)!}{(2\pi i)^n} \begin{cases} \widehat{L}_n(z_+, p, q) - \widehat{L}_n(z_-, p + \epsilon_1, q) & \text{if } \epsilon_1 z < 0, \quad \epsilon_2(1-z) > 0 \\ \widehat{L}_n(z_+, p, q) - \widehat{L}_n(z_-, p, q - \epsilon_2) & \text{if } \epsilon_1 z > 0, \quad \epsilon_2(1-z) < 0 \\ \widehat{L}_n(z_+, p, q) - \widehat{L}_n(z_-, p + \epsilon_1, q - \epsilon_2) & \text{if } \epsilon_1 z < 0, \quad \epsilon_2(1-z) < 0 \\ 0 & \text{if } \epsilon_1 z > 0, \quad \epsilon_2(1-z) > 0. \end{cases}$$

Clearly, $\Delta_{\epsilon_1 \epsilon_2}(z, p, q)$ only depends on the interval I (either $(-\infty, 0)$, $(0, 1)$, or $(1, \infty)$) where z belongs. We denote it by $\Delta_{\epsilon_1 \epsilon_2}^I(p, q)$ accordingly. Let

$$(5.9) \quad \delta(p, n) = (-1)^n((p-1)^{n-1} - p^{n-1}).$$

Lemma 5.1. We have

$$(5.10) \quad \begin{aligned} \Delta_{++}^{(-\infty, 0)}(p, q) &= q\delta(p+1, n), & \Delta_{++}^{(0, 1)}(p, q) &= \Delta_{++}^{(1, \infty)}(p, q) = 0 \\ \Delta_{-+}^{(-\infty, 0)}(p, q) &= q\delta(p + \frac{1}{2}, n), & \Delta_{-+}^{(0, 1)}(p, q) &= \Delta_{-+}^{(1, \infty)}(p, q) = 0 \\ \Delta_{+-}^{(-\infty, 0)}(p, q) &= (-p-1)^{n-1} + q\delta(p+1, n), & \Delta_{+-}^{(0, 1)}(p, q) &= \Delta_{+-}^{(1, \infty)}(p, q) = (-p)^{n-1} \\ \Delta_{--}^{(-\infty, 0)}(p, q) &= (-p - \frac{1}{2})^{n-1} + q\delta(p + \frac{1}{2}, n), & \Delta_{--}^{(0, 1)}(p, q) &= \Delta_{--}^{(1, \infty)}(p, q) = (\frac{1}{2} - p)^{n-1} \end{aligned}$$

Proof. Suppose $z < 0$. Then $\text{Li}_k(z_+) = \text{Li}_k(z_-)$ and we have

$$(5.11) \quad \begin{aligned} \widehat{L}_n(\langle z_+; p, q \rangle_{++}) - \widehat{L}_n(\langle z_-; p+1, q \rangle_{++}) &= \\ &= -\frac{2q\pi i}{(n-1)!} \left(\sum_{r=0}^{n-1} \binom{n-1}{r} (\text{Log}(z_+))^{n-r-1} (-\text{Log}(z_+; p))^r \right) - \\ &= \sum_{r=0}^{n-1} \binom{n-1}{r} \text{Log}(z_-)^{n-r-1} (-\text{Log}(z_-; p+1))^r \Big) = \\ &= -\frac{2q\pi i}{(n-1)!} ((-2p\pi i)^{n-1} - (-2(p+1)\pi i)^{n-1}) = \frac{(2\pi i)^n}{(n-1)!} q\delta(p+1, n). \end{aligned}$$

This proves the first equality. Some of the other ones make use of the identity

$$(5.12) \quad \text{Li}_n(z_+) - \text{Li}_n(z_-) = \frac{2\pi i \text{Log}(z)^{n-1}}{(n-1)!}, \quad z \in (1, \infty)$$

but are otherwise similar. We leave their verification to the reader. \square

It follows that \widehat{L}_n is holomorphic on $\widehat{\mathbb{C}}_{\epsilon_1 \epsilon_2}$ with values in $\frac{(\pi i)^n}{(n-1)!}$.

Lemma 5.2. \widehat{L}_n is a primitive for the one form ω_n defined in (2.3), i.e. $d\widehat{L}_n = \omega_n$.

Proof. By (1.1) one has $d\text{Li}_k(z) = \frac{\text{Li}_{k-1}(z)}{z} dz$, and it follows that $d\mathcal{L}i_k(z; q) = \frac{\mathcal{L}i_{k-1}(z; q)}{z} dz$. This holds for all $k \geq 1$ with the convention that $\mathcal{L}i_0(z; q) = \frac{z}{1-z}$. Letting $u = \text{Log}(\epsilon_1; p)$ and $v = \text{Log}(\epsilon_2(1-z); q)$ one has

$$(5.13) \quad \begin{aligned} d\widehat{L}_n &= \sum_{r=0}^{n-1} \frac{(-1)^r}{r!} (\mathcal{L}i_{n-r-1} u^r + r \mathcal{L}i_{n-r} u^{r-1}) \frac{dz}{z} - \frac{(-1)^n}{n!} ((n-1)u^{n-2} v du + u^{n-1} dv) \\ &= \frac{(-1)^{n-1}}{(n-1)!} u^{n-1} \mathcal{L}i_0(z; q) \frac{dz}{z} - \frac{(-1)^n}{n!} ((n-1)u^{n-2} v du + u^{n-1} dv) = \omega_n. \end{aligned}$$

The second equality follows by telescoping, and the third from the fact that $\text{Li}_0(z; q) \frac{dz}{z} = -dv$. \square

Theorem 5.3. *The form ω_n has periods in $\frac{(2\pi i)^n}{(n-1)!} \mathbb{Z}$ on $\widehat{\mathbb{C}}_{++}$ and $\widehat{\mathbb{C}}_{+-}$ and in $\kappa_n \frac{2\pi i}{(n-1)!}$ on $\widehat{\mathbb{C}}_{-+}$ and $\widehat{\mathbb{C}}_{--}$, where κ_n is defined in (2.4).*

Proof. It is enough to compute the integral of ω_n along a lift of the loop $aba^{-1}b^{-1}$ in $\mathbb{C} \setminus \{0, 1\}$, where a is a loop going counterclockwise around 0 and b is a loop going clockwise around 1. Since \widehat{L}_n is a primitive, this integral equals

$$(5.14) \quad \frac{(2\pi i)^n}{(n-1)!} \left(-\Delta_{\epsilon_1 \epsilon_2}^{(-\infty, 0)}(p, q) + \Delta_{\epsilon_1 \epsilon_2}^{(0, 1)}(p+1, q) - \Delta_{\epsilon_1 \epsilon_2}^{(1, \infty)}(p+1, q) + \Delta_{\epsilon_1 \epsilon_2}^{(-\infty, 0)}(p, q-1) - \Delta_{\epsilon_1 \epsilon_2}^{(0, 1)}(p, q-1) + \Delta_{\epsilon_1 \epsilon_2}^{(1, \infty)}(p, q) \right).$$

By Lemma 5.1 this equals $-\frac{(2\pi i)^n}{(n-1)!} \delta(p+1, n)$ when $\epsilon_1 = 1$ and $-\frac{(2\pi i)^n}{(n-1)!} \delta(p+\frac{1}{2}, n)$ when $\epsilon_1 = -1$. When expanding $(-p+\frac{1}{2})^{n-1} - (-p-\frac{1}{2})^{n-1}$ one easily verifies that the greatest common divisor of the denominators is 2^{n-2} if n is even and $2^{n-3-\nu_2(n-1)}$ if n is odd. This proves the result. \square

One can define (with the convention that $\text{Im}(z_+) > 0$ and $\text{Im}(z_-) < 0$)

$$(5.15) \quad \widehat{\mathcal{L}}_n = \widehat{L}_n + \frac{(2\pi i)^n}{(n-1)!} \begin{cases} 0 & \text{if } (\epsilon_1, \epsilon_2) \neq (-1, -1) \\ -(-p-\frac{1}{2})^{n-1} + (-\frac{1}{2})^{n-1} & \text{if } (\epsilon_1, \epsilon_2) = (-1, -1), \text{Im}(z) > 0 \\ (-\frac{1}{2})^{n-1} & \text{if } (\epsilon_1, \epsilon_2) = (-1, -1), \text{Im}(z) < 0 \end{cases}$$

We stress that this is well defined since the imaginary part is never zero on $\overline{\mathbb{C}}_{--}^{\text{cut}}$. The fact that $\widehat{\mathcal{L}}_n$ is defined modulo $\kappa_n \frac{2\pi i}{(n-1)!}$ on $\widehat{\mathbb{C}}_{--}$ is an easy consequence of Lemma 5.1.

We conclude with an elementary result whose purpose is to ensure that $\widehat{\mathcal{L}}_n$ takes relations of type (2) in Definition 2.30 to 0.

Lemma 5.4. Let $z_k \in \mathbb{C} \setminus \{0, 1\}$ be a sequence with $z_k \rightarrow 0$, and let $p_k \in \mathbb{Z}$ be a bounded sequence. We have

$$(5.16) \quad \widehat{\mathcal{L}}_n(\text{Log}(z_k) + p_k \pi i, \text{Log}(1-z_k)) \rightarrow 0 \quad \text{for } k \rightarrow \infty$$

Proof. By L'hospital's rule, the sequences $\text{Log}(1-z_k) \text{Log}(z_k)^r$ and $\text{Li}_s(z_k) \text{Log}(z_k)^r$ tend to 0 for any integers r and s . The result is an easy consequence. \square

6. CHARACTERIZATION OF DIFFERENTIAL $\widehat{\mathcal{L}}_n$ RELATIONS

We now give a proof of Theorem 2.27. As usual, α denotes an element as in (2.23).

Proposition 6.1. For $n > 2$, α is a differential $\widehat{\mathcal{L}}_n$ relation if and only if

$$(6.1) \quad \sum_{i=1}^K r_i w_{n-1}(u_i, v_i) \otimes u_i = 0 \in \Omega_{n-2}^1(\widetilde{S}) \otimes \widetilde{S}_1.$$

Proof. Let $V = \Omega_{n-1}^1(\widetilde{S})$, $W = \Omega_{n-2}^1(\widetilde{S}) \otimes \widetilde{S}_1$, and let

$$(6.2) \quad X = \sum_{i=1}^K r_i u_i^{n-2}(u_i dv_i - v_i du_i), \quad Y = \sum_{i=1}^K r_i u_i^{n-3}(u_i dv_i - v_i du_i) \otimes u_i.$$

We must show that X is zero in V if and only if Y is zero in W . For a symbolic log-pair (u, v) let

$$(6.3) \quad A = u^{n-2}(udv - vdu), \quad B = u^{n-3}(udv - vdu) \otimes u.$$

Note that V and W are free abelian groups with bases $f_I d\tilde{a}_i$ and $g_I d\tilde{a}_i \otimes \tilde{a}_j$, respectively, where f_I and g_I are monomials of degree $n-1$ and $n-2$, respectively. For a basis element v of V let $\text{Coeff}(A, v)$ be the coefficient of v when expanding A in the basis for V . We similarly define $\text{Coeff}(B, w)$ for a basis element w of W . A direct computation then shows that

$$(6.4) \quad \text{Coeff}(A, f_I d\tilde{a}_i) = \sum_{j \text{ where } \tilde{a}_j | f_I} \text{Coeff}(B, \frac{f_I}{\tilde{a}_j} d\tilde{a}_i \otimes \tilde{a}_j).$$

By linearity, this proves that $X = 0$ if $Y = 0$. A basis vector in W may be written uniquely in one of 3 ways: $\tilde{a}_i^k f_I d\tilde{a}_i \otimes \tilde{a}_i$ where \tilde{a}_i does not divide f_I , $\tilde{a}_i^k \tilde{a}_j^l d\tilde{a}_i \otimes \tilde{a}_j$ with $i \neq j$, or as $\tilde{a}_i^k \tilde{a}_j^l g_I d\tilde{a}_i \otimes \tilde{a}_j$ where $g_I \neq 1$ is not divisible by $\tilde{a}_i \neq \tilde{a}_j$. With this notation, another elementary computation shows that

$$(6.5) \quad \begin{aligned} (k + \deg(f_I)) \text{Coeff}(B, \tilde{a}_i^k f_I d\tilde{a}_i \otimes \tilde{a}_i) &= (k+1) \text{Coeff}(A, \tilde{a}_i^{k+1} f_I d\tilde{a}_i) \\ (k+l) \text{Coeff}(B, \tilde{a}_i^k \tilde{a}_j^l d\tilde{a}_i \otimes \tilde{a}_j) &= l \text{Coeff}(A, \tilde{a}_i^k \tilde{a}_j^{l+1} d\tilde{a}_i), \end{aligned}$$

and that

$$(6.6) \quad \begin{aligned} ((k+l+\deg(f_I)+1)^2 - 1) \text{Coeff}(B, f_I \tilde{a}_i^k \tilde{a}_j^l d\tilde{a}_i \otimes \tilde{a}_j) &= \\ (l+1)(k+l+\deg(f_I)+1) \text{Coeff}(A, f_I \tilde{a}_i^k \tilde{a}_j^{l+1} d\tilde{a}_i) &+ (k+1) \text{Coeff}(A, f_I \tilde{a}_i^{k+1} \tilde{a}_j^l d\tilde{a}_j). \end{aligned}$$

Again, by linearity, this shows that $Y = 0$ if $X = 0$. \square

By repeatedly applying Proposition 6.1 we obtain.

Corollary 6.2. If $n > 2$, α is a differential $\widehat{\mathcal{L}}_n$ relation if and only if

$$(6.7) \quad \sum_{i=1}^K r_i w_{n-l}(u_i, v_i) \otimes u_i^l = 0 \in \Omega_{n-l-1}^1(\widetilde{S}) \otimes \widetilde{S}_l$$

for all $1 \leq l \leq n-2$. Here \widetilde{S}_l denotes the group of degree l polynomials in \widetilde{S} .

Theorem 2.27 is an immediate consequence.

7. DEFINITION AND PROPERTIES OF $\tilde{R}_n(\mathbb{C})$

We now define the notions of *equivalence* of realizations, *proper realizations* and *proper ambiguity* needed to define $\tilde{R}_n(\mathbb{C})$ (see Section 2.3.1), and prove the theorems in Section 2.3.2. As usual, α denotes an element as in (2.23).

7.1. Equivalence of realizations. We shall consider realizations equivalent if the points are in the same component, and also if the points differ by simultaneously changing the sign of an a_i and adding πi to its logarithm. The formal definition is the following.

Definition 7.1. Two realizations $\tilde{p} = (\mathcal{V}_\epsilon, p, \tilde{p})$ and $\tilde{p}' = (\mathcal{V}_{\epsilon'}, p', \tilde{p}')$ of α are

- (1) *component-equivalent* if $\mathcal{V}_{\epsilon'} = \mathcal{V}_\epsilon$ and p and p' are in the same component of $X_\alpha^{\mathcal{V}_\epsilon}(\mathbb{C})$;
- (2) *sign-equivalent* if there exists a homomorphism $\phi: \tilde{S}_1 \rightarrow \mathbb{Z}$ such that

$$\epsilon_1^i = (-1)^{\phi(u_i)} \epsilon_1^i, \quad \epsilon_2^i = (-1)^{\phi(v_i)} \epsilon_2^i, \quad p'_U = p_U \bar{\phi}, \quad \tilde{p}' = \tilde{p} + \pi i \phi,$$

where $\bar{\phi}: U(S) \rightarrow \{\pm 1\}$ is the map induced by ϕ ;

- (3) *equivalent* if there is a realization $\tilde{p}'' = (\mathcal{V}_{\epsilon''}, p'', \tilde{p}'')$, such that \tilde{p}'' is sign-equivalent to \tilde{p}' and component equivalent to \tilde{p} .

Each of the three notions in Definition 7.1 forms an equivalence relation.

Example 7.2. If in Example 2.20 one simultaneously changes the sign of a_1 and adds πi to its logarithm \tilde{a}_1 , one obtains a new sign equivalent realization with sign determination $((1, 1), (1, -1))$.

Lemma 7.3. Changing a realization by a sign equivalence leaves the element $p(\alpha)$ in (2.25) unchanged.

Proof. A sign equivalence changes the sign of both ϵ_1^i and $p(\pi(u_i))$, so $p(\alpha)$ is unchanged. \square

7.2. Proper ambiguity.

Definition 7.4. A differential $\widehat{\mathcal{L}}_n$ relation α has proper ambiguity if $\sum r_i k_{j_i}^{n-1} l_{j_i}$ is divisible by n for every $j = 1, \dots, N$.

Recall the map ν from (2.34). When $n = 2$ one easily checks that a differential $\widehat{\mathcal{L}}_2$ relation α has proper ambiguity if and only if $\nu(\alpha) = 0$ (recall that $2\nu(\alpha) = 0$ by Lemma 2.25).

Remark 7.5. One may think of the notion of *proper ambiguity* as a generalization of Neumann's *parity condition* [Neu04, Sec. 4.2]. We shall not elaborate on this.

Example 7.6. The five term elements in Examples 2.19, 2.22, and 2.31 all have proper ambiguity. The element in Example 2.21 does not have proper ambiguity.

7.3. Proper realizations. This concept is defined inductively. If a notion of *proper realization* has been defined for n , one can define $\tilde{R}_n(\mathbb{C})$ as in Section 2.3.1.

Definition 7.7. A realization of a differential $\widehat{\mathcal{L}}_n$ relation α is *proper* if

$$(7.1) \quad \tilde{p}(\pi_{j_1, \dots, j_{n-l}}(\alpha)) \in \tilde{R}_l(\mathbb{C})$$

for all the lower level projections $\pi_{j_1, \dots, j_{n-l}}(\alpha)$ of α . Here $l \in \{2, \dots, n-1\}$.

When $n = 2$, (7.1) is vacuous, so all realizations are proper, starting the inductive definition.

7.4. Proof of Theorem 2.32. Recall that for any curve Y in $X_\alpha^{\mathcal{V}_\epsilon}(\mathbb{C})$ one has the element $p_Y(\alpha)$ from (2.26), which we shall regard as an element in $\mathbb{Z}[\mathbb{C}(Y) \setminus \{0, 1\}]$. Also recall the definition of \mathcal{A}_k from Section 3.3.2. We begin with a proof when $n = 2$.

Lemma 7.8. Suppose \tilde{p} is a realization of a differential $\widehat{\mathcal{L}}_2$ relation α . For any smooth curve in $X_\alpha^{\mathcal{V}_\epsilon}(\mathbb{C})$ containing p , $2p_Y(\alpha) \in \mathcal{A}_2(\mathbb{C}(Y))$.

Proof. The homomorphism $\pi: \tilde{S}_1 \rightarrow U(S)$ induces a homomorphism $\pi_*: \wedge^2(\tilde{S}_1) \rightarrow \wedge^2(U(S))$. Since each point $p \in Y$ restricts to a homomorphism $U(S) \rightarrow \mathbb{C}^*$, we have a homomorphism $p_Y: U(S) \rightarrow \mathbb{C}(Y)^*$. We now have

$$\begin{aligned} \nu_{2, \mathbb{C}(Y)}(2p_Y(\alpha)) &= 2 \sum r_i(\epsilon_1^i \pi(u_i)_Y) \wedge (1 - \epsilon_1^i \pi(u_i)_Y) \\ (7.2) \qquad \qquad \qquad &= 2 \sum r_i \pi(u_i)_Y \wedge \pi(v_i)_Y \\ &= 2p_{Y^*} \circ \pi_*(\nu(\alpha)). \end{aligned}$$

By Lemma 2.25, $2\nu(\alpha) = 0$. This proves the result. \square

Lemma 7.9. The covering map r takes $2\tilde{R}_2(\mathbb{C})$ to $R_2(\mathbb{C})$.

Proof. Recall that $\tilde{R}_2(\mathbb{C})$ is generated by the two types in Definition 2.30. Let α be a differential $\widehat{\mathcal{L}}_2$ relation with a (proper) realization $\tilde{p} = (\mathcal{V}_\epsilon, p, \tilde{p})$ and let \tilde{q} be a realization equivalent to \tilde{p} . After applying a sign equivalence to q (which by Lemma 7.3 does not change the image in $\mathbb{Z}[\mathbb{C} \setminus \{0, 1\}]$) we may assume that p and q are both in $X_\alpha^{\mathcal{V}_\epsilon}(\mathbb{C})$. We may also assume that $p \neq q$. Pick a smooth curve $Y \subset X_\alpha^{\mathcal{V}_\epsilon}(\mathbb{C})$ containing p and q . By Lemma 7.8, $2p_Y(\alpha) \in \mathcal{A}_2(\mathbb{C}(Y))$ and it follows that

$$(7.3) \qquad \qquad \qquad 2r(\tilde{p}(\alpha) - \tilde{q}(\alpha)) = 2p(\alpha) - 2q(\alpha) \in R_2(\mathbb{C}).$$

Similarly, if a curve Y in $X_\alpha^{\mathcal{V}_\epsilon}(\mathbb{C})$ containing p contains a point $q \in \overline{X_\alpha^{\mathcal{V}_\epsilon}(\mathbb{C})}$, we have $2r(\tilde{p}) = 2p(\alpha) \in R_2(\mathbb{C})$. \square

Now suppose $n > 2$. Assume by induction that r maps $2\tilde{R}_k(\mathbb{C})$ to $R_k(\mathbb{C})$ for all $k < n$. We begin with an elementary lemma, which holds for any integer $m \geq 2$.

Lemma 7.10. Let $\beta \in \mathbb{Z}[\text{SymbLogs}]$ and let $\tilde{p} = (\mathcal{V}_\epsilon, p, \tilde{p})$ be a realization of β . Suppose that for a smooth curve $Y \subset X_\beta^{\mathcal{V}_\epsilon}(\mathbb{C})$ containing p one has $p_Y(\beta) \in \mathcal{A}_m(\mathbb{C}(Y))$. If $p(\beta) \in R_m(\mathbb{C})$ then $p_Y(\beta) \in R_m(\mathbb{C}(Y))$.

Proof. Suppose $p_Y(\beta) = \sum_{i=1}^K r_i[x_i]$, where $x_i \in \mathbb{C}(Y)$. Then $X = Y(\mathbb{C}(Y))$ is a smooth curve in $X_\beta^{\mathcal{V}_\epsilon}(\mathbb{C}(Y))$ containing both $q = (x_1, \dots, x_K)$ and p . Moreover, $p_X(\beta) \in \mathcal{A}_m(\mathbb{C}(Y)(X))$. It follows that $p(\beta) - q(\beta) \in R_m(\mathbb{C}(Y))$ and since $p(\beta) \in R_m(\mathbb{C}) \subset R_m(\mathbb{C}(Y))$, it follows that $q(\beta) = p_Y(\beta) \in R_m(\mathbb{C}(Y))$. \square

Lemma 7.11. Suppose \tilde{p} is a proper realization of a differential $\widehat{\mathcal{L}}_n$ relation α . For any smooth curve Y in $X_\alpha^{\mathcal{V}_\epsilon}(\mathbb{C})$ containing p , $2p_Y(\alpha) \in \mathcal{A}_n(\mathbb{C}(Y))$.

Proof. By Theorem 2.27 the elements

$$(7.4) \qquad \qquad \qquad \beta_m = \beta_m(j_1, \dots, j_{n-m}) = \pi_{j_1, \dots, j_{n-m}}(\alpha),$$

are differential $\widehat{\mathcal{L}}_m$ relations for all $m = 2, \dots, n-1$ and all j_i . By our induction hypothesis that $2\tilde{R}_k(\mathbb{C})$ maps to $R_k(\mathbb{C})$ for all $k < n$, we see that $2p(\beta_m) \in R_m(\mathbb{C})$ for all $m < n$. We now prove by induction that $2p_Y(\beta_m) \in \mathcal{A}_m(\mathbb{C}(Y))$ for all $m = 2, \dots, n$. The case $m = n$ is the desired

statement. By Lemma 7.8 this holds when $m = 2$. Now suppose by induction that it holds for $m = k$ with $2 < k \leq n - 1$. We then have

$$(7.5) \quad \nu_{k+1, \mathbb{C}(Y)}(2p_Y(\beta_{k+1}(j_1, \dots, j_{n-k-1}))) = \sum_{j=1}^N 2p_Y(\beta_k(j, j_1, \dots, j_{n-k-1})) \otimes a_j,$$

which is zero by Lemma 7.10. Hence, $2p_Y(\beta_{k+1}) \in \mathcal{A}_{k+1}(\mathbb{C}(Y))$. This concludes the proof. \square

Theorem 7.12. *The covering map r takes $2\tilde{R}_n(\mathbb{C})$ maps to $R_n(\mathbb{C})$.*

Proof. The proof is the same as that of Lemma 7.9 except that Lemma 7.11 is used instead of Lemma 7.8. \square

Remark 7.13. In all of the above, we may replace \mathbb{C} by an arbitrary field F as long as the curve Y in $X_\alpha^{\mathcal{V}_\epsilon}(F)$ is geometrically irreducible. The fact that any two distinct points in a geometrically irreducible variety are connected by an irreducible curve can be found in [CP16, Cor. 1.9].

Recall the map $\hat{\nu}_n$ from (2.39).

Theorem 7.14. *The map $\hat{\nu}_n$ takes $\tilde{R}_n(\mathbb{C})$ to 0.*

Proof. Let α be a differential $\hat{\mathcal{L}}_n$ relation as in (2.23) and let \tilde{p} be a proper realization. The lift \tilde{p} induces a homomorphism $\tilde{p}_*: \wedge^2(\tilde{S}_1) \rightarrow \wedge^2(\mathbb{C})$, and the result for $n = 2$ now follows from the fact that $\tilde{p}_*(\nu(\alpha)) = \hat{\nu}_2(\tilde{p}(\alpha))$. For $n > 2$, we have

$$(7.6) \quad \hat{\nu}_n(\tilde{p}(\alpha)) = \sum_{i=1}^K r_i [(\tilde{p}(u_i), \tilde{p}(v_i))] \otimes \tilde{p}(u_i) = \sum_{j=1}^N \sum_{i=1}^K r_i k_{ji} [(\tilde{p}(u_i), \tilde{p}(v_i))] \otimes \tilde{p}(\tilde{a}_j).$$

Since \tilde{p} is proper, $\sum_{i=1}^K r_i k_{ji} [(\tilde{p}(u_i), \tilde{p}(v_i))] \in \tilde{R}_{n-1}(\mathbb{C})$ for all j , and the result follows. \square

7.5. Proof of Theorem 2.33. We now prove that $\hat{\mathcal{L}}_n(\beta) = 0$ if $\beta \in \tilde{R}_n(\mathbb{C})$. The fact that this holds for the relations of type (2) in Definition 2.30 is an elementary consequence of Lemma 5.4. Suppose α is a differential $\hat{\mathcal{L}}_n$ relation with proper ambiguity and that \tilde{p} is a proper realization. For $j \in \{1, \dots, N\}$ let $T_j(\tilde{p})$ be the (sign equivalent) realization obtained from \tilde{p} by changing the sign of a_j and adding πi to the logarithm of \tilde{a}_j . By Proposition 2.18 it is enough to prove the following.

Proposition 7.15. For all $j \in \{1, \dots, N\}$, we have $\hat{\mathcal{L}}_n(T_j(\tilde{p})(\alpha)) = \hat{\mathcal{L}}_n(\tilde{p}(\alpha))$ modulo $\frac{(\pi i)^n}{(n-1)!}$.

To prove Proposition 7.15 we start with a result comparing the values of $\hat{\mathcal{L}}_n$ at two points in $\hat{\mathbb{C}}_{\text{signs}}$ with the same image in $\mathbb{C} \setminus \{0, 1\}$.

Lemma 7.16. Let $\bar{k} = k\pi i$ and $\bar{l} = l\pi i$ denote multiples of πi . We have

$$(7.7) \quad \hat{\mathcal{L}}_n(u + \bar{k}, v + \bar{l}) - \hat{\mathcal{L}}_n(u, v) = \sum_{r=1}^{n-2} (-1)^r \frac{\bar{k}^r}{r!} \hat{\mathcal{L}}_{n-r}(u, v) + \frac{(-1)^n}{n!} \left(A(u, v; \bar{k}, \bar{l}) + \sum_{r=0}^{n-3} A_r(u, v; \bar{k}, \bar{l}) - \bar{k}^{n-1} \bar{l} \right),$$

modulo $\frac{(\pi i)^n}{(n-1)!}$ where

$$(7.8) \quad A(u, v; \bar{k}, \bar{l}) = (\bar{k}v - \bar{l}u)(u + \bar{k})^{n-2}, \quad A_r(u, v; \bar{k}, \bar{l}) = (\bar{k}v - \bar{l}u) \binom{n-2}{r+1} u^r \bar{k}^{n-2-r}.$$

Moreover, if $(u, v) \in \hat{\mathbb{C}}_{++}$ and \bar{k} and \bar{l} are even multiples of πi , then (7.7) holds modulo $\frac{(2\pi i)^n}{(n-1)!}$.

Proof. Let's define functions X , Y and Z of $(u, v) \in \widehat{\mathbb{C}}_{\text{signs}}$ as follows:

$$(7.9) \quad \begin{aligned} X &= \frac{n!}{(-1)^n} (\widehat{\mathcal{L}}_n(u + \bar{k}, v + \bar{l}) - \widehat{\mathcal{L}}_n(u, v)), & Y &= \frac{n!}{(-1)^n} \left(\sum_{r=1}^{n-2} \frac{(-1)^r}{r!} \bar{k}^r \widehat{\mathcal{L}}_{n-r}(u, v) \right) \\ Z &= A(u, v; \bar{k}, \bar{l}) + \sum_{r=0}^{n-3} A_r(u, v; \bar{k}, \bar{l}) - \bar{k}^{n-1} \bar{l} \end{aligned}$$

Note that when $(u, v) \in \widehat{\mathbb{C}}_{++}$ and \bar{k} and \bar{l} are even multiples of πi these functions are defined modulo $n(2\pi i)^n$. Otherwise, we shall regard them as functions modulo $n(\pi i)^n$. We must show that $X - Y = Z$. We first show that $dX - dY = dZ$. Using the fact that $d\widehat{\mathcal{L}}_k = \omega_k$, we obtain

$$(7.10) \quad \begin{aligned} dX &= (n-1) \left((u + \bar{k})^{n-2} ((u + \bar{k})dv - (v + \bar{l})du) - u^{n-2}(udv - vdu) \right) \\ dY &= \sum_{r=1}^{n-2} \bar{k}^r \binom{n}{r} (n-r-1) u^{r-2} (udv - vdu). \end{aligned}$$

It follows that $dX - dY$ equals

$$(7.11) \quad \sum_{r=1}^{n-2} \left(((n-1) \binom{n-2}{r} - \binom{n}{r} (n-r-1)) u^{n-r-2} \bar{k}^r \right) (udv - vdu) + (n-1)(u + \bar{k})^{n-2} (\bar{k}dv - \bar{l}du).$$

One now shows that $dX - dY = dZ$ by a term by term comparison. The coefficient of dv in dZ equals

$$(7.12) \quad \bar{k}(u + \bar{k})^{n-2} + \sum_{r=0}^{n-3} \bar{k} \binom{n-2}{r+1} u^r \bar{k}^{n-2-r}.$$

The coefficient of $u^s dv$ in $dX - dY$ is $n-1$ when $s=0$ and

$$(7.13) \quad (n-1) \binom{n-2}{n-1-s} - s \binom{n}{n-1-s} + (n-1) \binom{n-2}{s} = \binom{n-2}{s} + \binom{n-2}{s+1},$$

when $s \in \{1, \dots, n-1\}$. By (7.12) this agrees with the coefficient of $u^s dv$ in dZ . A similar consideration comparing coefficients of du completes the proof that $dX - dY = dZ$.

One now need only show that $X - Y - Z = 0$ for a single point in each of the 4 components of $\widehat{\mathbb{C}}_{\text{signs}}$. We choose points

$$(7.14) \quad (\pi i, \log(2)), \quad (0, \log(2)), \quad (0, \log(2) + \pi i), \quad (\pi i, \log(2) + \pi i)$$

in $\widehat{\mathbb{C}}_{++}$, $\widehat{\mathbb{C}}_{-+}$, $\widehat{\mathbb{C}}_{--}$, $\widehat{\mathbb{C}}_{+-}$, respectively. For $(u, v) = (\pi i, \log(2))$ we have

$$(7.15) \quad X = \frac{n!}{(-1)^n} \left(\sum_{r=0}^{n-1} \frac{(-1)^r}{r!} \text{Li}_{n-r}(-1) ((\pi i + \bar{k})^r - (\pi i)^r) \right) - (\pi i + \bar{k})^{n-1} (\log(2) + \bar{l}) + (\pi i)^{n-1} \log(2)$$

and

$$\begin{aligned}
(7.16) \quad Y &= \frac{n!}{(-1)^n} \sum_{t=1}^{n-2} \frac{(-1)^t}{t!} \bar{k}^t \left(\sum_{s=0}^{n-t-1} \frac{(-1)^s}{s!} \text{Li}_{n-t-s}(-1) (\pi i)^s - \frac{(-1)^{n-t}}{(n-t)!} (\pi i)^{n-t-1} \log(2) \right) \\
&= n \bar{k}^{n-1} \text{Li}_1(-1) + \frac{n!}{(-1)^n} \left(\sum_{r=1}^{n-1} ((\pi i + \bar{k})^r - (\pi i)^r) \frac{(-1)^r}{r!} \text{Li}_{n-r}(-1) \right) + \\
&\quad - \sum_{t=1}^{n-2} \binom{n}{t} \bar{k}^t (\pi i)^{n-t-1} \log(2).
\end{aligned}$$

Hence,

$$\begin{aligned}
(7.17) \quad X - Y &= (\pi i)^{n-1} \log(2) - (\pi i + \bar{k})^{n-1} (\log(2) + \bar{l}) + \sum_{t=1}^{n-2} \binom{n}{t} \bar{k}^t (\pi i)^{n-t-1} \log(2) - n \bar{k}^{n-1} \text{Li}_1(-1) \\
&= \log(2) \left((\pi i)^{n-1} - (\pi i + \bar{k})^{n-1} + \sum_{r=0}^{n-3} \binom{n}{r+2} \bar{k}^{n-2-r} (\pi i)^{r+1} + n \bar{k}^{n-1} \right) - \bar{l} (\pi i + \bar{k})^{n-1}.
\end{aligned}$$

Similarly, we obtain

$$(7.18) \quad Z = \log(2) \left(\bar{k} (\pi i + \bar{k})^{n-2} + \sum_{r=0}^{n-3} \bar{k} \binom{n-2}{r+1} (\pi i)^r \bar{k}^{n-2-r} \right) - \bar{l} (\pi i + \bar{k})^{n-1}.$$

Letting

$$(7.19) \quad a_r = \pi i \binom{n-2}{r+2} \bar{k}^{n-2-r} (\pi i)^r, \quad b_r = \pi i \binom{n-2}{r+1} \bar{k}^{n-2-r} (\pi i)^r,$$

(7.17) and (7.18) together with the equality $\binom{n}{r+2} = \binom{n-2}{r+2} + 2\binom{n-2}{r+1} + \binom{n-2}{r}$ imply that we have

$$\begin{aligned}
(7.20) \quad X - Y - Z &= \log(2) \left(n \bar{k}^{n-1} + \sum_{r=0}^{n-2} (a_r - a_{r-1}) + 2 \sum_{r=0}^{n-2} (b_r - b_{r-1}) \right) \\
&= \log(2) \left(n \bar{k}^{n-1} - a_{-1} - 2b_{-1} \right) = 0.
\end{aligned}$$

This concludes the proof when $(u, v) = (\pi i, \log(2))$. The computations for the other 3 points are similar (and much simpler for $(0, \log(2))$ for $(0, \log(2) + \pi i)$). \square

Let $\tilde{S}[\pi]$ be the polynomial ring obtained from \tilde{S} by adjoining a variable \tilde{a}_π , which we think of as a symbolic representation of $\pi i \in \mathbb{C}$. A homomorphism $\tilde{p}: \tilde{S}_1 \rightarrow \mathbb{C}$ extends canonically to $\tilde{S}[\pi]_1$ by taking \tilde{a}_π to πi . Fix $j \in \{1, \dots, N\}$, and consider the homomorphism

$$(7.21) \quad T_j: \tilde{S}_1 \rightarrow \tilde{S}[\pi]_1, \quad \tilde{a}_i \mapsto \begin{cases} \tilde{a}_i & \text{if } i \neq j \\ \tilde{a}_j + \tilde{a}_\pi & \text{if } i = j. \end{cases}$$

We then have

$$(7.22) \quad T_j(\tilde{p})(\alpha) = \tilde{p}(T_j(\alpha)).$$

Note that

$$(7.23) \quad T_j(\alpha) = \sum_{i=1}^K r_i (u_i + k_{ji} \tilde{a}_\pi, v_i + l_{ji} \tilde{a}_\pi).$$

Hence, by Lemma 7.16 we have

$$(7.24) \quad \widehat{\mathcal{L}}_n(\widetilde{T}_j(\widetilde{p})(\alpha)) - \widehat{\mathcal{L}}_n(\widetilde{p}(\alpha)) = \sum_{i=1}^K r_i \left(\sum_{r=1}^{n-2} (-1)^r \frac{\bar{k}_{ji}^r}{r!} \widehat{\mathcal{L}}_{n-r}(\widetilde{p}(u_i), \widetilde{p}(v_i)) \right) + \\ \frac{(-1)^n}{n!} \sum_{i=1}^K r_i A(\widetilde{p}(u_i), \widetilde{p}(v_i); \bar{k}_{ji}, \bar{l}_{ji}) + \frac{(-1)^n}{n!} \sum_{i=1}^K r_i \left(\sum_{r=0}^{n-3} A_r(\widetilde{p}(u_i), \widetilde{p}(v_i); \bar{k}_{ji}, \bar{l}_{ji}) - \right. \\ \left. \frac{(-1)^n}{n!} \sum_{i=1}^K r_i (\bar{k}_{ij})^{n-1} (\bar{l}_{ij}) \right).$$

If $n = 2$, this equals $\frac{1}{2} \sum r_i \bar{k}_{ij}^{n-1} \bar{l}_{ij}$, which is 0 modulo π^2 since α has proper ambiguity. This shows that Proposition 7.15 and therefore also Theorem 2.33 holds for $n = 2$. Assume that $n > 2$ and that Theorem 2.33 holds for all $k < n$. The fact that it holds for n as well is now an immediate consequence of Lemmas 7.17, 7.18, 7.19, and 7.20 below.

Lemma 7.17. For any $r = 1, \dots, n-2$ we have

$$(7.25) \quad \sum_{i=1}^K \bar{k}_{ji}^r \widehat{\mathcal{L}}_{n-r}(\widetilde{p}(u_i), \widetilde{p}(v_i)) = 0 \pmod{\frac{(\pi i)^n}{(n-r-1)!}}.$$

Proof. The lefthand side of (7.25) equals $\widehat{\mathcal{L}}_{n-r}(\widetilde{p}(\pi_{j, \dots, j}(\alpha)))$, and since \widetilde{p} is a proper realization, $\widetilde{p}(\pi_{j, \dots, j}(\alpha)) \in \widetilde{R}_{n-r}(\mathbb{C})$. The result then follows from the induction hypothesis that Theorem 2.33 holds for all $k < n$. \square

Lemma 7.18 (proof in Section 7.6). We have

$$(7.26) \quad \sum_{i=1}^K r_i A(\widetilde{p}(u_i), \widetilde{p}(v_i); \bar{k}_{ji}, \bar{l}_{ji}) = 0 \in \mathbb{C}.$$

Lemma 7.19 (proof in Section 7.6). For all $r = 0, \dots, n-3$, we have

$$(7.27) \quad \sum_{i=1}^K r_i A_r(\widetilde{p}(u_i), \widetilde{p}(v_i); \bar{k}_{ji}, \bar{l}_{ji}) = 0 \in \mathbb{C}.$$

Lemma 7.20. We have

$$(7.28) \quad \frac{1}{n!} \sum_{i=1}^K r_i (\bar{k}_{ji})^{n-1} (\bar{l}_{ji}) = 0 \pmod{\frac{(\pi i)^n}{(n-1)!}}.$$

Proof. This follows from the proper ambiguity condition. \square

7.6. Proof of Lemmas 7.18 and 7.19. Let $\widetilde{S}[\pi]_k$ and \widetilde{S}_k denote the subgroups generated by monomials of degree k . We shall identify $\text{Sym}^k(\widetilde{S}_1)$ with \widetilde{S}_k (and similarly for $\widetilde{S}[\pi]_k$). Note that a homomorphism $\widetilde{p}: \widetilde{S}[\pi]_1 \rightarrow \mathbb{C}$ extend to homomorphisms $\widetilde{p}: \widetilde{S}[\pi]_k \rightarrow \mathbb{C}$ by multiplication.

We have a homomorphism

$$(7.29) \quad \chi: \wedge^2(\widetilde{S}[\pi]_1) \rightarrow \widetilde{S}_1, \quad (u + p\widetilde{a}_\pi) \wedge (v + q\widetilde{a}_\pi) \mapsto pv - qu$$

and also projection homomorphisms

$$(7.30) \quad \Pi_k: \widetilde{S}[\pi]_{n-2} \rightarrow \widetilde{S}_k$$

defined by taking a monomial x to $x/\widetilde{a}_\pi^{n-2-k}$ if x is divisible by \widetilde{a}_π $n-2-k$ times and 0 otherwise.

Consider the map

$$(7.31) \quad \wedge^2 \text{Sym}^{n-2}: \mathbb{Z}[\text{SymbLogs}] \rightarrow \wedge^2(\tilde{\mathcal{S}}_1) \otimes \text{Sym}^{n-2}(\tilde{\mathcal{S}}_1), \quad (u, v) \mapsto (u \wedge v) \otimes u^{n-2}.$$

Note that by Corollary 6.2 we have $\wedge^2 \text{Sym}^{n-2}(\alpha) = 0$. Let $m: \tilde{\mathcal{S}}[\pi]_k \otimes \tilde{\mathcal{S}}[\pi]_l \rightarrow \tilde{\mathcal{S}}_{k+l}$ denote the canonical multiplication map. By Definition of the maps we have

$$(7.32) \quad m \circ (\chi \otimes \text{id}) \circ T_{j*}(\wedge^2 \text{Sym}^{n-2}(\alpha)) = \sum_{i=1}^K r_i (k_{ji} v_i - l_{ji} u_i) (u_i + k_{ji} \tilde{a}_\pi)^{n-2} \in \tilde{\mathcal{S}}[\pi]_{n-1}.$$

This proves Lemma 7.18. Similarly, for any r

$$(7.33) \quad m \circ (\chi \otimes \Pi_r) \circ T_{j*}(\wedge^2 \text{Sym}^{n-2}(\alpha)) = \sum_{i=1}^K r_i (k_{ji} v_i - l_{ji} u_i) \binom{n-2}{r} u_i^r k_{ji}^{n-2-r} \in \tilde{\mathcal{S}}_{r+1},$$

which proves Lemma 7.19.

Remark 7.21. Note that the proof above also shows that $\widehat{\mathcal{L}}_n(\beta) = 0$ modulo $\frac{(2\pi i)^n}{(n-1)!}$ if β is in the subgroup $\tilde{R}(\mathbb{C})_{++}$ defined in Section 2.3.4.

Remark 7.22. When $n = 3$, Proposition 7.15 also holds under the slightly weaker assumption that \tilde{p} is a proper realization of 2α . The proof of this is the same.

8. FUNCTIONAL RELATIONS AND QUIVER MUTATIONS

Our main inspiration for the results in this section is [GG⁺13]. In particular, we derive in Section 8.5 a lift of the 40 term relation for \mathcal{L}_3 from [GG⁺13].

Definition 8.1. A quiver Q is a finite oriented graph with no loops or 2-cycles together with a partition of its vertex set V_Q into *frozen vertices* and *mutable vertices*. The set of frozen vertices is denoted V_Q^0 . We also assume that V_Q is ordered.

Given two vertices i and j of a quiver Q , let ε_{ij} denote the number of edges from i to j (counting an edge from j to i as negative). Clearly, Q is uniquely determined by its vertex set, the set of frozen vertices, and the function $\varepsilon_Q: V_Q \times V_Q \rightarrow \mathbb{Z}$ taking (i, j) to ε_{ij} . Moreover, for any pair $V^0 \subset V$ of finite ordered sets, and any skew symmetric function $\varepsilon: V \times V \rightarrow \mathbb{Z}$ there is a unique quiver Q with $V_Q = V$, $V_Q^0 = V^0$, and $\varepsilon_Q = \varepsilon$.

Definition 8.2. A seed is a pair (Q, a) , where Q is a quiver with $V_Q = \{1, \dots, m\}$ and $u = \{a_1, \dots, a_m\}$ is an ordered set of generators of the quotient field $\mathbb{Q}(x_1, \dots, x_m)$ of a polynomial ring $\mathbb{Z}[x_1, \dots, x_m]$ in m -variables. We consider seeds only up to simultaneous reordering of the vertices and variables.

If (Q, a) is a seed and k a mutable vertex we can form a new seed $(\mu_k(Q), \mu_k(a))$, where $\mu_k(Q)$ is the quiver defined by $V_{\mu_k(Q)} = V_Q$, $V_{\mu_k(Q)}^0 = V_Q^0$ and

$$(8.1) \quad \varepsilon_{\mu_k(Q)}(i, j) = \begin{cases} -\varepsilon_{ij} & \text{if } k \in \{i, j\} \\ \varepsilon_{ij} & \text{if } \varepsilon_{ik}\varepsilon_{kj} \leq 0, \quad k \notin \{i, j\} \\ \varepsilon_{ij} + |\varepsilon_{ik}|\varepsilon_{kj} & \text{if } \varepsilon_{ik}\varepsilon_{kj} > 0, \quad k \notin \{i, j\}, \end{cases}$$

and $\mu_k(a) = (a'_1, \dots, a'_m)$ where

$$(8.2) \quad a'_k = \frac{1}{a_k} \left(\prod_{j|\varepsilon_{kj}>0} a_j^{\varepsilon_{kj}} + \prod_{j|\varepsilon_{kj}<0} a_j^{-\varepsilon_{kj}} \right), \quad a'_i = a_i \text{ for } i \neq j.$$

We say that $(\mu_k(Q), \mu_k(a))$ is obtained from (Q, a) by *mutation* at k .

Definition 8.3. Two seeds (Q, a) and (Q', a') are *mutation equivalent* if (Q', a') can be obtained from (Q, a) by a finite sequence of mutations. If so, we write $(Q, a) \sim (Q', a')$.

Remark 8.4. By [FZ03] the mutation class of a quiver is finite if and only if Q is an orientation of a Dynkin diagram of type A , D or E . If one, more generally, considers *valued quivers* (see e.g. [Mar13]) this extends to all finite Dynkin diagrams.

In the following we fix a quiver Q_0 with m vertices.

Definition 8.5. The *initial seed* of Q_0 is the seed (Q_0, a_0) with $a_0 = \{x_1, \dots, x_m\} \subset \mathbb{Q}(x_1, \dots, x_m)$.

Definition 8.6. An \mathcal{A} -coordinate of Q_0 is an element in the set A_{Q_0} defined as the union of all $a \subset \mathbb{Q}(x_1, \dots, x_m)$ where (Q, a) is a seed which is mutation equivalent to the initial seed of Q_0 .

Remark 8.7. It is known that each \mathcal{A} -coordinate is a Laurent polynomial in x_1, \dots, x_m [FZ02, FZ03].

Definition 8.8. The \mathcal{A} -variety of Q_0 is the scheme $\mathcal{A}_{Q_0} = \text{Spec}(\mathbb{Z}[A_{Q_0}])$, where $\mathbb{Z}[A_{Q_0}]$ is the \mathbb{Z} -algebra generated by the \mathcal{A} -coordinates of Q_0 . A *point* in $\mathcal{A}_{Q_0}(\mathbb{C})$ is a homomorphism $\mathbb{Z}[A_{Q_0}] \rightarrow \mathbb{C}$. A point is *non-degenerate* if it takes all \mathcal{A} -coordinates to non-zero complex numbers.

Definition 8.9. An \mathcal{X} -coordinate of Q_0 is an element in $\mathbb{Q}(x_1, \dots, x_m)$ of the form

$$(8.3) \quad X = \prod_{j=1}^m a_j^{\varepsilon_{kj}},$$

where (Q, a) is a seed equivalent to the initial seed of Q_0 and k is a mutable index of Q .

Remark 8.10. There is an \mathcal{X} -coordinate for each oriented edge in the *exchange graph* of Q_0 [Kel12]. We shall not need this notion here.

8.1. Grassmannians. Let $\text{Gr}(p, n)$ be the Grassmannian of p -planes in n -space. It is a projective scheme over \mathbb{Z} whose coordinate ring is generated by a *Plücker coordinate* a_I for each p -tuple I of elements in $\{1, \dots, n\}$ subject to the relations $a_{\sigma(I)} = \text{sgn}(\sigma)a_I$ as well as a *Plücker relation*

$$(8.4) \quad \sum_{k=1}^{p+1} a_{I \cup \{j_k\}} a_{J \setminus \{j_k\}} = 0$$

for each pair $I, J = (j_1, \dots, j_{p+1})$ of tuples of length $p-1$ and $p+1$, respectively. We may regard a point in $\text{Gr}(p, n)$ as a $p \times n$ matrix defined up to the action by $\text{GL}(p)$, and the Plücker coordinate a_I is simply the $p \times p$ minor determined by I . We are primarily interested in the affine cone $\widetilde{\text{Gr}}(p, n)$, which is simply the affine scheme determined by the above relations. A point may be regarded as a $p \times q$ matrix as above defined up to the action by $\text{SL}(p)$ instead of $\text{GL}(p)$.

The following result is essentially due to Scott [Sco06] (see also [GGS⁺13]).

Theorem 8.11. *For any $p, q > 1$, the Grassmannian $\widetilde{\text{Gr}}(p, p+q)$ is isomorphic to the \mathcal{A} -variety of the quiver $Q_{p,q}$ in Figure 1 with $1+pq$ vertices and an initial seed given by the Plücker coordinates $a_{1\dots p}$ and $a_{I_{p,q}}$ where*

$$(8.5) \quad I_{i,j} = \begin{cases} [i+1, p] \cup [n+1-j, n+i-j] & \text{if } i \leq j \\ [1, i-j] \cup [i+1, p] \cup [n+1-j, n] & \text{if } i > j. \end{cases}$$

Here $n = p+q$ and $[k, l]$ denotes the ordered set $\{k, \dots, l\}$. Also, every Plücker coordinate is an \mathcal{A} -coordinate.

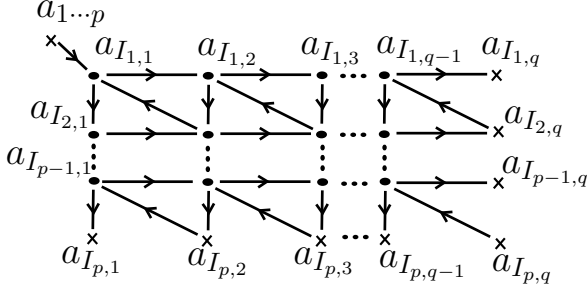


FIGURE 1. $Q_{p,q}$. The $p+q$ vertices marked by an x are frozen.

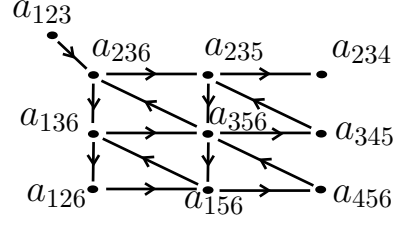


FIGURE 2. $Q_{3,6}$. Frozen vertices as in Figure 1.

In particular, all \mathcal{A} -coordinates are regular (not just rational) functions of the Plücker coordinates. For example, if one performs a mutation at the middle vertex of the seed in Figure 2 one obtains the \mathcal{A} -coordinate

$$(8.6) \quad \frac{a_{156}a_{236}a_{345} + a_{136}a_{235}a_{456}}{a_{356}} = a_{136}a_{245} - a_{126}a_{345} = a_{146}a_{235} - a_{156}a_{234} = a_{236}a_{145} - a_{123}a_{456}.$$

The equalities in (8.6) are all elementary consequences of the Plücker relations.

Remark 8.12. For $p \leq q$, the mutation class of the initial seeds for the quivers $Q_{p,q}$ are infinite unless $p = 2$ or $p = 3$ and $q \in \{3, 4, 5\}$.

8.2. The \mathcal{A} and \mathcal{X} -coordinates for $\widetilde{\text{Gr}}(3,6)$. The \mathcal{A} and \mathcal{X} -coordinates are well known for $\widetilde{\text{Gr}}(3,6)$ [GG⁺13]. The mutation class has 50 seeds, and there are 22 \mathcal{A} -coordinates consisting of the $\binom{6}{3} = 20$ Plücker coordinates as well as two additional coordinates

$$(8.7) \quad y_1 = \det(v_1 \times v_2, v_3 \times v_4, v_5 \times v_6), \quad y_2 = \det(v_2 \times v_3, v_4 \times v_5, v_6 \times v_1),$$

where the v_i are the column vectors of a representing matrix, and \times is the cross-product on 3-space. Clearly, y_1 and y_2 are regular functions (y_2 is the coordinate in (8.6)).

The symmetric group S_6 on six letters acts on $\widetilde{\text{Gr}}(3,6)$ by permuting the columns of a representing matrix. It also acts on the \mathcal{A} -variety by permuting the indices of the Plücker coordinates.

There are 104 \mathcal{X} -coordinates, which can all be obtained from the six \mathcal{X} -coordinates

$$(8.8) \quad \frac{a_{136}a_{235}}{a_{356}a_{123}}, \quad \frac{a_{126}a_{145}}{a_{124}a_{156}}, \quad \frac{a_{156}a_{236}a_{345}}{a_{136}a_{235}a_{456}}, \quad \frac{a_{123}a_{156}}{a_{126}a_{135}}, \quad \frac{a_{136}a_{145}a_{235}}{a_{123}a_{156}a_{345}}, \quad \frac{a_{123}a_{456}}{y_1}$$

by inversion $x \mapsto x^{-1}$ and the action by the (dihedral) group generated by

$$(8.9) \quad \sigma = (1, 2, 3, 4, 5, 6), \quad \tau = (1, 6)(2, 5)(3, 4).$$

Note that τ fixes y_1 and y_2 and σ flips them. The number of elements in the $\langle \sigma, \tau \rangle$ -orbits of the six \mathcal{X} -coordinates in (8.8) are 12, 12, 12, 6, 6, and 4, respectively.

8.3. Differential $\widehat{\mathcal{L}}_n$ relations from quivers. Let $\text{Mut}(Q_0)$ denote the set of triples (Q, a, k) , where (Q, a) is a seed mutation equivalent to the initial seed of Q_0 , and k is a mutable vertex. The \mathcal{X} -coordinate X of a triple (Q, a, k) satisfies

$$(8.10) \quad 1 + X = 1 + \prod_j a_j^{\varepsilon_{kj}} = \frac{\prod_{j|\varepsilon_{kj}>0} a_j^{\varepsilon_{kj}} + \prod_{j|\varepsilon_{kj}<0} a_j^{-\varepsilon_{kj}}}{\prod_{j|\varepsilon_{kj}<0} a_j^{-\varepsilon_{kj}}} = a_k a'_k \prod_{j|\varepsilon_{kj}<0} a_j^{\varepsilon_{kj}}.$$

Hence, if \tilde{a} denotes a logarithm of an \mathcal{A} -coordinate a , the element

$$(8.11) \quad \left(\sum_j \varepsilon_{kj} \tilde{a}_j, \tilde{a}_k + \tilde{a}'_k + \sum_{j|\varepsilon_{kj}<0} \varepsilon_{kj} \tilde{a}_j \right)$$

lies in $\widehat{\mathbb{C}}_{-+}$. Since Q_0 has countably many \mathcal{A} -coordinates, a choice of enumeration (the particular choice is inessential) allows us to associate a generator $a_i \in S$ and its lift $\tilde{a}_i \in \tilde{S}$ to each \mathcal{A} -coordinate. Hence, (8.11) can be regarded as a map

$$(8.12) \quad \widehat{\mathcal{X}}: \text{Mut}(Q_0) \rightarrow \text{SymbLogs}, \quad (Q, a, k) \mapsto \left(\sum_j \varepsilon_{kj} \tilde{a}_j, \tilde{a}_k + \tilde{a}'_k + \sum_{j|\varepsilon_{kj}<0} \varepsilon_{kj} \tilde{a}_j \right).$$

Let

$$(8.13) \quad \widehat{\mathcal{X}}(Q_0, n) = \frac{\mathbb{Z}[\widehat{\mathcal{X}}(\text{Mut}(Q_0))]}{\langle (u, v) + (-1)^n(-u, v - u) \rangle}$$

Recall the map

$$(8.14) \quad w_n: \mathbb{Z}[\text{SymbLogs}] \rightarrow \Omega_{n-1}^1(\tilde{S}), \quad (u, v) \mapsto u^{n-2}(udv - vdu).$$

One easily checks that $w_n(-u, v - u) = -(-1)^n w_n(u, v)$, so w_n induces a map

$$(8.15) \quad w_n: \widehat{\mathcal{X}}(Q_0, n) \rightarrow \Omega_{n-1}^1(\tilde{S}).$$

The following result is an immediate consequence of the definitions.

Proposition 8.13. Every element α in the kernel of $w_n: \widehat{\mathcal{X}}(Q_0, n) \rightarrow \Omega_{n-1}^1(\tilde{S})$ is a differential $\widehat{\mathcal{L}}_n$ relation. Any non-degenerate point in \mathcal{A}_{Q_0} together with a choice of logarithm of each \mathcal{A} -coordinate determines a realization where all sign pairs of \mathcal{V}_ε are $(-1, 1)$.

Remark 8.14. In all the examples below (although not in general), the symbolic log-pair associated to a triple only depends on the \mathcal{X} -coordinate, i.e. the map $\widehat{\mathcal{X}}: Q_0 \rightarrow \text{SymbLogs}$ factors through the set X_{Q_0} of \mathcal{X} -coordinates. Whenever this is the case, we may thus associate a symbolic log-pair $\widehat{\mathcal{X}}(X)$ to each \mathcal{X} -coordinate X . We stress that any realization of $\widehat{\mathcal{X}}(X)$ in $\widehat{\mathbb{C}}_{-+}$ maps to $-X$ in $\mathbb{C} \setminus \{0, 1\}$.

8.4. Basic examples.

8.4.1. *Two vertices joined by an edge.* Let Q_0 denote the quiver with two mutable vertices joined by an edge. The mutation class contains 5 seeds shown in Figure 3.

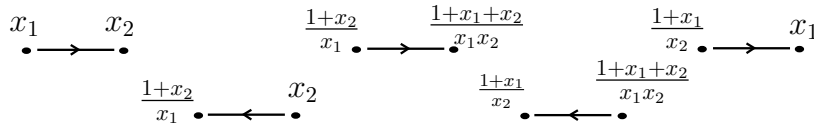


FIGURE 3. The five seeds for Q_0 .

There are thus five \mathcal{A} -coordinates

$$(8.16) \quad a_1 = x_1, \quad a_2 = x_2, \quad a_3 = \frac{1+x_2}{x_1}, \quad a_4 = \frac{1+x_1+x_2}{x_1 x_2}, \quad a_5 = \frac{1+x_1}{x_2},$$

and each of these is also an \mathcal{X} -coordinate. The group $\widehat{\mathcal{X}}(Q_0, 2)$ is generated by the five symbolic log-pairs

$$(8.17) \quad (\tilde{a}_2, \tilde{a}_3 + \tilde{a}_1), \quad (\tilde{a}_3, \tilde{a}_4 + \tilde{a}_2), \quad (\tilde{a}_4, \tilde{a}_5 + \tilde{a}_3), \quad (\tilde{a}_5, \tilde{a}_1 + \tilde{a}_4), \quad (\tilde{a}_1, \tilde{a}_2 + \tilde{a}_5),$$

and the kernel of w_2 is generated by the symbolic $\widehat{\mathcal{L}}_2$ relation in Example 2.22. The corresponding relation in $\mathbb{Z}[\mathbb{C} \setminus \{0, 1\}]$ is the element $FT^+[x_1, x_2]$ from 3.6.

8.4.2. $\widetilde{\text{Gr}}(2, 5)$. Let $Q_0 = Q_{2,3}$. The part of Q_0 involving only mutable vertices is isomorphic to the quiver in Example 8.4.1 above, so there are 5 mutation classes. One checks that $\widehat{\mathcal{X}}(Q_0, 2)$ is generated by

$$(8.18) \quad \widehat{\mathcal{X}}\left(\frac{a_{15}a_{24}}{a_{12}a_{45}}\right), \quad \widehat{\mathcal{X}}\left(\frac{a_{14}a_{23}}{a_{12}a_{34}}\right), \quad \widehat{\mathcal{X}}\left(\frac{a_{13}a_{45}}{a_{15}a_{34}}\right), \quad \widehat{\mathcal{X}}\left(\frac{a_{12}a_{35}}{a_{15}a_{23}}\right), \quad \widehat{\mathcal{X}}\left(\frac{a_{25}a_{34}}{a_{23}a_{45}}\right)$$

and that the kernel of w_2 is generated by the sum of the above 5 symbolic log-pairs. Its image in $\mathbb{Z}[\mathbb{C} \setminus \{0, 1\}]$ is

$$(8.19) \quad \left[-\frac{a_{15}a_{24}}{a_{12}a_{45}}\right] + \left[-\frac{a_{14}a_{23}}{a_{12}a_{34}}\right] + \left[-\frac{a_{13}a_{45}}{a_{15}a_{34}}\right] + \left[-\frac{a_{12}a_{35}}{a_{15}a_{23}}\right] + \left[-\frac{a_{25}a_{34}}{a_{23}a_{45}}\right] = FT^+\left[\frac{a_{15}a_{24}}{a_{12}a_{45}}, \frac{a_{14}a_{23}}{a_{12}a_{34}}\right].$$

Remark 8.15. More generally, for any quiver Q_0 and any embedding of the quiver in Example 8.4.1 into a quiver mutation equivalent to Q_0 , one gets a five term relation in the kernel of w_2 .

8.5. **A 40 term relation $\widehat{\mathcal{L}}_3$ relation on $\widetilde{\text{Gr}}(3, 6)$.** In [GG⁺13] the authors found a 40 term relation R_{40} for \mathcal{L}_3 expressed entirely in terms of the \mathcal{X} -coordinates on $\widetilde{\text{Gr}}(3, 6)$. We describe it below and note that our R_{40} has the opposite sign and also differs by inverting a few of the \mathcal{X} -coordinates (which is inconsequential since $\mathcal{L}_3(x) = \mathcal{L}_3(x^{-1})$).

For a subgroup G of a symmetric group and a G -set X we have a homomorphism

$$(8.20) \quad \text{Alt}_G: \mathbb{Z}[X] \rightarrow \mathbb{Z}[X], \quad [x] \mapsto \sum_{\sigma \in G} \text{sgn}(\sigma)[\sigma(x)].$$

If we let

$$(8.21) \quad \eta = \text{Alt}_{\langle \sigma^2, \tau \rangle} \left(\left[-\frac{a_{156}a_{236}a_{345}}{a_{136}a_{235}a_{456}} \right] - \left[-\frac{a_{126}a_{145}}{a_{124}a_{156}} \right] + \left[-\frac{a_{136}a_{145}}{a_{134}a_{156}} \right] \right) - \text{Alt}_{\langle \tau \rangle} \left(\left[-\frac{a_{136}a_{145}a_{235}}{a_{123}a_{156}a_{345}} \right] \right).$$

we can then write the relation as

$$(8.22) \quad \mathcal{L}_3(R_{40}) = 0, \quad R_{40} = \sigma(\eta) + \eta.$$

We show below that this relation lifts to a relation \widetilde{R}_{40} for $\widehat{\mathcal{L}}_3$. Given our theory this is a fairly elementary extension of the work in [GG⁺13], so we shall omit a few computational details.

For each k , the group $\widehat{\mathcal{X}}(Q_0, k)$ is generated by 52 symbolic log-pairs, one for each of the \mathcal{X} coordinates in the $\langle \sigma, \tau \rangle$ orbit of (8.8). Using that $1 + X$ is given by

$$(8.23) \quad \frac{a_{135}a_{236}}{a_{123}a_{356}}, \quad \frac{a_{125}a_{146}}{a_{124}a_{156}}, \quad \frac{a_{356}y_2}{a_{136}a_{235}a_{456}}, \quad \frac{a_{125}a_{136}}{a_{126}a_{135}}, \quad \frac{a_{135}y_2}{a_{123}a_{156}a_{345}}, \quad \frac{a_{124}a_{356}}{y_1}$$

for the \mathcal{X} -coordinates in (8.8), one can explicitly compute each of the 52 symbolic log-pairs. For example, one has

$$(8.24) \quad \widehat{\mathcal{X}}\left(\frac{a_{156}a_{236}a_{345}}{a_{136}a_{235}a_{456}}\right) = (\widetilde{a}_{156} + \widetilde{a}_{236} + \widetilde{a}_{345} - \widetilde{a}_{136} - \widetilde{a}_{235} - \widetilde{a}_{456}, \widetilde{a}_{356} + \widetilde{y}_2 - \widetilde{a}_{136} - \widetilde{a}_{235} - \widetilde{a}_{456}).$$

Let $\widetilde{R}_{40} = \sigma(\widetilde{\eta}) + \widetilde{\eta}$, were

$$(8.25) \quad \widetilde{\eta} = \widehat{\mathcal{X}} \left(\text{Alt}_{\langle \sigma^2, \tau \rangle} \left(\left[\frac{a_{156}a_{236}a_{345}}{a_{136}a_{235}a_{456}} \right] - \left[\frac{a_{126}a_{145}}{a_{124}a_{156}} \right] + \left[\frac{a_{136}a_{145}}{a_{134}a_{156}} \right] \right) - \text{Alt}_{\langle \tau \rangle} \left(\left[\frac{a_{136}a_{145}a_{235}}{a_{123}a_{156}a_{345}} \right] \right) \right).$$

Clearly the covering map r takes \widetilde{R}_{40} to R_{40} . The following is straightforward.

Lemma 8.16. The element \widetilde{R}_{40} is a symbolic $\widehat{\mathcal{L}}_3$ relation, which generates the kernel of w_3 .

Proposition 8.17. We have $2\tilde{R}_{40} \in \tilde{R}_3(\mathbb{C})$ for all realizations in $\mathbb{Z}[\widehat{\mathbb{C}}_{-+}]$.

Proof. By inspecting the terms one checks that all integers $\sum_i r_i k_{ji}^2 l_{ji}$ are 0, proving that \tilde{R}_{40} has proper ambiguity. We next show that all realizations of $2\tilde{R}_{40}$ are proper. There are 18 non-zero level 2 projections of \tilde{R}_{40} , namely $\pi_{a_{123}}(\tilde{R}_{40})$, $\pi_{a_{124}}(\tilde{R}_{40})$, and $\pi_{a_{136}}(\tilde{R}_{40})$ together with their orbits under σ . We wish to prove that their realizations in $\mathbb{Z}[\widehat{\mathbb{C}}_{-+}]$ are linear combinations of lifted non-alternating five term relations. The element $\pi_{a_{123}}(\tilde{R}_{40})$ has 14 terms, and a straightforward computation shows that it is a lift of

$$(8.26) \quad \begin{aligned} & FT^+ \left[\frac{a_{126}a_{135}}{a_{123}a_{156}}, \frac{a_{125}a_{345}}{a_{145}a_{235}} \right] + FT^+ \left[\frac{a_{125}a_{134}}{a_{123}a_{145}}, \frac{a_{135}a_{234}}{a_{123}a_{345}} \right] + FT^+ \left[\frac{a_{126}a_{235}}{a_{123}a_{256}}, \frac{a_{236}a_{345}}{a_{234}a_{356}} \right] - \\ & FT^+ \left[\frac{a_{135}a_{234}}{a_{123}a_{345}}, \frac{a_{136}a_{235}}{a_{123}a_{356}} \right] - FT^+ \left[\frac{a_{126}a_{134}}{a_{123}a_{146}}, \frac{a_{124}a_{345}}{a_{145}a_{234}} \right] - FT^+ \left[\frac{a_{126}a_{135}}{a_{123}a_{156}}, \frac{a_{136}a_{345}}{a_{134}a_{356}} \right]. \end{aligned}$$

Similarly, $\pi_{a_{124}}(\tilde{R}_{40})$ and $\pi_{a_{136}}(\tilde{R}_{40})$ both have nine terms and are linear combinations of 3 lifted non-alternating five term relations. Since such are 2-torsion in $\widehat{\mathcal{P}}'(\mathbb{C})$, it follows that all realizations of $2\tilde{R}_{40}$ are proper. One now has that the difference between any two realizations is in $\tilde{R}_3(\mathbb{C})$. We thus need only find a realization where all the 40 terms cancel out. This happens e.g. for the point represented by

$$(8.27) \quad \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & -2 \\ 0 & 1 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & \frac{1}{4} & 1 & 1 \end{pmatrix}.$$

This concludes the proof. \square

Corollary 8.18. One has $\widehat{\mathcal{L}}_3(\tilde{R}_{40}) = 0 \in \mathbb{C}/\frac{(\pi i)^3}{2}\mathbb{Z}$ for all realizations in $\mathbb{Z}[\widehat{\mathbb{C}}_{-+}]$.

Proof. It follows from Proposition 8.17 that $2\widehat{\mathcal{L}}_3(\tilde{R}_{40}) = 0 \in \mathbb{C}/\frac{(\pi i)^3}{2}\mathbb{Z}$. The fact that we can remove the 2 follows from Remark 7.22. \square

Remark 8.19. Experimental evidence suggests that $\widehat{\mathcal{L}}_3(\tilde{R}_{40})$ is in fact zero modulo $(2\pi i)^3$.

Remark 8.20. One can similarly prove that w_2 is generated by 25 non-alternating 5 term relations that all come from Remark 8.15.

Remark 8.21. For $\widetilde{\text{Gr}}(3, 7)$ one can show that w_3 is generated by 22 relations that are each an instance of \tilde{R}_{40} .

9. A LIFT OF GONCHAROV'S REGULATOR

Let $\widetilde{\text{Gr}}(p, q)^*(\mathbb{C})$ denote the subset of $\widetilde{\text{Gr}}(p, q)(\mathbb{C})$ consisting of points where all \mathcal{A} -coordinates are non-zero. Recall the complex $G_*(3)$ from Section 2.5. We shall instead use the subcomplex $G_*^{A \neq 0}(3)$ using $\widetilde{\text{Gr}}(3, k+1)^*(\mathbb{C})$ instead of $\widetilde{\text{Gr}}(3, k+1)(\mathbb{C})$.

9.1. Goncharov's regulator. Goncharov [Gon94] (later modified in [Gon96, Gon05b, Gon05a]) constructed a commutative diagram

$$(9.1) \quad \begin{array}{ccccccc} \mathbb{Z}[\widetilde{\text{Gr}}(3, 7)^*(\mathbb{C})] & \xrightarrow{\partial} & \mathbb{Z}[\widetilde{\text{Gr}}(3, 6)^*(\mathbb{C})] & \xrightarrow{\partial} & \mathbb{Z}[\widetilde{\text{Gr}}(3, 5)^*(\mathbb{C})] & \xrightarrow{\partial} & \mathbb{Z}[\widetilde{\text{Gr}}(3, 4)^*(\mathbb{C})] \\ \downarrow & & \downarrow g_5 & & \downarrow g_4 & & \downarrow g_3 \\ 0 & \longrightarrow & \mathcal{P}_3(\mathbb{C})_{\mathbb{Q}} & \xrightarrow{\delta_1} & (\mathcal{P}_2(\mathbb{C}) \otimes \mathbb{C}^*)_{\mathbb{Q}} & \xrightarrow{\delta_1} & \wedge^3(\mathbb{C}^*)_{\mathbb{Q}}. \end{array}$$

Defining $\Gamma_i(\mathbb{C}, 3) = \Gamma^{6-i}(\mathbb{C}, 3)$ we may regard it as a chain map $G_*^{A \neq 0}(3) \rightarrow \Gamma_*(\mathbb{C}, 3)$. The maps are defined as follows (see [Gon96]):

$$(9.2) \quad \begin{aligned} g_3 &= \frac{1}{6} \text{Alt}_4(a_{134} \wedge a_{124} \wedge a_{123}), \\ g_4 &= \frac{1}{12} \text{Alt}_5 \left([r(\bar{v}_1 | \bar{v}_2, \bar{v}_3, \bar{v}_4, \bar{v}_5)] \otimes a_{123} \right), \\ g_5 &= \frac{1}{90} \text{Alt}_6 \left(\begin{bmatrix} a_{124} a_{235} a_{136} \\ a_{125} a_{236} a_{134} \end{bmatrix} \right) \end{aligned}$$

Here Alt_n is shorthand for Alt_{S_n} . Also, a quadruple of points in $(x_1, x_2, x_3, x_4) \in P_{\mathbb{C}}^1 = \mathbb{C} \cup \{\infty\}$ has a *cross-ratio* $\frac{(x_1-x_3)(x_2-x_4)}{(x_1-x_4)(x_2-x_3)}$, and $r(\bar{v}_1 | \bar{v}_2, \bar{v}_3, \bar{v}_4, \bar{v}_5)$ denotes the cross-ratio of the projection of the quadruple (v_2, v_3, v_4, v_5) to $P(\mathbb{C}^3 / \langle v_1 \rangle) = P_{\mathbb{C}}^1$. We note that in [Gon96], g_4 has opposite sign due to v_2 being defined as $(1-x) \wedge x$ and not $x \wedge (1-x)$.

Let $C_*^{\text{gen}}(\text{SL}(3, \mathbb{C}))$ denote the chain complex generated in degree k by $k+1$ -tuples (g_0, \dots, g_k) of elements in $\text{SL}(3, \mathbb{C})$ satisfying that the $3 \times (k+1)$ matrix whose i th column is the first column of g_i represents a point in $\widetilde{\text{Gr}}(3, k+1)^*(\mathbb{C})$. A standard argument (as in [DZ06, Lemma 1.3]) shows that $C_*^{\text{gen}}(\text{SL}(3, \mathbb{C}))$ is acyclic, so we have a canonical map

$$(9.3) \quad \Gamma: H_*(\text{SL}(3, \mathbb{C})) = H_*(C_*^{\text{gen}}(\text{SL}(3, \mathbb{C})) \otimes_{\text{SL}(3, \mathbb{C})} \mathbb{Z}) \rightarrow H_*(G_*^{A \neq 0}(3))$$

induced by taking a generic tuple (g_0, \dots, g_k) to the point in $\widetilde{\text{Gr}}(3, k+1)^*(\mathbb{C})$ represented by the matrix with columns $g_0 e_1, \dots, g_k e_1$. Goncharov shows that the composition

$$(9.4) \quad H_5(\text{SL}(3, \mathbb{C})) \xrightarrow{\Gamma} H_5(G_*^{A \neq 0}(3)) \xrightarrow{g_5} \mathcal{B}_3(\mathbb{C}) \xrightarrow{\mathcal{L}_3} \mathbb{R}$$

is a non-zero rational multiple of the Borel regulator.

9.2. A lift of Goncharov's regulator. Since this is a straightforward improvement of Goncharov's result we shall omit some details. Recall the notation $\widehat{\mathcal{X}}(X)$ from Remark 8.14.

Consider the map $f_*: G_*^{A \neq 0}(3) \rightarrow \widehat{\Gamma}_*(\mathbb{C}, 3) = \widehat{\Gamma}^{6-*}(\mathbb{C}, 3)$ defined by (all other f_k are 0)

$$(9.5) \quad \begin{aligned} f_3 &= \text{Alt}_{\langle(1,2,3,4)\rangle}(\tilde{a}_{134} \wedge \tilde{a}_{124} \wedge \tilde{a}_{123}), \\ f_4 &= \text{Alt}_{\langle(1,2,3,4,5)\rangle} \left(\widehat{\mathcal{X}} \left(\frac{a_{125} a_{134}}{a_{123} a_{145}} \right) \otimes (\tilde{a}_{123} + \tilde{a}_{145}) \right), \\ f_5 &= \widehat{\mathcal{X}} \left(\text{Alt}_{\langle\sigma^2, \tau\rangle} \left(\left[\frac{a_{156} a_{236} a_{345}}{a_{136} a_{235} a_{456}} \right] - \left[\frac{a_{126} a_{145}}{a_{124} a_{156}} \right] + \left[\frac{a_{136} a_{145}}{a_{134} a_{156}} \right] \right) - \text{Alt}_{\langle\tau\rangle} \left(\left[\frac{a_{136} a_{145} a_{235}}{a_{123} a_{156} a_{345}} \right] \right) \right). \end{aligned}$$

Note that f_5 is the element $\tilde{\eta}$ from (8.25).

Theorem 9.1. *The map f_* is a chain map modulo 2-torsion. Moreover, the composition*

$$(9.6) \quad H_5(G_*^{A \neq 0}(3)) \xrightarrow{f_5} \widehat{\mathcal{B}}_3(\mathbb{C}) \xrightarrow{r} \mathcal{B}_3(\mathbb{C})_{\mathbb{Q}}$$

agrees with Goncharov's g_5 .

Proof. Commutativity of the rightmost square (visualized as in (9.1)) is elementary. By a direct computation one checks that

$$(9.7) \quad \delta_1 f_5 - f_4 \partial = \text{Alt}_{\langle\sigma^2, \tau\rangle} \left(\widehat{\mathcal{X}}(R_1) \otimes \tilde{a}_{123} + \widehat{\mathcal{X}}(R_2) \otimes \tilde{a}_{145} \right),$$

where R_1 and R_2 in the free abelian group on the \mathcal{X} -coordinates of $\widetilde{\text{Gr}}(3, 6)$ are given by

$$(9.8) \quad \begin{aligned} R_1 &= \left[\frac{a_{123}a_{245}}{a_{125}a_{234}} \right] + \left[\frac{a_{125}a_{134}}{a_{123}a_{145}} \right] + \left[\frac{a_{135}a_{234}}{a_{123}a_{345}} \right] + \left[\frac{a_{126}a_{135}}{a_{123}a_{156}} \right] - \left[\frac{a_{126}a_{134}}{a_{123}a_{146}} \right] \\ &\quad + \left[\frac{a_{123}a_{156}a_{245}}{a_{126}a_{145}a_{235}} \right] + \left[\frac{a_{136}a_{145}a_{235}}{a_{123}a_{156}a_{345}} \right] + \left[\frac{a_{123}a_{146}a_{345}}{a_{136}a_{145}a_{234}} \right] + \left[\frac{a_{126}a_{145}a_{234}}{a_{123}a_{146}a_{245}} \right] \\ R_2 &= \left[\frac{a_{145}a_{235}}{a_{125}a_{345}} \right] + \left[\frac{a_{136}a_{145}}{a_{134}a_{156}} \right] - \left[\frac{a_{126}a_{145}}{a_{124}a_{156}} \right] + \left[\frac{a_{125}a_{134}}{a_{123}a_{145}} \right] + \left[\frac{a_{124}a_{345}}{a_{145}a_{234}} \right] \\ &\quad - \left[\frac{a_{123}a_{156}a_{245}}{a_{126}a_{145}a_{235}} \right] - \left[\frac{a_{136}a_{145}a_{235}}{a_{123}a_{156}a_{345}} \right] - \left[\frac{a_{123}a_{146}a_{345}}{a_{136}a_{145}a_{234}} \right] - \left[\frac{a_{126}a_{145}a_{234}}{a_{123}a_{146}a_{245}} \right] \end{aligned}$$

Using (4.9) one checks that $\mathcal{X}(R_1)$ and $\mathcal{X}(R_2)$ are linear combinations of 3 lifted non-alternating five term relations up to six torsion. The images $x = r(\mathcal{X}(R_1))$ and $y = r(\mathcal{X}(R_2))$ are given by

$$(9.9) \quad \begin{aligned} x &= FT^+ \left[\frac{a_{125}a_{134}}{a_{123}a_{145}}, \frac{a_{126}a_{135}}{a_{123}a_{156}} \right] + FT^+ \left[\frac{a_{134}a_{156}}{a_{136}a_{145}}, \frac{a_{135}a_{234}}{a_{123}a_{345}} \right] - FT^+ \left[\frac{a_{124}a_{156}}{a_{126}a_{145}}, \frac{a_{125}a_{234}}{a_{123}a_{245}} \right] \\ y &= FT^+ \left[\frac{a_{125}a_{134}}{a_{123}a_{145}}, \frac{a_{126}a_{135}}{a_{123}a_{156}} \right] - FT^+ \left[\frac{a_{126}a_{135}}{a_{123}a_{156}}, \frac{a_{125}a_{345}}{a_{145}a_{235}} \right] + FT^+ \left[\frac{a_{126}a_{134}}{a_{123}a_{146}}, \frac{a_{124}a_{345}}{a_{145}a_{234}} \right]. \end{aligned}$$

This proves commutativity of the middle square. To prove commutativity of the leftmost square one first checks that $f_5\partial$ (regarded as an element in $\mathbb{Z}[\text{SymbLogs}]$) is in the kernel of w_3 . It then follows from Remark 8.21 that all realizations are linear combinations of instances of \widetilde{R}_{40} , which are 2-torsion in $\widehat{\mathcal{P}}_3(\mathbb{C})$ by Proposition 8.17. The last statement follows by showing that $\frac{1}{720} \text{Alt}_6(r(f_5)) = g_5$. This is a direct term by term comparison that does not use any relations in $\mathcal{P}_3(\mathbb{C})_{\mathbb{Q}}$. \square

Remark 9.2. One may define $f_*: G_*^{A \neq 0}(F, 3) \rightarrow \widehat{\Gamma}_*(F, 3)$ as above for an arbitrary field, but it is only a chain map up to 6-torsion.

10. COMPARING $\widehat{\mathcal{L}}_n$ AND \mathcal{L}_n

We now prove Theorem 2.10 and Theorem 2.36. Recall that

$$(10.1) \quad \mathcal{L}_n(z) = \mathfrak{R}_n \left(\sum_{r=0}^{n-1} \beta_r \text{Li}_{n-r}(z) \text{Log}(|z|) \right), \quad \beta_r = \frac{2^r}{r!} B_r.$$

Define signs

$$(10.2) \quad \eta_j = \begin{cases} (-1)^{\frac{j(j-1)}{2}} & n \text{ even} \\ (-1)^{\frac{j(j+1)}{2}} & n \text{ odd} \end{cases}, \quad \epsilon_j = \begin{cases} (-1)^{\frac{j}{2}} & j \text{ even} \\ (-1)^{\frac{j+1}{2}} & j \text{ odd.} \end{cases}$$

For non-negative integers i and j let

$$(10.3) \quad c_i = (1 - 2^{1-i})\beta_i, \quad c_{i,j} = \frac{c_i}{j!} \eta_j, \quad d_{i,j} = (-1)^{\frac{i+j}{2}} \epsilon_n \sum_{r=0}^i \frac{c_r}{(i+j+2-r)!}.$$

In particular, $c_0 = -1$. Note that up to a sign, the $c_{i,j}$ and $d_{i,j}$ are independent of n , and 0 when i is odd. We wish to prove that

$$(10.4) \quad \begin{aligned} \mathfrak{R}_n(\widehat{\mathcal{L}}_n(u, v)) - \mathcal{L}_n(z) &= \sum_{s=1}^{n-2} \left(\mathfrak{R}_{n-s}(\widehat{\mathcal{L}}_{n-s}(u, v)) \sum_{i=0}^s c_{i,s-i} \text{Re}(u)^i \text{Im}(u)^j \right) \\ &\quad + \det(u \wedge v) \sum_{i=0}^{n-2} d_{i,n-2-i} \text{Re}(u)^i \text{Im}(u)^j, \end{aligned}$$

where $z = r(u, v) \in \mathbb{C} \setminus \{0, 1\}$. We do this directly by expanding both sides of (10.4) and comparing terms. For notational simplicity let $\text{Li}_r(z) = x_r + iy_r$ and $\text{Log}(z) = a + bi$. We shall only compare the terms involving x_m and leave the analogous comparison of y_m terms and terms not involving any x_m or y_m to the reader. Letting $\text{Coeff}_{LHS}(x_m)$ and $\text{Coeff}_{RHS}(x_m)$ denote the coefficients of x_m when expanding the lefthand, respectively, righthand side of (10.4), we thus wish to prove that $\text{Coeff}_{LHS}(x_m) = \text{Coeff}_{RHS}(x_m)$ for all m . We assume for notational simplicity that $(u, v) = (a + bi + 2p\pi i, -x_1 - iy_1 + 2q\pi i) \in \widehat{\mathbb{C}}_{++}$.

10.1. The lefthand side. As a simple consequence of the formula (2.7) we have

$$(10.5) \quad \widehat{\mathcal{L}}_n(u, v) = \sum_{r=0}^{n-1} \frac{1}{r!} (-1)^r x_{n-r} (a + (b + 2p\pi)i)^r + i \sum_{r=0}^{n-1} \frac{1}{r!} (-1)^r y_{n-r} (a + (b + 2p\pi)i)^r \\ - (-1)^n \frac{n-1}{n!} x_1 (a + (b + 2p\pi)i)^{n-1} - i (-1)^n \frac{n-1}{n!} y_1 (a + (b + 2p\pi)i)^{n-1} \\ - \frac{2q\pi i}{(n-1)!} (-2p\pi i)^{n-1} + \frac{2q\pi i}{n!} (-1)^{n-1} (a + (b + 2p\pi)i)^{n-1}.$$

Using (10.5) and (10.1) we see that

$$(10.6) \quad \text{Coeff}_{LHS}(x_m) = \frac{(-1)^{n-m}}{(n-m)!} \mathfrak{R}_n \left((a + (b + 2p\pi)i)^{n-m} \right) - a^{n-m} \beta_{n-m} \mathbf{1}_{\text{odd}}(n) \quad \text{for } 1 < m \leq n, \\ \text{Coeff}_{LHS}(x_1) = (-1)^{n-1} \frac{n-1}{n!} \mathfrak{R}_n \left((a + (b + 2p\pi)i)^{n-1} \right) - a^{n-1} \beta_{n-1} \mathbf{1}_{\text{odd}}(n).$$

Lemma 10.1. We have

$$(10.7) \quad \mathfrak{R}_s((a + (b + 2p\pi)i)^{s-m}) = \sum_{k+l=s-m} \binom{s-m}{k} (-1)^{\frac{m+k+1}{2}} \epsilon_s \mathbf{1}_{\text{odd}}(m+k) a^k (b + 2p\pi)^l.$$

Proof. This is an elementary consequence of the binomial theorem. \square

It thus follows that $\text{Coeff}_{LHS}(x_m)$ can be written as sums of terms of the form $a^k (b + 2p\pi)^l$ where $k+l+m = n$. Let $\text{Coeff}_{LHS}(x_m, k, l)$ denote the coefficient of $a^k (b + 2p\pi)^l$ in $\text{Coeff}_{LHS}(x_m)$. By Lemma 10.1 it follows from (10.6) that

$$(10.8) \quad \text{Coeff}_{LHS}(x_m, k, l) = \begin{cases} \frac{(-1)^{n-m}}{k!l!} (-1)^{\frac{m+k+1}{2}} \epsilon_n \mathbf{1}_{\text{odd}}(m+k) & \text{for } l > 0 \\ \frac{(-1)^{n-m}}{k!l!} (-1)^{\frac{m+k+1}{2}} \epsilon_n \mathbf{1}_{\text{odd}}(m+k) - \mathbf{1}_{\text{odd}}(n) \beta_k & \text{for } l = 0 \end{cases}$$

for $m > 1$ and that

$$(10.9) \quad \text{Coeff}_{LHS}(x_1, k, l) \begin{cases} \frac{(-1)^{n-1}}{k!l!} \frac{n-1}{n} (-1)^{\frac{k+2}{2}} \epsilon_n \mathbf{1}_{\text{even}}(k) & \text{for } l > 0 \\ \frac{(-1)^{n-1}}{k!l!} \frac{n-1}{n} (-1)^{\frac{k+2}{2}} \epsilon_n \mathbf{1}_{\text{even}}(m+k) - \mathbf{1}_{\text{odd}}(n) \beta_k & \text{for } l = 0. \end{cases}$$

10.2. The righthand side. We shall need the following technical lemmas

Lemma 10.2. For any non-negative integers l and s we have

$$(10.10) \quad \sum_{j=0}^l (-1)^j \binom{l}{j} = 0 \text{ for } l > 0, \quad \sum_{j=0}^l \frac{(-1)^j \binom{l}{j}}{s+l-j} = \frac{(-1)^l}{s \binom{l+s}{l}}.$$

Proof. The first is elementary and the second can be found in [SWZ04, eq. (5)]. \square

Lemma 10.3. Let k and l be non-negative integers with $l > 1$ odd.

$$(10.11) \quad \sum_{i=0}^k \frac{c_i}{(k-i)!} = (-1)^{k-1} \beta_k, \quad \sum_{i=0}^{l-1} \frac{l-1-i}{(l-i)!} c_i = -\beta_{l-1}$$

Proof. Since $\sum_{r=0}^{\infty} \frac{B_r}{r!} x^r = \frac{x}{e^x-1}$ it follows that the generating function for β_i is the function $f(x) = \frac{2x}{e^{2x}-1}$. The first equation now follows from the fact that $e^x(f(x) - 2f(x/2)) = -f(-x)$ and the second from the fact that $(\cosh(x) - \frac{\sinh(x)}{x})(f(x) - 2f(x/2)) = 1 - x - f(x)$. \square

Since $(u, v) = (a + bi + 2p\pi i, -x_1 - iy_1 + 2q\pi i)$ it follows that

$$(10.12) \quad \det(u \wedge v) = x_1(b + 2p\pi) - y_1 a + 2\pi q a.$$

The right hand side of (10.4) expands to

$$(10.13) \quad \sum_{s=2}^{n-1} \left(\mathfrak{R}_s(\widehat{\mathcal{L}}_s) \sum_{i+j=n-s} c_{i,j} a^i (b+2p\pi)^j \right) + (x_1(b+2p\pi) - y_1 a + 2\pi q a) \sum_{i+j=n-2} d_{i,j} a^i (b+2p\pi)^j$$

and it follows from (10.5) that we have

$$(10.14) \quad \begin{aligned} \text{Coeff}_{RHS}(x_m) &= \sum_{s=m}^{n-1} \left(\frac{(-1)^{s-m}}{(s-m)!} \mathfrak{R}_s \left((a + (b + 2p\pi)i)^{s-m} \right) \sum_{i+j=n-s} c_{i,j} a^i (b + 2p\pi)^j \right) \\ \text{Coeff}_{RHS}(x_1) &= \sum_{s=2}^{n-1} \left((-1)^{s-1} \frac{s-1}{s!} \mathfrak{R}_s \left((a + (b + 2p\pi)i)^{s-1} \right) \sum_{i+j=n-s} c_{i,j} a^i (b + 2p\pi)^j \right) \\ &\quad + (b + 2p\pi) \sum_{i+j=n-2} d_{i,j} a^i (b + 2p\pi)^j. \end{aligned}$$

We can thus define $\text{Coeff}_{RHS}(x_m, k, l)$ for $k+l+m = n$ as above. Let's first assume that $m > 1$. By Lemma 10.1 $\text{Coeff}_{RHS}(x_m, k, l)$ is given by

$$(10.15) \quad \begin{aligned} &\sum_{i=0}^k \sum_{\substack{j=0 \\ (i,j) \neq (0,0)}}^l \frac{(-1)^{n-i-j-m}}{(n-i-j-m)!} \binom{n-i-j-m}{k-i} (-1)^{\frac{m+k-i+1}{2}} \epsilon_{n-i-j} \mathbf{1}_{\text{odd}}(m+k-i) c_{i,j} \\ &= (-1)^{n-m} (-1)^{\frac{m+k+1}{2}} \epsilon_n \mathbf{1}_{\text{odd}}(m+k) \left(\sum_{i=0}^k \sum_{j=0}^l \frac{c_i (-1)^{-j} \binom{l}{j}}{(k-i)! j!} - \frac{c_0}{k! l!} \right), \end{aligned}$$

which follows from the fact that $(-1)^{\frac{i}{2}} \epsilon_{n-i-j} = \epsilon_{n-j} = \epsilon_n \eta_j$ whenever i is even. By Lemma 10.2 and Lemma 10.3 it follows that this agrees with (10.8). We have thus proved that $\text{Coeff}_{LHS}(x_m) = \text{Coeff}_{RHS}(x_m)$ for $m > 1$.

Now let $m = 1$ and suppose $l > 0$. By (10.14) we see that $\text{Coeff}(x_1, k, l)$ equals

$$\begin{aligned}
(10.16) \quad & d_{k,l-1} + \sum_{\substack{i=0 \\ i \text{ even}}}^k \sum_{\substack{j=0 \\ (i,j) \neq (0,0)}}^l \frac{c_{i,j}(-1)^{n-j-1}(n-i-j-1)(-1)^{\frac{k+2}{2}} \epsilon_{n-j} \mathbf{1}_{\text{odd}}(1+k)}{(n-i-j)(k-i)!(l-j)!} \\
& = d_{k,l-1} + (-1)^{n-1}(-1)^{\frac{k+2}{2}} \epsilon_n \mathbf{1}_{\text{even}}(k) \left(\sum_{\substack{i=0 \\ i \text{ even}}}^k \sum_{j=0}^l \frac{c_i(-1)^{-j}(n-i-j-1) \binom{l}{j}}{(n-i-j)(k-i)!l!} - \frac{c_0(n-1)}{nk!l!} \right) \\
& = d_{k,l-1} + (-1)^{n-1}(-1)^{\frac{k+2}{2}} \epsilon_n \mathbf{1}_{\text{even}}(k) \left(- \sum_{\substack{i=0 \\ i \text{ even}}}^k \sum_{j=0}^l \frac{c_i(-1)^{-j} \binom{l}{j}}{(n-i-j)(k-i)!l!} - \frac{c_0(n-1)}{nk!l!} \right) \\
& = d_{k,l-1} - (-1)^{n-1}(-1)^{\frac{k+2}{2}} \epsilon_n \mathbf{1}_{\text{even}}(k) (-1)^l \sum_{\substack{i=0 \\ i \text{ even}}}^k \frac{c_i}{(n-i)!} + (-1)^{n-1}(-1)^{\frac{k+2}{2}} \epsilon_n \mathbf{1}_{\text{even}}(k) \\
& = (-1)^{n-1}(-1)^{\frac{k+2}{2}} \epsilon_n \mathbf{1}_{\text{even}}(k),
\end{aligned}$$

where the second last equality follows from Lemma 10.2. The fact that this equals (10.9) follows from Lemma 10.3.

Finally, when $l = 0$ (so that $k = n - 1$) a similar computation shows that $\text{Coeff}_{RHS}(x_1, k, 0)$ is given by

$$(10.17) \quad (-1)^{n-1}(-1)^{\frac{k+2}{2}} \epsilon_n \mathbf{1}_{\text{even}}(k) \left(\sum_{\substack{i=0 \\ i \text{ even}}}^{n-1} \frac{n-1-i}{(n-i)!} c_i - \frac{c_0(n-1)}{nk!l!} \right).$$

The fact that this agrees with (10.9) follows from Lemma 10.3. This concludes the proof of Theorem 2.10.

10.3. Proof of Theorem 2.36. For $s = 1, \dots, n - 2$, let

$$(10.18) \quad \Psi_s : \mathbb{Z}[\widehat{\mathbb{C}}_{\text{signs}}] \rightarrow \widehat{\mathcal{P}}_{n-s}(\mathbb{C}) \otimes \mathbb{C}^{\otimes s}, \quad \Psi_s = (\delta_1 \otimes \text{id})^{s-1} \circ \delta_1.$$

It takes $[(u, v)]$ to $[(u, v)] \otimes u^{\otimes s}$. Also, let (for $i = 0, \dots, s$)

$$(10.19) \quad \text{ReIm}_i : \mathbb{C}^{\otimes s} \rightarrow \mathbb{R}, \quad z_1 \otimes \cdots \otimes z_s \mapsto \prod_{k=1}^i \text{Re}(z_k) \prod_{k=i+1}^s \text{Im}(z_k)$$

Define

$$(10.20) \quad C_s = \sum_{i=0}^s c_{i,s-i} \text{ReIm}_i : \mathbb{C}^{\otimes s} \rightarrow \mathbb{R}, \quad D = \sum_{i=0}^{n-2} d_{i,n-2-i} \text{ReIm}_i : \mathbb{C}^{\otimes n-2} \rightarrow \mathbb{R}.$$

Let

$$(10.21) \quad \Delta : \widehat{\mathcal{P}}_n(\mathbb{C}) \rightarrow \mathbb{R}, \quad \alpha \mapsto \mathfrak{R}_n \circ \widehat{\mathcal{L}}_n(\alpha) - \mathcal{L}_n \circ r(\alpha).$$

It then follows from Theorem 2.10 that (where $m : \mathbb{R} \otimes \mathbb{R} \rightarrow \mathbb{R}$ is multiplication)

$$(10.22) \quad \Delta = m \circ \left(\sum_{s=1}^{n-2} (\mathfrak{R}_{n-s} \circ \widehat{\mathcal{L}}_{n-s}) \otimes C_s \right) \circ \Psi_s + (\det \circ D) \circ \Psi_{n-2}.$$

Since δ_1 takes $\widetilde{R}_n(\mathbb{C})$ to 0, the result follows.

REFERENCES

- [Blo86] Spencer Bloch. Algebraic cycles and the Beilinson conjectures. In *The Lefschetz centennial conference, Part I (Mexico City, 1984)*, volume 58 of *Contemp. Math.*, pages 65–79. Amer. Math. Soc., Providence, RI, 1986.
- [CP16] François Charles and Bjorn Poonen. Bertini irreducibility theorems over finite fields. *J. Amer. Math. Soc.*, 29(1):81–94, 2016.
- [CS85] Jeff Cheeger and James Simons. Differential characters and geometric invariants. In *Geometry and topology (College Park, Md., 1983/84)*, volume 1167 of *Lecture Notes in Math.*, pages 50–80. Springer, Berlin, 1985.
- [DS82] Johan L. Dupont and Chih Han Sah. Scissors congruences. II. *J. Pure Appl. Algebra*, 25(2):159–195, 1982.
- [DZ06] Johan L. Dupont and Christian K. Zickert. A dilogarithmic formula for the Cheeger-Chern-Simons class. *Geom. Topol.*, 10:1347–1372, 2006.
- [EV88] Hélène Esnault and Eckart Viehweg. Deligne-Beilinson cohomology. In *Beilinson’s conjectures on special values of L-functions*, volume 4 of *Perspect. Math.*, pages 43–91. Academic Press, Boston, MA, 1988.
- [FZ02] Sergey Fomin and Andrei Zelevinsky. Cluster algebras. I. Foundations. *J. Amer. Math. Soc.*, 15(2):497–529, 2002.
- [FZ03] Sergey Fomin and Andrei Zelevinsky. Cluster algebras. II. Finite type classification. *Invent. Math.*, 154(1):63–121, 2003.
- [Gan16] Herbert Gangl. Multiple polylogarithms in weight 4. *arXiv:1609.05557*, 2016.
- [GG⁺13] John Golden, Alexander B. Goncharov, Marcus Spradlin, Cristian Vergu, and Anastasia Volovich. Motivic amplitudes and cluster coordinates. *arXiv:1305.1617*, 2013.
- [Gon94] A. B. Goncharov. Polylogarithms and motivic Galois groups. In *Motives (Seattle, WA, 1991)*, volume 55 of *Proc. Sympos. Pure Math.*, pages 43–96. Amer. Math. Soc., Providence, RI, 1994.
- [Gon95] A. B. Goncharov. Geometry of configurations, polylogarithms, and motivic cohomology. *Adv. Math.*, 114(2):197–318, 1995.
- [Gon96] A. B. Goncharov. Deninger’s conjecture of L -functions of elliptic curves at $s = 3$. *J. Math. Sci.*, 81(3):2631–2656, 1996. Algebraic geometry, 4.
- [Gon05a] A. B. Goncharov. Polylogarithms, regulators, and Arakelov motivic complexes. *J. Amer. Math. Soc.*, 18(1):1–60, 2005.
- [Gon05b] Alexander B. Goncharov. Regulators. In *Handbook of K-theory. Vol. 1, 2*, pages 295–349. Springer, Berlin, 2005.
- [GR18] Alexander B. Goncharov and Daniil Rudenko. Motivic correlators, cluster varieties and zagier’s conjecture on $\zeta(f, 4)$. *arXiv:1803.08585*, 2018.
- [GZ07] Sebastian Goette and Christian Zickert. The extended Bloch group and the Cheeger-Chern-Simons class. *Geom. Topol.*, 11:1623–1635 (electronic), 2007.
- [Kel12] Bernhard Keller. Cluster algebras and derived categories. In *Derived categories in algebraic geometry*, EMS Ser. Congr. Rep., pages 123–183. Eur. Math. Soc., Zürich, 2012.
- [Lev97] Marc Levine. Lambda-operations, K -theory and motivic cohomology. In *Algebraic K-theory (Toronto, ON, 1996)*, volume 16 of *Fields Inst. Commun.*, pages 131–184. Amer. Math. Soc., Providence, RI, 1997.
- [Lew81] Leonard Lewin. *Polylogarithms and associated functions*. North-Holland Publishing Co., New York-Amsterdam, 1981. With a foreword by A. J. Van der Poorten.
- [Lew91] Leonard Lewin, editor. *Structural properties of polylogarithms*, volume 37 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1991.
- [Mar13] Robert J. Marsh. *Lecture notes on cluster algebras*. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2013.
- [MVW06] Carlo Mazza, Vladimir Voevodsky, and Charles Weibel. *Lecture notes on motivic cohomology*, volume 2 of *Clay Mathematics Monographs*. American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2006.
- [Neu04] Walter D. Neumann. Extended Bloch group and the Cheeger-Chern-Simons class. *Geom. Topol.*, 8:413–474 (electronic), 2004.
- [Sah89] Chih-Han Sah. Homology of classical Lie groups made discrete. III. *J. Pure Appl. Algebra*, 56(3):269–312, 1989.

- [Sco06] Joshua S. Scott. Grassmannians and cluster algebras. *Proc. London Math. Soc. (3)*, 92(2):345–380, 2006.
- [Sus87] A. A. Suslin. Algebraic K -theory of fields. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986)*, pages 222–244. Amer. Math. Soc., Providence, RI, 1987.
- [Sus90] A. A. Suslin. K_3 of a field, and the Bloch group. *Trudy Mat. Inst. Steklov.*, 183:180–199, 229, 1990. Translated in *Proc. Steklov Inst. Math.* **1991**, no. 4, 217–239, Galois theory, rings, algebraic groups and their applications (Russian).
- [SWZ04] B. Sury, Tianming Wang, and Feng-Zhen Zhao. Identities involving reciprocals of binomial coefficients. *J. Integer Seq.*, 7(2):Article 04.2.8, 12, 2004.
- [Voe02] Vladimir Voevodsky. Motivic cohomology groups are isomorphic to higher Chow groups in any characteristic. *Int. Math. Res. Not.*, (7):351–355, 2002.
- [Wei05] Charles Weibel. Algebraic K -theory of rings of integers in local and global fields. In *Handbook of K -theory. Vol. 1, 2*, pages 139–190. Springer, Berlin, 2005.
- [Zag91] Don Zagier. Polylogarithms, Dedekind zeta functions and the algebraic K -theory of fields. In *Arithmetic algebraic geometry (Texel, 1989)*, volume 89 of *Progr. Math.*, pages 391–430. Birkhäuser Boston, Boston, MA, 1991.
- [Zag07] Don Zagier. The dilogarithm function. In *Frontiers in number theory, physics, and geometry. II*, pages 3–65. Springer, Berlin, 2007.
- [Zic15] Christian K. Zickert. The extended Bloch group and algebraic K -theory. *J. Reine Angew. Math.*, 704:21–54, 2015.

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