TRIANGULATION INDEPENDENT PTOLEMY VARIETIES

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Abstract. The Ptolemy variety for $\mathrm{SL}(2,\mathbb{C})$ is an invariant of a topological ideal triangulation of a compact 3-manifold M. It is closely related to Thurston's gluing equation variety. The Ptolemy variety maps naturally to the set of conjugacy classes of boundary-unipotent $\mathrm{SL}(2,\mathbb{C})$-representations, but (like the gluing equation variety) it depends on the triangulation, and may miss several components of representations. In this paper, we define a Ptolemy variety, which is independent of the choice of triangulation, and detects all boundary-unipotent irreducible $\mathrm{SL}(2,\mathbb{C})$-representations. We also define variants of the Ptolemy variety for $\mathrm{PSL}(2,\mathbb{C})$-representations, and representations that are not necessarily boundary-unipotent. In particular, we obtain an algorithm to compute the full A-polynomial. All the varieties are topological invariants of M.

1. Introduction

The Ptolemy variety $P_n(\mathcal{T})$ a topologically ideal triangulation $\mathcal{T}$ of a compact 3-manifold M was defined by Garoufalidis, Thurston and Zickert [8]. It gives coordinates (called Ptolemy coordinates) for boundary-unipotent $\mathrm{SL}(n,\mathbb{C})$-representations of $\pi_1(M)$ in the sense that each point in $P_n(\mathcal{T})$ determines a representation (up to conjugation). The Ptolemy variety is explicitly computable for many census manifolds when $n = 2$ or 3 (see [5, 4] for a database), and invariants such as volume and Chern-Simons invariant can be explicitly computed from the Ptolemy coordinates. The Ptolemy variety, however, depends on the triangulation and may miss several components of representations.

We focus exclusively on the case when $n = 2$, so we omit the subscript n on the Ptolemy variety. Our goal is to define a refined Ptolemy variety $\overline{P}(\mathcal{T})$, which is a topological invariant of M and is guaranteed to detect all irreducible boundary-unipotent $\mathrm{SL}(2,\mathbb{C})$-representations. We also define refined variants of the Ptolemy variety for $\mathrm{PSL}(2,\mathbb{C})$-representations [8, 7], and for the enhanced Ptolemy variety [11] for $\mathrm{SL}(2,\mathbb{C})$-representations that are not necessarily boundary-unipotent. The refined variant of the latter detects all irreducible $\mathrm{SL}(2,\mathbb{C})$-representations.

Thurston's gluing equation variety also computes $\mathrm{PSL}(2,\mathbb{C})$-representations and the issue of triangulation dependence has existed since its inception. Segerman [9] defined a generalization of Thurston's gluing equation variety which detects at least the irreducible representations that don't have image in a generalized dihedral group (we can detect these as well). It is unclear to us whether Segerman's description provides an algorithm to compute it (it relies on deciding whether the lift of a normal surface is connected in the universal cover). Using Ptolemy coordinates, the resulting generalized variety has a more explicit description, is efficiently computable, and can easily be proven to be independent of the triangulation.

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1.1. **The Ptolemy variety and decorated representations.** The Ptolemy variety $P(\mathcal{T})$ is given by *non-zero* complex variables assigned to the edges of $\mathcal{T}$ subject to a *Ptolemy relation* for each simplex (see Section 3 for a review). The Ptolemy variety parametrizes so-called *generically decorated representations*, and thus only detects representations admitting a generic decoration (see Section 3). We stress that the definition of a decoration is independent of the ideal triangulation $\mathcal{T}$, but the notion of genericity depends on $\mathcal{T}$.

Any decorated boundary-unipotent representation defines Ptolemy coordinates satisfying the Ptolemy relations, but if the decoration is not generic, some Ptolemy coordinates are 0. In this case the Ptolemy relations alone are not enough to determine a (decorated) representation.

Remark 1.1. If all Ptolemy coordinates are nonzero, they define ideal simplex shapes satisfying Thurston's gluing equations. If some Ptolemy coordinates are 0 the simplices are degenerate.

We call a decorated representation *totally degenerate* if all Ptolemy coordinates are zero. This is equivalent to the condition that the image of the developing map (3.3) is a single point in $\mathrm{SL}(2,\mathbb{C})/B$, and implies that the representation is reducible. This notion is independent of the triangulation.

1.2. **Statement of results.** In Section 8 below we define a generalization $\overline{P}(\mathcal{T})$ of $P(\mathcal{T})$. This is defined via the usual Ptolemy coordinates and Ptolemy relations, but also involves additional coordinates and additional relations. The standard Ptolemy variety $P(\mathcal{T})$ embeds as a component of $\overline{P}(\mathcal{T})$, the one where all the standard coordinates are non-zero.

Recall that although the set of (boundary-unipotent) representations of $\pi_1(M)$ is a variety (algebraic set), the set of representations up to conjugation is not a variety in general.

Theorem 1.2. *The set of non totally degenerate decorated boundary-unipotent representations modulo conjugation is a variety. For each triangulation $\mathcal{T}$ of M, the generalized Ptolemy variety $\overline{P}(\mathcal{T})$ provides coordinates, i.e. there is an isomorphism of varieties*

$$\overline{P}(\mathcal{T}) \cong \left\{ \begin{matrix} \textit{Decorated, boundary-unipotent} \\ \pi_1(M) \to \mathrm{SL}(2,\mathbb{C}) \end{matrix} \right\} / \, \mathrm{Conj}\,. \tag{1.1}$$

Under this isomorphism, the standard Ptolemy variety $P(\mathcal{T})$ corresponds to the set of generically decorated representations. ☐

Since the righthand side is independent of the triangulation, we have:

Corollary 1.3. The Ptolemy variety $\overline{P}(\mathcal{T})$ is up to isomorphism independent of $\mathcal{T}$, i.e. it is a topological invariant of M. ☐

There is an action of the complex torus $(\mathbb{C}^*)^c$ on $\overline{P}(\mathcal{T})$, where c is the number of boundary components of M. The quotient $\overline{P}(\mathcal{T})_{\mathrm{red}}$ is called the *reduced Ptolemy variety.* A representation is *boundary-nondegenerate* if its restriction to $\pi_1(\partial_i M)$ is non-trivial for each boundary component $\partial_i M$ of M. Note that if M has a spherical boundary component no representation is boundary-nondegenerate.

Theorem 1.4. *The map*

$$\overline{P}(\mathcal{T})_{\mathrm{red}} \to \left\{ \begin{matrix} \textit{Boundary-unipotent} \\ \pi_1(M) \to \mathrm{SL}(2,\mathbb{C}) \end{matrix} \right\} / \, \mathrm{Conj} \tag{1.2}$$

induced from (1.1) by ignoring the decoration has image containing all irreducible representations and is one-to-one over the set of irreducible boundary-nondegenerate representations. ☐

Corollary 1.5. The set of conjugacy classes of irreducible, boundary-nondegenerate, boundary-unipotent representations is a variety. ☐

Remark 1.6. The preimage over a representation which is not boundary-nondegenerate is typically (although not always; see Remark 3.15) higher dimensional.

Remark 1.7. The image of (1.2) may contain reducible representations as well (see e.g. the example in Section 11.4.1), but such are necessarily boundary-degenerate (see Proposition 7.7).

Remark 1.8. One can detect all representations (including all reducible ones) by performing a single barycentric subdivision [8]. This is not feasible in practice since both the number of simplices and the expected dimension grow (generically, each non-ideal vertex increases the dimension by 1).

1.2.1. *The* $\mathrm{PSL}(2,\mathbb{C})$*-Ptolemy variety.* A manifold may have boundary-unipotent representations in $\mathrm{PSL}(2,\mathbb{C})$ that do not lift to boundary-unipotent representations in $\mathrm{SL}(2,\mathbb{C})$. For example, the geometric representation of a one-cusped hyperbolic manifold has no boundary-unipotent lifts (any lift of a longitude has trace -2, not 2 [2]). The obstruction to existence of a boundary-unipotent lift is a class $\sigma \in H^2(\widehat{M};\mathbb{Z}/2\mathbb{Z})$, where $\widehat{M}$ is the space obtained from M by collapsing each boundary component to a point. For each such class there is a Ptolemy variety $P^\sigma(\mathcal{T})$ (see [8, 7]). In Section 8 we define a generalization of $\overline{P}^\sigma(\mathcal{T})$. The analogue of Theorem 1.4 is the following.

Theorem 1.9. *Let* $k = \left|H^1(\widehat{M};\mathbb{Z}/2\mathbb{Z})\right|$. *The map*

$$\overline{P}^\sigma(\mathcal{T})_{\mathrm{red}} \to \left\{\begin{matrix} \textit{Boundary-unipotent} \\ \pi_1(M)\to \mathrm{PSL}(2,\mathbb{C}) \\ \textit{with obstruction class } \sigma \end{matrix}\right\} / \operatorname{Conj} \tag{1.3}$$

has image containing all irreducible representations and is $k:1$ *over the set of irreducible, boundary-nondegenerate representations.*

1.2.2. *The enhanced Ptolemy variety.* There is an enhanced Ptolemy variety $\mathcal{E}P(\mathcal{T})$ for $\mathrm{SL}(2,\mathbb{C})$-representations that are not necessarily boundary-unipotent [11]. It is defined for manifolds with torus boundary with a fixed choice of meridian μ_s and longitude λ_s for each boundary component $\partial_s M$. It involves the ususal Ptolemy coordinates together with additional coordinates m_s and l_s. We define a generalization $\mathcal{E}\overline{P}(\mathcal{T})$ of $\mathcal{E}P(\mathcal{T})$ in Section 8.

Theorem 1.10. *Suppose* M *has* c *torus boundary components. There is a map*

$$\mathcal{E}\overline{P}(\mathcal{T})_{\mathrm{red}} \to \left\{\begin{matrix} \textit{Boundary-Borel} \\ \pi_1(M)\to \mathrm{SL}(2,\mathbb{C}) \end{matrix}\right\} / \operatorname{Conj} \tag{1.4}$$

with image containing all irreducible representations. It is generically $2^c:1$ *over the irreducible, boundary-nondegenerate representations. Moreover, the projection to the* (m_s, l_s) *coordinates is the variety of eigenvalues of* μ_s *and* λ_s.

This can be used to compute the A-polynomial; see Section 11.3 for an example.

Remark 1.11. Theorems 1.9 and 1.10 were known for $\mathcal{P}^\sigma(\mathcal{T})$ and $\mathcal{E}P(\mathcal{T})$ and representations admitting a generic decoration (see [7, 8, 11]).

Remark 1.12. There is also a variant $\mathcal{E}P^\sigma(\mathcal{T})$ for boundary-Borel $\mathrm{PSL}(2,\mathbb{C})$-representations defined for each element σ in the cokernel of $H^1(\partial M;\mathbb{Z}/2\mathbb{Z}) \to H^2(\widehat{M};\mathbb{Z}/2\mathbb{Z})$. We will not develop this theory here. See Remark 11.1 for a brief discussion.

2. Description of the algorithm

We describe an algorithm that is guaranteed to find all boundary-unipotent irreducible representations of a manifold, no matter what triangulation $\mathcal{T}_0$ is used as input. The steps involve triangulations obtained from $\mathcal{T}_0$ by performing 2-3 moves, but these are purely auxiliary. We shall regard a transitive partition (Definition 7.1) as a pair $(\mathcal{T}, E)$.

Step 0: Compute all (non-zero) transitive partitions $(\mathcal{T}_0, E_0)$.

Step 1: If a pair $(\mathcal{T}, E)$ with $E \in \mathcal{E}(\mathcal{T})$ has a degenerate simplex, replace it by a descendant (Definition 8.9) $(\mathcal{T}', E')$ after performing a 2-3 move at a face adjacent to a non-degenerate and a degenerate simplex. This procedure reduces the number of degenerate simplices by one. Repeat until all simplices are non-degenerate.

Step 2: If a pair $(\mathcal{T}, E)$ has a degenerate face, replace it by the set of descendants $(\mathcal{T}', E')$ where $\mathcal{T}'$ is the triangulation obtained from $\mathcal{T}$ by performing a 2-3 move on each degenerate face. No descendants have degenerate faces.

Step 3: For each $E \in \mathcal{E}(\mathcal{T})$ from step 2 compute the primary decomposition of the reduced Ptolemy variety $P(\mathcal{T}, E)_{\mathrm{red}}$.

Step 4: For each zero dimensional component of $P(\mathcal{T}, E)_{\mathrm{red}}$, compute the corresponding boundary-unipotent $\mathrm{SL}(2, \mathbb{C})$-representations via the Bruhat cocycle (Section 5.2). The output is a matrix of exact algebraic expressions for each generator of $\pi_1(M)$ in the face pairing presentation.

Remark 2.1. The computation of $\overline{P}^{\sigma}(\mathcal{T})_{\mathrm{red}}$ and $\mathcal{E}\overline{P}(\mathcal{T})_{\mathrm{red}}$ follow the same steps.

Remark 2.2. If a Ptolemy variety contains a higher dimensional component C, one can compute the tautological representation $\pi_1(M) \to \mathrm{SL}(2, F(C))$, where $F(C)$ is the function field of C. See Section 11.4.3 for an example.

Remark 2.3. Among the 61911 manifolds in the SnapPy census `OrientableCuspedCensus` there is no manifold with more than 53 edge partitions (average 16). Most of the edge partitions are only mildly degenerate (72%), so steps 1 and 2 above are rarely needed.

3. Ptolemy coordinates, decorations and cocycles

We refer to [8] and [6] for the general theory, and to [7] for a review of the case when $n = 2$ with many worked out examples.

Let M be a compact manifold with non-empty boundary and let $\widetilde{M}$ be the universal cover of M. Let $\widehat{M}$ and $\widehat{\widetilde{M}}$ denote the spaces obtained from M and $\widetilde{M}$, respectively, by collapsing each boundary component to a point. Given a topologically ideal triangulation $\mathcal{T}$ of M we refer to the cells as *vertices*, *edges*, *faces* and *simplices* of $\mathcal{T}$.

3.1. Ptolemy assignments.

Fix a (topologically ideal) triangulation $\mathcal{T}$ of M.

Definition 3.1. A *Ptolemy assignment* on an ordered simplex Δ is an assignment of a non-zero complex number c_{ij} to each oriented edge ε_{ij} of Δ satisfying the *Ptolemy relation*

$$c_{03}c_{12} + c_{01}c_{23} = c_{02}c_{13}, \tag{3.1}$$

and the *edge orientation relations* $c_{ji} = -c_{ij}$.

Definition 3.2. A *Ptolemy assignment* on $\mathcal{T}$ is a Ptolemy assignment on each simplex Δ_k such that the Ptolemy coordinates $c_{ij,k}$ satisfy the *identification relations*

$$c_{ij,k} = c_{i'j',k'}, \qquad \text{when} \quad \varepsilon_{ij,k} \sim \varepsilon_{i'j',k'} \tag{3.2}$$

where $\sim$ denotes identification of oriented edges.

Definition 3.3. The *Ptolemy variety* $P(\mathcal{T})$ is the Zariski open subset where $c_{ij,k} \neq 0$ of the zero set of the ideal in $\mathbb{Q}[\{c_{ij,k}\}]$ generated by the Ptolemy relations, the identification relations, and the edge orientation relations. As a set it is equal to the set of Ptolemy assignments on $\mathcal{T}$.

Remark 3.4. The purpose of the edge orientation relations is to make the Ptolemy variety "order agnostic", i.e. independent of the choice of vertex orderings. Whenever convenient we shall eliminate the edge orientation relations and only consider $c_{ij,k}$ with $i < j$. By choosing once and for all representatives for each (oriented) edge of $\mathcal{T}$, the Ptolemy variety can be given by a variable for each edge and a relation for each simplex.

3.2. **Decorations.** Let $B \subset \mathrm{SL}(2,\mathbb{C})$ denote the subgroup of upper triangular matrices, and P the subgroup of upper triangular matrices with 1's on the diagonal. Let $G = \mathrm{SL}(2,\mathbb{C})$ and let $H \subset G$ denote either P or B.

Definition 3.5. A (G,H)-representation is a representation $\pi_1(M) \to G$ taking each peripheral subgroup $\pi_1(\partial_i M)$ to a conjugate of H. Such are called *boundary-unipotent* for $H = P$ and *boundary-Borel* for $H = B$.

Definition 3.6. A (G,H)-representation is *boundary-nondegenerate* if its restriction to $\pi_1(\partial M_i)$ is non-trivial for each boundary component $\partial_i M$ of M.

Let $I(\widetilde{M})$ denote the set of ideal vertices of $\widetilde{M}$, i.e. vertices of $\widehat{\widetilde{M}}$ with the triangulation induced by $\mathcal{T}$. Note that each ideal point corresponds to a boundary component of $\widetilde{M}$, so $I(\widetilde{M})$ is independent of $\mathcal{T}$.

Definition 3.7. A *decoration* of a simplex Δ is an assignment of a coset g_iH to each vertex v_i of Δ. A decoration is thus a tuple (g_0H, g_1H, g_2H, g_3H), and we consider two decorations to be equal if the tuples differ by multiplication by an element in $\mathrm{SL}(2,\mathbb{C})$. A decoration is *generic* if the cosets are distinct as B-cosets (distinct if $H = B$).

Definition 3.8. Let ρ be a (G,H)-representation. A *decoration* of ρ is a ρ-equivariant map

$$D\colon I(\widetilde{M}) \to G/H, \tag{3.3}$$

i.e. an equivariant assignment of H-cosets to the ideal points of $\widetilde{M}$. When the representation plays no role, we shall refer to a decorated representation simply as a decoration. Since gD is a decoration of $g\rho g^{-1}$ if D is a decoration of ρ, we consider two decorations to be equal if they differ by left multiplication by an element in G.

Remark 3.9. A boundary-unipotent representation is also boundary-Borel. We shall refer to decorations as P-decorations or B-decorations depending on context.

Definition 3.10. A decoration is *generic* if the induced decoration of each simplex of $\mathcal{T}$ is generic.

Remark 3.11. Every (G,H)-representation has a decoration, but whether a decoration is generic depends on $\mathcal{T}$.

Remark 3.12. A B-coset determines a point in $\partial\overline{\mathbb{H}}^3 = \mathbb{C} \cup \{\infty\}$, the boundary of hyperbolic 3-space, via $gB \leftrightarrow g\infty$. A decoration of a boundary-Borel representation thus determines a developing map (see e.g. Zickert [12]) assigning ideal simplex shapes to the simplices of $\mathcal{T}$. A decoration is generic if and only if all shapes are non-degenerate. We shall not need this here.

3.2.1. *Freedom in the choice of decoration.* If we choose points $e_i \in I(\widetilde{M})$, one for each boundary component $\partial_i M$ of M, a decoration is uniquely determined by the cosets $D(e_i)$. The freedom in the choice of $D(e_i)$ is determined by the image of the boundary components.

Proposition 3.13. Let ρ be a (G,B)-representation and let e_i be as above.

(i) If $\rho(\pi_1(\partial_i(M))$ is trivial, $D(e_i)$ may be chosen arbitrarily.
(ii) If $\rho(\pi_1(\partial_i(M))$ is non-trivial and unipotent, $D(e_i)$ is uniquely determined by ρ.
(iii) If $\rho(\pi_1(\partial_i(M))$ is non-trivial and diagonalizable, $D(e_i)$ is determined by ρ up to a $\mathbb{Z}/2\mathbb{Z}$-action.

Corollary 3.14. A boundary-nondegenerate (G,P)-representation has a unique B-decoration. A boundary-nondegenerate (G,B)-representation generically has 2^c decorations, where c is the number of boundary components.

Remark 3.15. Different choices of $D(e_i)$ may give rise to equal decorations, i.e. decorations differing only by left multiplication by an element in G. Hence, even when a boundary-component is collapsed, there may be only finitely many decorations (see e.g. the example in Section 11.4.1).

3.3. **Natural cocycles.** The triangulation $\mathcal{T}$ of M induces a decomposition $\overline{\mathcal{T}}$ of M by truncated simplices.

Definition 3.16. A *natural cocycle* on a truncated simplex $\overline{\Delta}$ is a labeling of the oriented edges by elements in $\mathrm{SL}(2,\mathbb{C})$ such that

(i) The product around each face (triangular and hexagonal) is I, the identity matrix.
(ii) Flipping the orientation replaces a labeling by its inverse, i.e. $\alpha_{ij}\alpha_{ji} = I = \beta^k_{ij}\beta^k_{ji}$.
(iii) Short edges are labeled by elements β^k_{ij} in P.
(iv) Long edges are labeled by counter diagonal elements α_{ij}.

The indexing is such that α_{ij} is the labeling of the long edge from vertex i to j, and β^k_{ij} is the labeling of the short edge near vertex k parallel to the edge from i to j; see Figure 2.

Definition 3.17. A natural cocycle on $\overline{\mathcal{T}}$ is a natural cocycle on each truncated simplex such that the labelings of identified edges agree.

Remark 3.18. Decorations, (G,H)-representations, and natural cocycles are also defined for $G = \mathrm{PSL}(2,\mathbb{C})$. The analogue of Corollary 3.14 holds as well.

3.4. **The diagonal action.** If c is the number of boundary components and $T \subset G$ is the subgroup of diagonal matrices, the torus T^c acts on P-decorations, Ptolemy assignments and natural cocycles. This action is called the *diagonal action* and is illustrated in Figure 1.

Definition 3.19. The quotient of $P(\mathcal{T})$ by the diagonal action is called the *reduced Ptolemy variety* $P(\mathcal{T})_{\mathrm{red}}$.

3.5. **The fundamental correspondence.** The following result is proved in [8] for $\mathrm{SL}(n,\mathbb{C})$. For $n = 2$ the correspondence is particularly simple, and is illustrated in Figure 2.

Theorem 3.20. *We have a one-to-one-correspondence*

$$\{\textit{Generic } P\textit{-decorations}\} \overset{1:1}{\longleftrightarrow} P(\mathcal{T}) \overset{1:1}{\longleftrightarrow} \{\textit{Natural cocycles on } \overline{\mathcal{T}}\} \tag{3.4}$$

respecting the diagonal action.

Remark 3.21. A decoration $(g_0P,\dots,g_3P)$ is generic if and only if $c_{ij} \neq 0$, where c_{ij} are defined as in Figure 2. Even if a decoration is not generic, the Ptolemy relation is still satisfied.

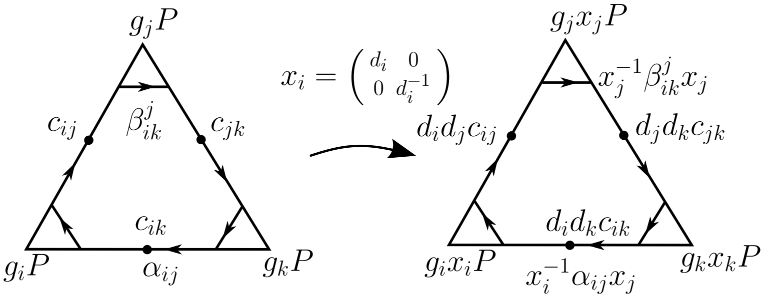


FIGURE 1. The diagonal action on decorations, Ptolemy assignments and natural cocycles.

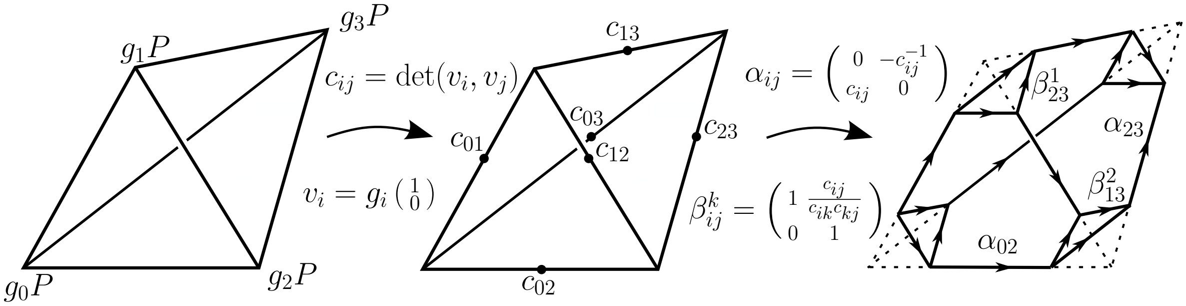


FIGURE 2. The fundamental correspondence.

Remark 3.22. Given a decoration D, one can define for each pair (x, y) of ideal points a complex number $c_{(x,y)} = \det(g_x \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), g_y \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right))$ where $D(x) = g_xP$ and $D(y) = g_yP$. The set of decorations thus embeds into an infinite dimensional affine space $A(M)$ with a variable for each pair of ideal points. As we shall see the image is always finite dimensional, and each triangulation provides explicit coordinates.

4. OTHER VARIANTS OF THE PTOLEMY VARIETY

References for Section 4.1 are [8, 7], and the reference for Section 4.2 is [11].

4.1. **Ptolemy coordinates for** $\mathrm{PSL}(2,\mathbb{C})$**-representations.** Let $\sigma \in C^2(\widehat{M}; \mathbb{Z}/2\mathbb{Z})$ be a cellular cocycle representing a class in $H^2(\widehat{M}; \mathbb{Z}/2\mathbb{Z})$. Although σ may not be a coboundary, its restriction σ_k to each simplex Δ_k is a coboundary. Fix $\eta_k \in C^1(\Delta_k; \mathbb{Z}/2\mathbb{Z})$ such $\delta(\eta_k) = \sigma_k$. We identify $\mathbb{Z}/2\mathbb{Z}$ with $\{\pm 1\}$.

Definition 4.1. The Ptolemy variety $P^\sigma(\mathcal{T})$ is the variety generated by the Ptolemy relations, the edge orientation relations, and the *modified edge identification relations*

$$c_{ij,k} = (\eta_{ij,k}\eta_{i'j',k'})c_{i'j',k'}, \qquad \text{if } \varepsilon_{ij,k} = \varepsilon_{i'j',k'}. \tag{4.1}$$

Here $\eta_{ij,k}$ denotes the value of η_k on the ij-edge of Δ_k.

Up to canonical isomorphism this is independent of the choices of η_k, and only depends on the class of σ in $H^2(\widehat{M}; \mathbb{Z}/2\mathbb{Z})$.

Remark 4.2. In [7], the Ptolemy relations are modified, but not the identification relations. One easily checks that the two definitions agreee.

Theorem 4.3. *The analogue of the fundamental correspondence* (3.4) *is*

(4.2)
$$\begin{tikzcd}\left\{\begin{matrix}\textit{Generically decorated}\\ (\mathrm{PSL}(2,\mathbb{C}),P)\textit{-representations}\\ \textit{with obstruction class } \sigma\end{matrix}\right\} & P^\sigma(\mathcal{T}) \arrow[l, "1:z"'] \arrow[r, "z:1"] & \left\{\begin{matrix}\textit{Natural } \mathrm{PSL}(2,\mathbb{C})\textit{-cocycles on } \overline{\mathcal{T}}\\ \textit{with obstruction class } \sigma\end{matrix}\right\} \arrow[ll, leftrightarrow, bend left, "1:1"]\end{tikzcd},$$

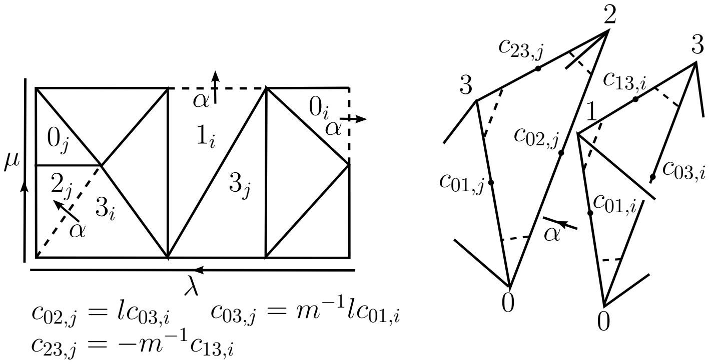
where z is the order of $Z^1(\widehat{M};\mathbb{Z}/2\mathbb{Z})$, the group of $\mathbb{Z}/2\mathbb{Z}$-valued 1*-cocycles.*

Remark 4.4. A boundary in $Z^1(\widehat{M};\mathbb{Z}/2\mathbb{Z})$ acts trivially on the reduced Ptolemy variety, so the map from $P^\sigma(\mathcal{T})_{\mathrm{red}}$ to the set of B-decorations is $k:1$, where $k=\left|H^1(\widehat{M};\mathbb{Z}/2\mathbb{Z})\right|$.

4.2. **Non-boundary-unipotent representations.** Assume that each boundary component $\partial_s M$ is a torus, and that we have fixed a meridian μ_s and a longitude λ_s in $H_1(\partial_s M)$. The definition of $\mathcal{E}P(\mathcal{T})$ involves a choice of *fundamental rectangle* R_s for each boundary component $\partial_s M$, such that the triangulation induced on the torus obtained by identifying the sides of R_s agrees with the triangulation of $\partial_s M$ induced by $\mathcal{T}$. It is given by the usual variables $c_{ij,k}$ as well as additional variables m_s, l_s indexed by the boundary components.

Definition 4.5. The *enhanced Ptolemy variety* $\mathcal{E}P(\mathcal{T})$ is the Zarisky open subset, where all $c_{ij,k}$ and all m_s, l_s are non-zero, of the zero set of the ideal in $\mathbb{Q}[\{c_{ij,k}\}\cup\{m_s,l_s\}]$ generated by the Ptolemy relations, the edge orientation relations together with *modified identification relations* of the form $c_{ij,k}=pc_{i'j',k'}$, where p is a monomial in the m_s and l_s.

We refer to [11] for the precise definition of the modified identification relations. They are illustrated in Figure 3 in the case where there is a single boundary torus, and where the sides μ' and λ' of the fundamental rectangle agree with μ and λ (if they don't agree perform the appropriate coordinate change).



$c_{02,j}=lc_{03,i}$ $c_{03,j}=m^{-1}lc_{01,i}$
$c_{23,j}=-m^{-1}c_{13,i}$

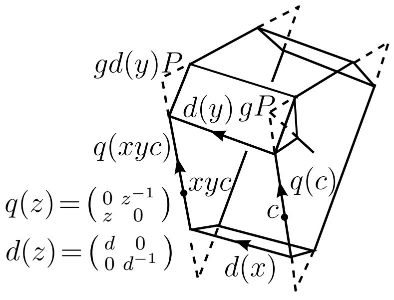


FIGURE 3. The identification relations for a face pairing α.

FIGURE 4. Face pairing edges and the fundamental correspondence.

Definition 4.6. Let $\rho\colon \pi_1(M)\to \mathrm{SL}(2,\mathbb{C})$ be boundary-Borel. A *P-decoration* of ρ is a ρ-equivariant assignment of P-cosets to each triangular face of $\widetilde{\overline{\mathcal{T}}}$.

Definition 4.7. The *fattened decomposition* of M is the decomposition obtained by thickening each hexagonal face. The edges consist of the usual long and short edges together with *face* *pairing edges* (see Figure 4). A *fattened natural cocycle* is a cocycle on the fattened decomposition restricting to a natural cocycle on each truncated simplex and labeling face pairing edges by diagonal elements.

Theorem 4.8. *The analogue of the fundamental correspondence is*

$$\{\textit{Generic } P\textit{-decorations}\} \overset{1:1}{\longleftrightarrow} \mathcal{E}P(\mathcal{T}) \overset{1:1}{\longleftrightarrow} \{\textit{Fattened natural cocycles on } \overline{\mathcal{T}}\}\,. \tag{4.3}$$

Moreover, the projection of $\mathcal{E}P(\mathcal{T})$ onto the m_s, l_s coordinates is the eigenvalue variety. In particular, if M has a single torus boundary component, the one-dimensional components give rise to factors of the A-polynomial.

5. GENERALIZATION OF THE FUNDAMENTAL CORRESPONDENCE

The fundamental correspondence between *generic* decorations, Ptolemy assignments, and natural cocycles plays an important role. The Ptolemy variety gives explicit coordinates enabling concrete computations, the decorations establish the link to group homology allowing for explicit computation of the Cheeger-Chern-Simons class [8], and the natural cocycles allow us to explicitly recover a representation from its coordinates.

As mentioned in Remark 3.21 a non-generic decoration still determines a Ptolemy assignment (where some coordinates are allowed to be zero) via the map in Figure 2, but this map is neither injective nor surjective. For example, if three Ptolemy coordinates on a face are zero, the Ptolemy relation becomes $0 = 0$, and we can not recover the decoration. Also, a Ptolemy assignment where all but one of the Ptolemy coordinates are 0, can never arise from a decoration. In this section we generalize the one-to-one correspondence between decorations and natural cocycles. The definition of the generalized Ptolemy variety is carried out in Section 8.

5.1. Cocycles, and the action by coboundaries.

Let $G = \mathrm{SL}(2, \mathbb{C})$. Recall that the triangulation $\mathcal{T}$ induces a decomposition $\overline{\mathcal{T}}$ of M by truncated simplices.

Definition 5.1. A *G-cocycle* on M is a labeling of the oriented edges of M satisfying (i) and (ii) of Definition 3.16. A G-cocycle satisfying (iii) as well is called a (G, P)-cocycle.

A G-cocycle τ determines (up to conjugation) a representation $\pi_1(M) \to G$ by taking products along edge paths. Note that if τ is a (G, P)-cocycle this representation is boundary-unipotent and is canonically decorated. Hence, a (G, P)-cocycle determines a decoration.

Definition 5.2. If D is a decoration and τ a (G, P)-cocycle, we say that τ is compatible with D if the decoration determined by τ equals D.

Let $V(\overline{\mathcal{T}})$ denote the set of vertices of $\overline{\mathcal{T}}$. Given an oriented edge e of $\overline{\mathcal{T}}$, let $e_0, e_1 \in V(\overline{\mathcal{T}})$, denote the starting and ending vertex of e, respectively.

Definition 5.3. A *zero-cochain* is a map $\eta\colon V(\overline{\mathcal{T}}) \to G$. The *coboundary* of a zero-cochain η is the G-cocycle labeling an oriented edge e by $\eta(e_0)^{-1}\eta(e_1)$.

Definition 5.4. The *coboundary action* of a *P-valued* zero-cochain $\eta\colon V(\overline{\mathcal{T}}) \to P$ on a (G, P)-cocycle τ replaces τ by $\eta\tau$, the cocycle defined by

$$\eta\tau(e) = \eta(e_0)^{-1}\tau(e)\eta(e_1). \tag{5.1}$$

Note that the coboundary action does not change the decorated representation.

5.2. Bruhat cocycles.

Definition 5.5. A *Bruhat cocycle* is a natural cocycle as in Definition 3.16, but where long edges may be either diagonal or counter-diagonal. A zero-cochain $\eta\colon V(\overline{\mathcal{T}})\to P$ *preserves* a Bruhat cocycle τ if $\eta\tau$ is again a Bruhat cocycle.

The motivation for our definition is the following corollary of Bruhat decomposition.

Lemma 5.6. Let gP and hP be cosets. If $gB\neq hB$ there are unique coset representatives gx_0 and hx_1 such that $(gx_0)^{-1}hx_1$ is counterdiagonal. If $gB=hB$ there are (not unique) coset representatives such that $(gx_0)^{-1}hx_1$ is diagonal.

Proof. By Bruhat decomposition, every $g\in G$ can be decomposed as b_1wb_2, where w is either $\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ and $b_i\in B$. The result is an elementary consequence. ☐

Corollary 5.7. For every decoration $D=(g_0P,g_1P,g_2P,g_3P)$ of a simplex Δ there exists a Bruhat cocycle τ on the corresponding truncated simplex $\overline{\Delta}$ compatible with D. Moreover τ is unique up to coboundaries preserving τ.

Proof. Lemma 5.6 provides the existence of a map $\eta\colon V(\overline{\Delta})\to G$ such that $\delta\eta$ is a Bruhat cocycle (see Figure 6). Uniqueness up to coboundaries and compatiblity with D is immediate from the construction. ☐

The following is the global analogue of Corollary 5.7.

Theorem 5.8. *There is a one-to-one correspondence*

$$\left\{\begin{matrix}\textit{Decorated boundary-unipotent}\\ \mathrm{SL}(2,\mathbb{C})\textit{-representations}\end{matrix}\right\}\xleftrightarrow{1:1}\left\{\begin{matrix}\textit{Bruhat cocycles } \tau \textit{ up to}\\ \textit{coboundaries preserving } \tau\end{matrix}\right\}. \tag{5.2}$$

Proof. We must show that given a decoration D there exists a Bruhat cocycle τ compatible with D and that τ is unique up to the action by coboundaries preserving τ. A decoration $D\colon I(\widetilde{M})\to\mathrm{SL}(2,\mathbb{C})/P$ of a representation ρ determines a map $\Gamma\colon V(\widetilde{\overline{\mathcal{T}}})\to\mathrm{SL}(2,\mathbb{C})/P$ defined by taking a vertex near an ideal vertex v to the coset assigned to v. We shall construct a ρ-equivariant lift of Γ to a map $\eta\colon V(\widetilde{\overline{\mathcal{T}}})\to\mathrm{SL}(2,\mathbb{C})$ such that $\tilde{\tau}=\delta\eta$ is a natural cocycle on $\widetilde{\overline{\mathcal{T}}}$ descending to a natural cocycle τ on $\overline{\mathcal{T}}$. Any cocycle compatible with D arises from this construction (since $\widetilde{M}$ is simply connected all lifts are coboundaries). Fix an orientation of each long edge e of $\overline{\mathcal{T}}$, and a lift $\tilde{e}$ of e. Each vertex of $\widetilde{\overline{\mathcal{T}}}$ is then in the π_1 orbit of exactly one of the vertices $\tilde{e}_0$ and $\tilde{e}_1$. By ρ-equivariance it is thus enough to define η on the set of endpoints $\tilde{e}_0$ and $\tilde{e}_1$. Now define $\eta(\tilde{e}_i)=g_ix_i$, where $\Gamma(\tilde{e}_i)=g_iP$ and $x_i\in P$ is an element as provided by Lemma 5.6. By construction, $\delta\eta$ is a Bruhat cocycle descending to a Bruhat cocycle on $\overline{\mathcal{T}}$. The freedom in the choice of η is exactly the action by coboundaries preserving τ. This proves the result. ☐

5.2.1. *Explicit formulas via Ptolemy coordinates.* Note that the labelings of the long edges are canonically determined, whereas a short edge is canonically determined if and only if it connects two counter diagonal long edges. The result below, generalizing the correspondence in Figure 2, gives explicit formulas for the canonically determined edges in terms of Ptolemy coordinates when at most one Ptolemy coordinate per face is zero (the mildly degenerate case; see Definition 7.4). For $a\in\mathbb{C}$ and $b\in\mathbb{C}\setminus\{0\}$ let

$$x(a)=\begin{pmatrix}1&a\\0&1\end{pmatrix},\qquad q(b)=\begin{pmatrix}0&-b^{-1}\\b&0\end{pmatrix},\qquad d(b)=\begin{pmatrix}b&0\\0&b^{-1}\end{pmatrix}. \tag{5.3}$$

Proposition 5.9. Let α_{ij} and β_{ij}^k be the labelings of long and short edges of a Bruhat cocycle coming from a decoration (g_0P, g_1P, g_2P, g_3P). If c_{ij} are the Ptolemy coordinates, we have

$$\begin{aligned} &\alpha_{ij} = q(c_{ij}) \text{ if } c_{ij} \neq 0.\\ &\beta_{ij}^k = x(\frac{c_{ij}}{c_{ik}c_{kj}}) \text{ if } c_{ik}c_{kj} \neq 0.\\ &\alpha_{ij} = -d(c_{ik}^{-1}c_{kj}) \text{ if } c_{ij} = 0 \text{ and } c_{ik}c_{kj} \neq 0. \end{aligned} \tag{5.4}$$

Proof. The first formula only involves a single edge and is thus independent of whether or not the remaining Ptolemy coordinates are zero. The second only involves the face ijk and holds by the fundamental correspondence if $c_{ij} \neq 0$ and for $c_{ij} = 0$ as well by analytic continuation. The third is a consequence of the first two using the cocycle condition. □

Remark 5.10. Identifying P with $\mathbb{C}$ via $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \leftrightarrow x$ allows us to view the labelings of the short edges near an ideal vertex v of M as complex vectors. The action of a coboundary taking a vertex v to $x \in \mathbb{C}$ (keeping all other vertices fixed) then corresponds to moving v by x (see Figure 5).

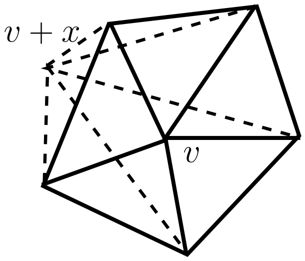


FIGURE 5. The action by coboundary corresponds to moving the "center" point.

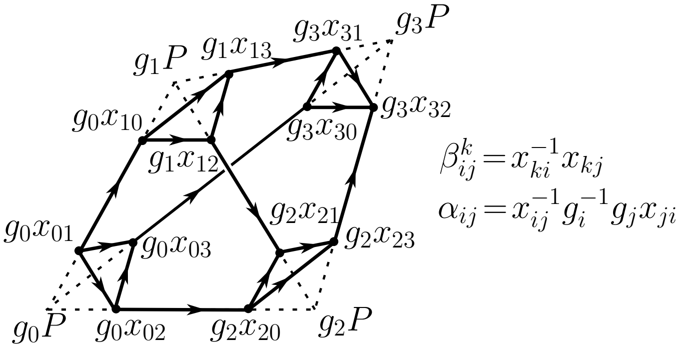


FIGURE 6. The natural cocycle from a decoration.

5.3. $\mathrm{PSL}(2,\mathbb{C})$**-representations.** The analogue of Theorem 5.8 for $\mathrm{PSL}(2,\mathbb{C})$-representations and $\mathrm{PSL}(2,\mathbb{C})$ Bruhat cocycles also holds. The proof is identical to that of Theorem 5.8 using the obvious analogues of Lemma 5.6 and Corollary 5.7 for $\mathrm{PSL}(2,\mathbb{C})$.

5.4. **Non-boundary-unipotent representations.** Let M be as in Section 4.2. Recall that a decoration of $\rho\colon \pi_1(M) \to \mathrm{SL}(2,\mathbb{C})$ is a ρ-equivariant assignment of P-cosets to the triangular faces of $\widetilde{\widetilde{\mathcal{T}}}$. A *fattened Bruhat cocycle* is a cocycle on the fattened decomposition restricting to a Bruhat cocycle on each truncated simplex and labeling each face pairing edge by diagonal elements.

Theorem 5.11. *There is a one-to-one correspondence between decorated* $\mathrm{SL}(2,\mathbb{C})$*-representations and fattened Bruhat cocycles.*

Proof. The proof is similar to that of Theorem 5.8 using the fattened decomposition instead of the regular one. □

6. The edge relations

We now define a relation among the Ptolemy coordinates of a decorated (boundary-unipotent) representation. This relation is a consequence of the Ptolemy relations in the case when all Ptolemy coordinates are non-zero, but if some Ptolemy coordinates are zero, this relation is independent of the Ptolemy relations. Hence, when defining the refined Ptolemy variety this relation must be imposed in addition to the Ptolemy relations.

Let K be the space obtained by cyclically gluing together ordered simplices $\Delta_0, \dots, \Delta_{N-1}$ along a common edge. We order the vertices of each simplex such that the common edge is the 01 edge of each simplex, and such that the orientations induced by the orderings agree (see Figure 7). We refer to the edges (other than the common edge) as *top*, *bottom*, and *center* edges respectively.

Lemma 6.1. Let c be a Ptolemy assignment on K where all the Ptolemy coordinates of the top and and bottom edges of K are non-zero. We then have

$$\sum_{k=0}^{N-1} \frac{c_{23,k}}{c_{12,k}c_{13,k}} = 0 \iff \sum_{k=0}^{N-1} \frac{c_{23,k}}{c_{02,k}c_{03,k}} = 0. \tag{6.1}$$

Moreover, if the Ptolemy coordinate c_{01} of the interior edge is also non-zero, both equations are satisfied.

Proof. We first assume that $c_{01} \neq 0$. We have

$$\begin{aligned} c_{01}\sum_{k=0}^{N-1} \frac{c_{23,k}}{c_{12,k}c_{13,k}} &= \sum_{k=0}^{N-1} \frac{c_{02,k}c_{13,k} - c_{03,k}c_{12,k}}{c_{12,k}c_{13,k}} \\ &= \sum_{k=0}^{N-1} \frac{c_{03,k-1}c_{13,k} - c_{03,k}c_{13,k-1}}{c_{13,k-1}c_{13,k}} \\ &= \sum_{k=0}^{N-1} \Big(\frac{c_{03,k-1}}{c_{13,k-1}} - \frac{c_{03,k}}{c_{13,k}}\Big) \\ &= 0. \end{aligned} \tag{6.2}$$

Here, the first equality follows from the Ptolemy relations (3.1), and the second from the identification relations $c_{12,k} = c_{13,k-1}$ and $c_{02,k} = c_{03,k-1}$ (indices modulo N). By a similar computation, one also has

$$c_{01}\sum_{k=0}^{N-1} \frac{c_{23,k}}{c_{02,k}c_{03,k}} = 0. \tag{6.3}$$

Since we are assuming that $c_{01} \neq 0$, (6.2) and (6.3) together prove the second statement.

If $c_{01} = 0$, the Ptolemy relations become $c_{03,k}c_{12,k} = c_{02,k}c_{13,k}$, which together with the identification relations $c_{12,k} = c_{13,k-1}$ and $c_{02,k} = c_{03,k-1}$ imply that the ratio between $c_{12,k}c_{13,k}$ and $c_{02,k}c_{03,k}$ is independent of k. This concludes the proof. □

Definition 6.2. We call the (equivalent) relations (6.1) *edge relations* around the center edge of K.

Remark 6.3. When all Ptolemy coordinates are non-zero, the fundamental correspondence provides a natural cocycle on the space $\overline{K}$ obtained from K by truncating the simplices. In this case, the edge relatations are equivalent to the relation coming from the fact that the product

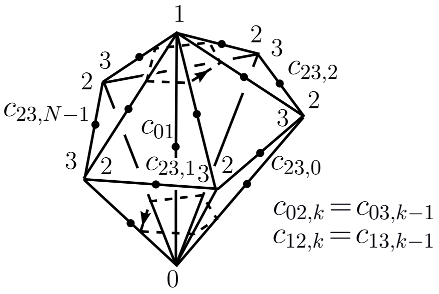


FIGURE 7. A Ptolemy assignment on K.

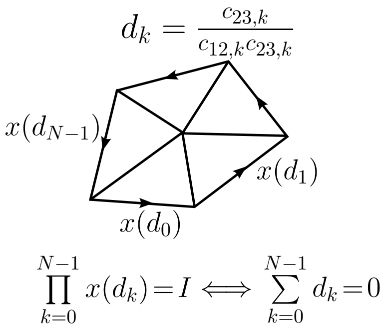


FIGURE 8. Bruhat cocycle on $\overline{K}$ (top view).

of short edges around the top and bottom is I (see Figure 8). When some of the Ptolemy coordinates are zero this relation is no longer automatically satisfied and must be imposed.

6.1. **Edge relations for** $\mathrm{PSL}(2,\mathbb{C})$ **Ptolemy assignments.** The obvious analogue of Lemma 6.1 for $\mathrm{PSL}(2,\mathbb{C})$ Ptolemy assignments still holds and we define the edge relations for $\mathrm{PSL}(2,\mathbb{C})$ Ptolemy assignments by the exact same formula 6.1.

6.2. **Edge relations for enhanced Ptolemy assignments.** Let c be an enhanced Ptolemy assignment on K where all the Ptolemy coordinates of the top and and bottom edges of K are non-zero. The identification relations are given by $c_{12,k} = t_k m_k c_{13,k-1}$, $c_{02,k} = b_k m_k c_{03,k-1}$ and $c_{01,k} = c_{01,k-1} t_k b_k$, where the t_k, b_k and m_k are monomials in the m_s and l_s (see Figure 9). An elementary modification (we leave the details to the reader) of the proof of Lemma 6.1 shows that we have

$$\sum_{k=0}^{N-1}\Big(\frac{c_{23,k}}{c_{12,k}c_{13,k}}\prod_{j=1}^{k} t_j^2\Big) = 0 \iff \sum_{k=0}^{N-1}\Big(\frac{c_{23,k}}{c_{02,k}c_{03,k}}\prod_{j=1}^{k} b_k^2\Big) = 0, \tag{6.4}$$

and that both are satisfied if the $c_{01,k}$ are non-zero. We refer to these as *edge relations*. The geometric interpretation of the edge relations is given in Figure 10.

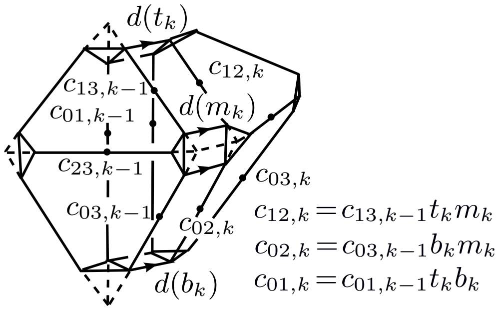


FIGURE 9. Fattened version of Figure 7.

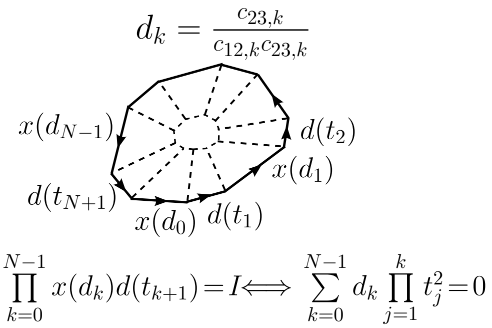


FIGURE 10. Fattened Bruhat cocycle (top view).

7. Transitive partitions and degeneracy types

Note that a decoration determines a partition of the edge set of $\mathcal{T}$ into those whose Ptolemy coordinates are zero, and those whose Ptolemy coordinates are nonzero. We refer to this partition as the *Ptolemy partition*. We shall consider such partitions in more detail.

Definition 7.1. A *transitive partition* is a partitioning of the edges of $\mathcal{T}$ into *zero-edges* and *non-zero* edges such that if two edges on a face are zero-edges, so is the third. The set of transitive partitions is denoted by $\mathcal{E}(\mathcal{T})$.

A transitive partition canonically lifts to a partition of the edges of $\widetilde{\mathcal{T}}$ and of the long edges of $\overline{\mathcal{T}}$ and $\widetilde{\overline{\mathcal{T}}}$.

Lemma 7.2. The Ptolemy partition of a decoration is transitive.

Proof. If g_0P, g_1P and g_2P are the cosets assigned to the vertices of a face, the Ptolemy coordinates are given by $c_{ij} = \det(v_i, v_j)$, where $v_i = g_i \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$. Hence, a Ptolemy coordinate c_{ij} is zero if and only if v_i and v_j are linearly dependent. The result now follows from the fact that linear dependence of vectors is transitive. □

Definition 7.3. Let $E \in \mathcal{E}(\mathcal{T})$ be a transitive partition. A face of $\mathcal{T}$ is *degenerate* if all its three edges are zero-edges. A simplex of $\mathcal{T}$ is degenerate if all of its six edges are zero-edges.

Definition 7.4. We divide the transitive partitions into the following types:

Non-degenerate: No zero-edges.
Mildly degenerate: Some zero-edges, but no degenerate faces.
Moderately degenerate: Some degenerate faces, but no degenerate simplices.
Wildly degenerate: Some, but not all, simplices are degenerate.
Totally degenerate: All simplices are degenerate (there is a unique such).

Clearly, any $E \in \mathcal{E}(\mathcal{T})$ falls into exactly one of these types.

7.1. Decorations, Ptolemy assignments and natural cocycles.

Definition 7.5. Let $E \in \mathcal{E}(\mathcal{T})$. A decoration is of *type* E if its Ptolemy partition equals E. A natural cocycle τ has *type* E if the set of long edges that are diagonal agrees with the zero-edges of E. A Ptolemy assignment has type E if the set of edges whose Ptolemy coordinate is zero agrees with the zero-edges of E.

Note that the one-to-one-correspondence in Theorem 5.8 preserves the type. We stress that the type depends on the triangulation.

Remark 7.6. Decorations of type E may not exist. For example, if there is an edge loop where all but one edge is zero, the argument in the proof of Lemma 7.2 shows that E cannot be the Ptolemy partition of a decoration.

7.2. The totally degenerate partition.

Proposition 7.7. If a decoration of a representation ρ is totally degenerate, then ρ is reducible.

Proof. If all Ptolemy coordinates are zero, the decoration must take all ideal vertices of $\widetilde{M}$ to the same B-coset. This is only possible if ρ is reducible. □

Proposition 7.8. If a decoration of a reducible representation ρ is not totally degenerate, then ρ is boundary-degenerate.

Proof. A reducible representation has a totally degenerate decoration. The only way it can also have a decoration which is not totally degenerate is if a boundary component is a sphere or collapsed. ☐

8. The generalized Ptolemy varieties

We now define a Ptolemy variety $P(\mathcal{T}, E)$ for each transitive partition E, which is not totally degenerate. We shall see that this variety parametrizes decorations with Ptolemy partition E. The representation corresponding to an element in $P(\mathcal{T}, E)$ can be recovered explicitly via the corresponding Bruhat cocycle.

8.1. **Mildly degenerate partitions.** Let $E \in \mathcal{E}(\mathcal{T})$ be mildly degenerate. Then each face of $\mathcal{T}$ has at most one zero-edge.

Proposition 8.1. Let c be a Ptolemy assignment on a simplex Δ. If each face has at most one Ptolemy coordinate which is zero, then c is the Ptolemy assignment of a unique decoration.

Proof. By reordering the vertices if necessary, we may assume that c_{01}, c_{12} and c_{23} are non-zero. Letting

$$g_0 = I, \quad g_1 = q(c_{01}), \quad g_2 = g_1 x(\frac{c_{02}}{c_{01}c_{12}})q(c_{12}), \quad g_3 = g_2 x(\frac{c_{13}}{c_{12}c_{23}})q(c_{23}). \tag{8.1}$$

The Ptolemy coordinates of the decoration (g_0P, g_1P, g_2P, g_3P) then agree with c as is shown by explicitly computing $\det\big(g_i\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), g_j\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\big)$. Uniqueness follows from Corollary 5.7 since the natural cocycle is determined up to coboundaries by the Ptolemy coordinates (Proposition 5.9). ☐

Let $C(E)$ be a union of disjoint cylindrical neighborhoods of the zero-edges of E. The space $M \setminus C(E)$ decomposes as a union of truncated simplices with the zero-edges being "chopped" (see Figure 11).

Corollary 8.2. A Ptolemy assignment c of type E canonically determines a G-cocycle on the space $M \setminus C(E)$.

Proof. By Proposition 8.1 and Corollary 5.7, c determines up to coboundaries a Bruhat cocycle on each truncated simplex. By Proposition 5.9, all the long edges are canonically determined by the Ptolemy coordinates, but a short edge near a zero-edge is only determined up to the coboundary action. However, as shown in Figure 11, each chopped truncated simplex inherits canonical edge labelings. The fact that the chopped cocycles match up follows from the identification relations. ☐

The link of each zero-edge is a complex K as in Section 6 (embedded in $\widehat{\widetilde{M}}$) and since E is only mildly degenerate, the top and bottom edges of K are all non-zero. Hence, for a Ptolemy assignment of type E the edge relations (Definition 6.2) around each zero-edge are well defined.

Definition 8.3. The Ptolemy variety $P(\mathcal{T}, E)$ is the variety defined by the usual relations (as in Definition 3.3) together with the edge relations around zero edges and the relations $c_e = 0$ if and only if e is a zero edge.

Note that each element in $P(\mathcal{T}, E)$ is a Ptolemy assignment of type E.

Theorem 8.4. *Let $E \in \mathcal{E}(\mathcal{T})$ be mildly degenerate. There is a one-to-one correspondence*

$$\left\{\begin{matrix}\textit{Decorated, boundary-unipotent}\\ \textit{representations of type } E\end{matrix}\right\} \overset{1:1}{\longleftrightarrow} P(\mathcal{T}, E) \overset{1:1}{\longleftrightarrow} \left\{\begin{matrix}\textit{Bruhat cocycles of type } E\\ \textit{up to coboundaries}\end{matrix}\right\} \tag{8.2}$$

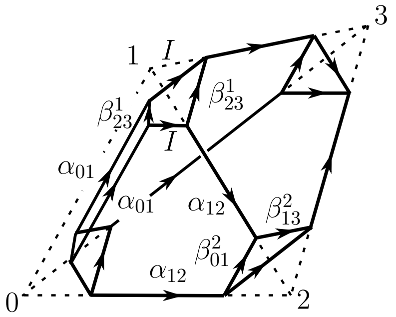


FIGURE 11. Natural cocycle after removing a neigborhood of a zero edge.

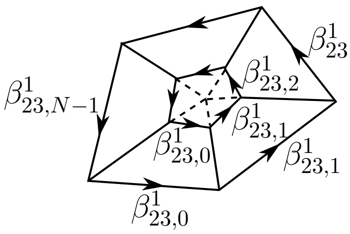


FIGURE 12. The cocycle extends if and only if the edge relation holds.

Proof. By Theorem 5.8 all we need to prove is that a Ptolemy assignment of type E determines a natural cocycle. By the Corollary 8.2, we have a canonical cocycle on $M \setminus C(E)$. By the van Kampen theorem this extends to a natural cocycle on M if and only if the product of the labelings around each cylinder is I, which is a consequence of the edge relations. The freedom in the choice of extension is exactly the coboundary action (see e.g. Remark 5.10). ☐

Definition 8.5. For $\sigma \in H^2(\widehat{M};\mathbb{Z}/2\mathbb{Z})$ define $P^\sigma(\mathcal{T},E)$ as in Definition 8.3, but using the identification relations (4.1).

Theorem 8.6. *Let $E \in \mathcal{E}(\mathcal{T})$ be mildly degenerate. There is a one-to-one correspondence*

$$\begin{tikzcd}\left\{\begin{matrix}\textit{Type } E \textit{ decorated}\\ \textit{boundary-unipotent}\\ \pi_1(M)\to \mathrm{PSL}(2,\mathbb{C})\end{matrix}\right\} \arrow[rr, leftrightarrow, bend right, "1{:}1"'] & P^\sigma(\mathcal{T},E) \arrow[l, "1{:}k"'] \arrow[r, "k{:}1"] & \left\{\begin{matrix}\textit{Bruhat } \mathrm{PSL}(2,\mathbb{C})\textit{-cocycles}\\ \textit{type } E\textit{, obstruction class } \sigma\\ \textit{up to coboundaries}\end{matrix}\right\},\end{tikzcd} \tag{8.3}$$

Proof. As in the proof of Theorem 8.4 an element in $P^\sigma(\mathcal{T},E)$ determines a Bruhat cocycle up to coboundaries. The fact that this map is $k:1$ follows from the elementary fact that two Ptolemy assignments determine the same Bruhat cocycle if and only if they differ by an element in $Z^1(\widehat{M};\mathbb{Z}/2\mathbb{Z})$. ☐

Definition 8.7. For M as in Section 4.2 define the enhanced Ptolemy variety $\mathcal{E}P(\mathcal{T},E)$ as in Definition 8.3 using the identification relations from Definition 4.5 and the edge relations (6.4).

Theorem 8.8. *Let $E \in \mathcal{E}(\mathcal{T})$ be mildly degenerate and M as in Section 4.2. There is a one-to-one correspondence*

$$\left\{\begin{matrix}\textit{Decorated representations}\\ \textit{of type } E\end{matrix}\right\} \overset{1:1}{\longleftrightarrow} \mathcal{E}P(\mathcal{T},E) \overset{1:1}{\longleftrightarrow} \left\{\begin{matrix}\textit{Fattened Bruhat cocycles of type } E\\ \textit{up to coboundaries}\end{matrix}\right\} \tag{8.4}$$

Proof. The proof is identical to that of Theorem 8.4 using the fattened decomposition and the geometric interpretation of the edge relations (6.4) given in Figure 10. ☐

8.2. Moderately degenerate partitions.

Definition 8.9. Let $\mathcal{T}'$ be a triangulation whose edge set contains that of $\mathcal{T}$. A *descendant* of $E \in \mathcal{E}(\mathcal{T})$ is an element $E' \in \mathcal{E}(\mathcal{T}')$ such that E' agrees with E on the edges of $\mathcal{T}$. If $E' \in \mathcal{E}(\mathcal{T}')$ is a descendent of $E \in \mathcal{E}(\mathcal{T})$ we write $E < E'$.

Lemma 8.10. Let $E \in \mathcal{E}(\mathcal{T})$ be moderately degenerate and suppose that there are k degenerate faces. If $\mathcal{T}'$ is the triangulation obtained from $\mathcal{T}$ by performing a 2-3 move at each degenerate face then E has 2^k descendants each of which is mildly degenerate.

Proof. Performing a 2-3 move adds an edge which can be either zero or non-zero without violating transitivity. In either case, none of the new faces are degenerate (see Figure 13). This concludes the proof. ☐

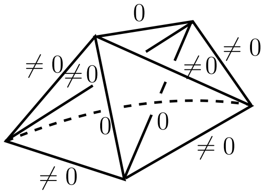


Figure 13. A 2-3 move, moderately degenerate partition.

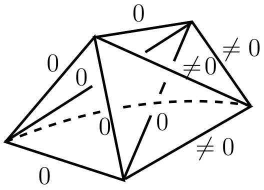


Figure 14. A 2-3 move, wildly degenerate partition.

Definition 8.11. Let $E \in \mathcal{E}(\mathcal{T})$ be moderately degenerate and let $\mathcal{T}'$ be the triangulation obtained from $\mathcal{T}$ by performing 2-3 moves between all degenerate faces. Define

$$P(\mathcal{T}, E) = \coprod_{E < E'} P(\mathcal{T}', E') \tag{8.5}$$

and define $P^\sigma(\mathcal{T}, E)$ and $\mathcal{E}P(\mathcal{T}, E)$ similiarly.

Remark 8.12. We may regard the disjoint union 8.5 as a union inside the affine space $A(M)$ as in Remark 3.22.

Theorem 8.13. *The one-to-one correspondences* (8.2), (8.3), *and* (8.4) *still hold if E is moderately degenerate.*

Proof. If $\mathcal{T}'$ is a decoration with edge set containing $\mathcal{T}$, the set of decorations of type $E \in \mathcal{E}(\mathcal{T})$ is the disjoint union, over the descendants E' of E, of the sets of decorations of type E'. The result now follows from Theorem 8.4. ☐

8.3. **Wildly degenerate partitions.** Given a wildly degenerate partition $E \in \mathcal{E}(\mathcal{T})$. Let $d(E)$ denote the number of degenerate simplices.

Lemma 8.14. Let $E \in \mathcal{E}(\mathcal{T})$ be wildly degenerate and let $\mathcal{T}'$ be a triangulation obtained from $\mathcal{T}$ be performing a 2-3 moves at a face between a degenerate and a non-degenerate simplex. Then E has a unique descendant $E' \in \mathcal{E}(\mathcal{T})$. Furthermore $d(E') = d(E) - 1$.

Proof. Transitivity forces the new edge to be non-zero, so none of the new simplices are degenerate (see Figure 14). This proves the result. ☐

Corollary 8.15. Let $E \in \mathcal{E}(\mathcal{T})$ be wildly degenerate. There exists a triangulation $\mathcal{T}'$ obtained from $\mathcal{T}$ by performing 2-3 moves such that E has a unique descendant $E' \in \mathcal{E}(\mathcal{T}')$, which is moderately degenerate.

Proof. Repeatedly perform 2-3 moves at a face between a degenerate and a non-degenerate simplex until there are no more degenerate simplices. ☐

Definition 8.16. Let $E \in \mathcal{E}(\mathcal{T})$ be wildly degenerate and let E' and $\mathcal{T}'$ be as in Corollary 8.15. Define

$$P(\mathcal{T}, E) = P(T', E') \tag{8.6}$$

and define $P^\sigma(\mathcal{T}, E)$ and $\mathcal{E}P(\mathcal{T}, E)$ similarly.

We must prove that this is independent of the choice of $\mathcal{T}'$ up to canonical isomorphism. Recall that a decoration determines a Ptolemy assignment on each triangulation $\mathcal{T}$. We refer to the Ptolemy coordinates as Ptolemy coordinates on $\mathcal{T}$.

Lemma 8.17. Let D be a decoration and $\mathcal{T}$ a triangulation such that D is mildly degenerate. For any triangulation $\mathcal{T}'$, each Ptolemy coordinate on $\mathcal{T}'$ is a regular function of the Ptolemy coordinates on $\mathcal{T}$.

Proof. Since the Ptolemy partition E of D on $\mathcal{T}$ is non-degenerate, Corollary 8.2 provides a canonical cocycle τ on $M \setminus C(E)$ compatible with D. If e is an oriented edge of $\mathcal{T}'$ then e is homotopic to an edge path α in $M \setminus C(E)$, so the Ptolemy coordinate c_e equals $\det\big(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), g\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\big)$, where g is the product of the labelings along α. By Proposition 5.9 all labelings are regular functions of the Ptolemy coordinates on $\mathcal{T}$. This proves the result. □

Corollary 8.18. The definition 8.16 of the Ptolemy varieties for wildly degenerate partitions is independent of the choice of $\mathcal{T}'$ up to canonical biregular isomorphism.

Theorem 8.19. *The one-to-one correspondences* (8.2), (8.3), *and* (8.4) *still hold if* E *is wildly degenerate.*

Proof. This follows from Theorem 8.13. □

8.4. **The refined Ptolemy varieties.**

We denote the totally degenerate partition where all edges are zero-edges by $0 \in \mathcal{E}(\mathcal{T})$.

Definition 8.20. The *generalized Ptolemy variety* $\overline{P}(\mathcal{T})$ is defined by

$$\overline{P}(\mathcal{T}) = \coprod_{E \in \mathcal{E}(\mathcal{T}) \setminus \{0\}} P(\mathcal{T}, E). \tag{8.7}$$

We similarly define $\overline{P}^\sigma(\mathcal{T})$ and $\mathcal{E}\overline{P}(\mathcal{T})$.

9. THE REDUCED PTOLEMY VARIETIES

The diagonal action on decorations, Ptolemy assignments, and natural cocycles extends canonically to the case where some of the Ptolemy coordinates are zero and some of the long edges are diagonal instead of counterdiagonal. We define the reduced Ptolemy variety $\overline{P}_{\mathrm{red}}(\mathcal{T})$ to be the quotient of $\overline{P}(\mathcal{T})$ by the diagonal action. In [7] we showed that the reduced Ptolemy variety $P(\mathcal{T})_{\mathrm{red}}$ can be computed by setting appropriately chosen Ptolemy coordinates equal to 1:

Theorem 9.1 ([7, Prop. 1.16])**.** *Let* G *be a graph in the one-skeleton of* $\widehat{M}$, *which contains a (possibly empty) maximal tree, and has fundamental group* $\mathbb{Z}$. *The reduced Ptolemy variety* $P(\mathcal{T})_{\mathrm{red}}$ *is isomorphic to the subvariety of* $P(\mathcal{T})$ *where the Ptolemy coordinates of all edges in* G *are* 1. *The same result holds for* $P^\sigma(\mathcal{T}, E)$

Remark 9.2. In [7] this is only proved when G is a so-called "maximal tree with 1- or 3-cycle", but the proof can be trivially extended to any graph with fundamental group $\mathbb{Z}$ containing a maximal tree.

Theorem 9.3. *Let $E \in \mathcal{E}(\mathcal{T})$ be mildly degenerate. Then, there exists at least one graph G of non-zero edges which has fundamental group $\mathbb{Z}$ and contains a maximal tree. The reduced Ptolemy variety $P(\mathcal{T},E)_{\mathrm{red}}$ is isomorphic to the subvariety of $P(\mathcal{T},E)$ where the Ptolemy coordinates of all edges in G are 1. The same result holds for $P^{\sigma}(\mathcal{T},E)_{\mathrm{red}}$ and $\mathcal{E}P(\mathcal{T},E)_{\mathrm{red}}$.*

Proof. Since E is mildly degenerate, the set of non-zero edges connects all vertices of $\mathcal{T}$ and contains cycles. Thus, we can choose a graph G with the above properties. Hence, the result follows from Theorem 9.1. The proofs trivially extend to the case of $\mathcal{E}P(\mathcal{T},E)_{\mathrm{red}}$. □

10. SUMMARY OF THE PROOFS OF MAIN RESULTS

Theorem 1.2 is an immediate consequence of Theorem 8.19. The fact that the transition maps when changing the triangulation are biregular follows from Lemma 8.17. Theorem 1.4 follows from Corollary 3.14. Corollary 1.5 follows from the fact that the image of each peripheral curve is given by regular functions in the Ptolemy coordinates. Theorem 1.9 follows from Theorem 8.19 and the analogue of Corollary 3.14 for $\mathrm{PSL}(2,\mathbb{C})$. Finally, Theorem 1.10 follows from Theorem 8.19 and Corollary 3.14.

11. EXAMPLES

Let M be the manifold `m009` from the SnapPy census [3]. The census triangulation $\mathcal{T}$ of M is shown in Figure 15. We refer to the three edges as *edge 1*, *edge 2*, and *edge 3*, according to the number of arrow heads. By inspecting the figure, one checks that there are four edge partitions: The non-degenerate partition where all edges are non-zero, the mildly degenerate partition where only edge 1 is zero, the mildly degenerate partition where only edge 2 is zero, and the totally degenerate partition (which we ignore, see Section 7.2). The edge relation around edge 1 is given by

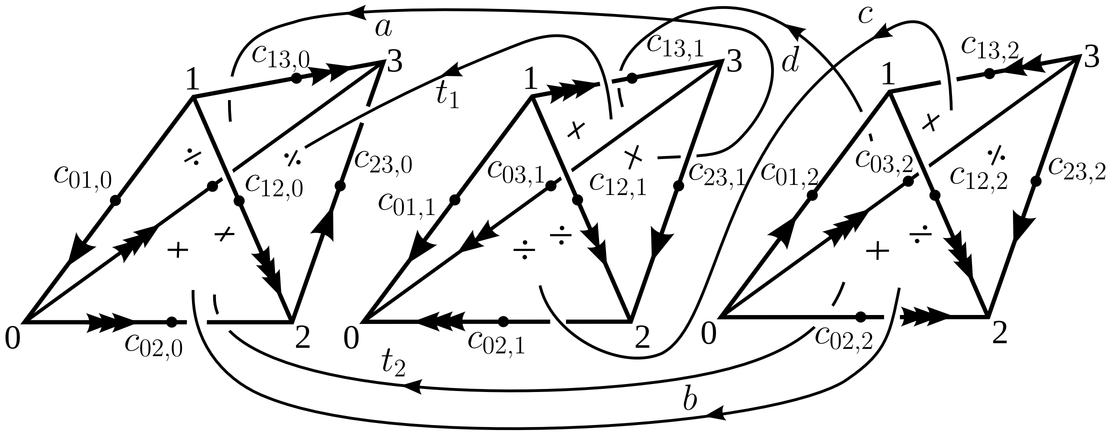


FIGURE 15. Census triangulation of `m009`. The signs on the faces indicate the obstruction cocycle σ^3.

$$\frac{c_{23,0}}{c_{21,0}c_{13,0}}+\frac{c_{01,1}}{c_{03,1}c_{31,1}}+\frac{c_{10,0}}{c_{12,0}c_{20,0}}+\frac{c_{01,2}}{c_{03,2}c_{31,2}}+\frac{c_{23,1}}{c_{21,1}c_{13,1}}+\frac{c_{32,2}}{c_{30,2}c_{02,2}}=0 \tag{11.1}$$

and the edge relation around edge 2 is given by

$$\frac{c_{02,0}}{c_{01,0}c_{12,0}}+\frac{c_{30,1}}{c_{31,1}c_{10,1}}+\frac{c_{20,2}}{c_{23,2}c_{30,2}}+\frac{c_{12,1}}{c_{13,1}c_{32,1}}=0. \tag{11.2}$$

Since there is no edge partition where edge 3 is zero (except the totally degenerate partition), Theorem 9.3 implies that the reduced Ptolemy variety (all variants) is given by setting the Ptolemy coordinate of edge 3 equal to 1.

11.1. The Ptolemy variety $\overline{P}(\mathcal{T})$.

The identification relations corresponding to the three edges are

$$\begin{gathered} c_{23,0}=-c_{23,2}=-c_{01,1}=c_{01,2}=-c_{01,0}=-c_{23,1} \\ c_{13,0}=c_{12,1}=-c_{13,2}=-c_{03,1}=c_{13,0} \\ c_{13,1}=c_{03,2}=c_{02,0}=c_{12,2}=-c_{02,1}=c_{03,0}=c_{02,2}=c_{12,0}. \end{gathered} \tag{11.3}$$

Letting $x=c_{23,0}$, $y=c_{13,0}$ and $z=c_{13,1}$, the Ptolemy relations $c_{03,i}c_{12,i}+c_{01,i}c_{23,i}=c_{02,i}c_{13,i}$ become

$$z^2-x^2=zy, \qquad -y^2+x^2=-z^2, \qquad z^2-x^2=-zy. \tag{11.4}$$

One easily checks that the only solution to this is $x=y=z=0$, so the Ptolemy variety $\overline{P}(\mathcal{T})$ is empty.

11.2. The Ptolemy varieties $\overline{P}^{\sigma}(\mathcal{T})$.

An elementary cohomology computation shows that $H^2(\widehat{M};\mathbb{Z}/2\mathbb{Z})=\mathbb{Z}/4\mathbb{Z}$ and that the three non-trivial classes are represented by cocycles $\sigma^i\in C^2(\widehat{M};\mathbb{Z}/2\mathbb{Z})$ whose restrictions σ^i_k to the kth simplex of $\mathcal{T}$ are given by

$$\begin{array}{lll} \sigma^1_0=f_0^*+f_2^*, & \sigma^1_1=f_0^*+f_1^*, & \sigma^1_2=0, \\ \sigma^2_0=f_0^*+f_1^*, & \sigma^2_1=f_0^*+f_3^*, & \sigma^2_2=f_0^*+f_1^*, \\ \sigma^3_0=f_1^*+f_2^*, & \sigma^3_1=f_1^*+f_3^*, & \sigma^3_2=f_0^*+f_1^*, \end{array} \tag{11.5}$$

where $f_i^*\in C^2(\Delta;\mathbb{Z}/2\mathbb{Z})$ denotes the cochain taking the face f_i opposite vertex i to -1 and all other faces to 1. The cocycle σ^3 is indicated in Figure 15.

One easily checks that $\sigma^i_k=\delta(\eta^i_k)$, where η^i_k is given by

$$\begin{array}{lll} \eta^1_0=\varepsilon^*_{13}, & \eta^1_1=\varepsilon^*_{23}, & \eta^1_2=0, \\ \eta^2_0=\varepsilon^*_{23}, & \eta^2_1=\varepsilon^*_{12}, & \eta^2_2=\varepsilon^*_{23}, \\ \eta^3_0=\varepsilon^*_{03}, & \eta^3_1=\varepsilon^*_{02}, & \eta^3_2=\varepsilon^*_{23}, \end{array} \tag{11.6}$$

where $\varepsilon^*_{ij}\in C^1(\Delta;\mathbb{Z}/2\mathbb{Z})$ is the cochain taking ε_{ij} to -1 and all other edges to 1. The Ptolemy variety for the trivial obstruction class is equal to $\overline{P}(\mathcal{T})$, which is trivial, as we saw earlier. Recall that the reduced Ptolemy variety is obtained by setting the Ptolemy coordinate z of edge 3 equal to 1.

11.2.1. *Ptolemy variety for* σ^1.

The identification relations (4.1) are

$$\begin{gathered} c_{23,0}=-c_{23,2}=-c_{01,1}=c_{01,2}=-c_{01,0}=c_{23,1} \\ c_{13,0}=-c_{12,1}=c_{13,2}=c_{03,1}=c_{13,0} \\ c_{13,1}=c_{03,2}=c_{02,0}=c_{12,2}=-c_{02,1}=c_{03,0}=c_{02,2}=c_{12,0}. \end{gathered} \tag{11.7}$$

Again, letting $x=c_{23,0}$, $y=c_{13,0}$ and $z=c_{13,1}$, the Ptolemy relations become

$$z^2-x^2=zy, \qquad -y^2-x^2=-z^2, \qquad z^2-x^2=zy. \tag{11.8}$$

Setting $z=1$, the equations then have the three solutions

$$(x,y,z)=(0,1,1), \qquad (x,y,z)=(-1,0,1), \qquad (x,y,z)=(1,0,1). \tag{11.9}$$

However, not all of these solutions are valid, since the edge equations may not be satisfied. The edge relation (11.1) around edge 1 (defined when y and z are non-zero) is

$$\frac{x}{-zy}+\frac{-x}{y(-z)}+\frac{x}{z(-z)}+\frac{x}{z(-y)}+\frac{x}{yz}+\frac{x}{-zz}=\frac{-2x}{z^2}=0 \tag{11.10}$$

which is satisfied when x is zero. Similarly, the edge relation (11.2) around edge 2 (defined when z and x are non-zero) is

$$\frac{z}{-xz}+\frac{-y}{-zx}+\frac{-z}{-x(-z)}+\frac{-y}{z(-x)}=\frac{2y}{zx}-\frac{2}{x}=0, \tag{11.11}$$

which is not satisfied. Hence, the reduced Ptolemy variety for σ^1 consists of a single point given by $(x, y, z) = (0, 1, 1)$.

11.2.2. *Ptolemy variety for* σ^2. The Ptolemy relations are

$$z^2+x^2=yz, \qquad y^2+x^2=-z^2, \qquad z^2+x^2=-yz, \tag{11.12}$$

which have no non-trivial solution. Hence, the Ptolemy variety $\overline{P}^{\sigma^2}(\mathcal{T})$ is empty.

11.2.3. *Ptolemy variety for* σ^3. The Ptolemy relations are

$$z^2+x^2=yz, \qquad y^2+x^2=-z^2, \qquad z^2+x^2=-yz, \tag{11.13}$$

and setting $z = 1$ they are equivalent to

$$x^2+y+1=0, \qquad y^2+y+2=0, \qquad z=1 \tag{11.14}$$

Hence, there are no solutions with x or y being 0, and $\overline{P}^{\sigma^3}(\mathcal{T})_{\mathrm{red}}$ is defined over the number field $\mathbb{Q}(w)$, where $w^4 + w^2 + 2 = 0$, and is given by $x = w$, $y = -w^2 - 1$, $z = 1$.

11.3. **The Ptolemy variety $\mathcal{E}\overline{P}(\mathcal{T})$ and the A-polynomial.** A fundamental rectangle for the boundary of M is shown in Figure 16. Using the rules illustrated in Figure 3 we obtain that the identification relations are given by

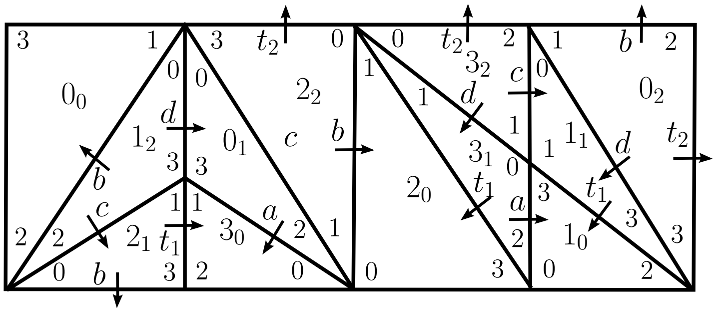


Figure 16. Fundamental rectangle for `m009`.

$$\begin{gathered}
c_{23,0} \overset{t_2}{=} -m^2c_{23,2} \overset{c}{=} -m^2c_{01,1} \overset{d}{=} m^2c_{01,2} \overset{b}{=} -mc_{01,0} \overset{a}{=} -m^2c_{23,1}, \\
c_{13,0} \overset{t_1}{=} c_{12,1} \overset{c}{=} -c_{13,2} \overset{d}{=} -c_{03,1} \overset{t_1}{=} c_{13,0}, \\
c_{13,1} \overset{d}{=} c_{03,2} \overset{t_2}{=} m^{-1}lc_{02,0} \overset{b}{=} m^{-1}lc_{12,2} \overset{c}{=} -m^{-1}lc_{02,1} \overset{a}{=} lc_{03,0} \overset{t_2}{=} mc_{02,2} \overset{b}{=} c_{12,0},
\end{gathered} \tag{11.15}$$

where the symbol $\overset{\alpha}{=}$ indicates that the identification is via the face pairing α. Hence, the Ptolemy relations become

$$mz^2 - lx^2 - m^2yz, \qquad m^2ly^2 - lx^2 - m^3z^2, \qquad m^5z^2 - lx^2 + m^3lyz. \tag{11.16}$$

One easily checks that there are no non-trivial solutions with $x = 0$. The edge relation (11.2) around edge 2 is

$$\frac{ml^{-1}z}{-m^{-1}xz} + \frac{y}{(-z)m^{-2}x} + \frac{-m^{-1}z}{-m^{-2}x(-z)} + \frac{y}{zx} = 0 \iff m^2z + m^2ly + mlz - ly = 0 \tag{11.17}$$

Adding this equation to the Ptolemy relations (11.16) and substituting $z = 1$, one can check (using magma [1]) that the system is equivalent to

$$\begin{gathered} x^2 + yl - m^8 + 3m^6 + m^5l + m^4 - m^3l - 3m^2 - ml = 0, \\ y^2l + yl^2 + m^4l - m^3 - m^2l - ml^2 - m - l = 0, \\ ym + yl + m^4 - m^2 - ml - 1 = 0, \\ yl^3 - yl - m^5l + m^4l^2 + m^3l + m^2 - ml^3 + 2ml - l^2 = 0, \\ m^6l - 2m^4l - m^3l^2 - m^3 - 2m^2l + l = 0. \end{gathered} \tag{11.18}$$

This shows that the A-polynomial of `m009` is given by

$$A(m, l) = m^6l - 2m^4l - m^3l^2 - m^3 - 2m^2l + l. \tag{11.19}$$

It also follows that $\mathcal{E}\overline{P}(\mathcal{T})$ is a branched cover over the A-polynomial curve of degree 2. Another magma computation shows that it is given explicitly by

$$x^2 = -\frac{-m^4 - 2m^3l + ml}{l^2(m^2 - 1)}, \qquad y = -\frac{m^2 + ml}{m^2l - l}, \qquad z = 1, \qquad A(m, l) = 0. \tag{11.20}$$

Note that y and x^2 are regular functions on the A-polymial curve.

11.4. **Recovering the representations.** The dual triangulation of `m009` has an oriented edge for each of the face pairings a,b,c,d,t_1, and t_2. The edges t_1 and t_2 form a maximal tree, so the fundamental group of `m009` is generated by a, b, c, and d. By inspecting Figure 16 we see that

$$\pi_1(M) = \langle a, b, c, d \mid cd^{-1}a^{-1}, cb^{-1}d^{-1}ba, ca^{-1}bd^{-1} \rangle, \tag{11.21}$$

and that the meridian μ and longitude λ are given by

$$\mu = acb^{-1}, \qquad \lambda = d^{-1}cd^{-1}bc^{-1}db^{-1}. \tag{11.22}$$

One also checks that the generators may be represented by edge paths in the truncated complex as follows:

$$\begin{aligned} a &= \beta^0_{23,0}\alpha_{03,0}\beta^0_{23,1}\alpha_{03,1}\beta^3_{01,1}(\beta^2_{01,0})^{-1}\alpha^{-1}_{02,0} \\ b &= \alpha_{02,0}\beta^2_{01,0}\alpha^{-1}_{12,0}(\beta^0_{23,0})^{-1} \\ c &= \beta^0_{23,0}\alpha_{03,0}\beta^3_{01,0}\alpha^{-1}_{12,1}(\beta^3_{01,2})^{-1}\alpha^{-1}_{03,2} \\ d &= \alpha_{02,0}\beta^2_{01,0}\alpha^{-1}_{12,0}. \end{aligned} \tag{11.23}$$

One can then compute the representations explicitly using the formulas for α_{ij} and β^k_{ij} given by Proposition 5.9.

11.4.1. *The representation with obstruction class* σ^1. We obtain

$$a = c = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad b = d = I, \qquad \mu = -\lambda = I. \tag{11.24}$$

This shows that the Ptolemy variety can detect reducible representations. It is boundary-degenerate as it must be by Proposition 7.8.

11.4.2. *The representations with obstruction class* σ^3. We obtain

$$\begin{aligned} a &= \begin{pmatrix} w^3+w & 1 \\ 1 & -w \end{pmatrix}, \qquad b = \begin{pmatrix} 1 & -w \\ -w & w^2+1 \end{pmatrix}, \qquad c = \begin{pmatrix} w^3 & 1 \\ w^2+1 & -w \end{pmatrix} \\ d &= \begin{pmatrix} 1 & 0 \\ -w & 1 \end{pmatrix}, \qquad \mu = \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}, \qquad \lambda = \begin{pmatrix} -1 & 2w^3+w \\ 0 & -1 \end{pmatrix}. \end{aligned} \tag{11.25}$$

One easily checks that $H^1(\widehat{M};\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. Hence, by Theorem 1.9 there should be only two $\mathrm{PSL}(2,\mathbb{C})$ representations, not four. Indeed, replacing w by its Galois conjugate $-w$ corresponds to conjugating the representation by the diagonal matrix with entries $\sqrt{-1}$ and $-\sqrt{-1}$. Note that the fixed field of the Galois isomorphism $w \mapsto -w$ is $\mathbb{Q}(\sqrt{-7})$, which is the shape field of `m009`.

11.4.3. *The non-boundary-unipotent representations.* Representing the generators by edge paths in the fattened truncated complex, we have

$$\begin{aligned} a &= \beta^0_{23,0}\alpha_{03,0}\beta^0_{23,1}\alpha_{03,1}\beta^3_{01,1}(\beta^2_{01,0})^{-1}\alpha^{-1}_{02,0} \\ b &= \alpha_{02,0}\beta^2_{01,0}\alpha^{-1}_{12,0}M^{-1}L^{-1}(\beta^0_{23,0})^{-1} \\ c &= \beta^0_{23,0}\alpha_{03,0}\beta^3_{01,0}\alpha^{-1}_{12,1}(\beta^3_{01,2})^{-1}\alpha^{-1}_{03,2}L^{-1} \\ d &= \alpha_{02,0}\beta^2_{01,0}\alpha^{-1}_{12,0}L^{-1}, \end{aligned} \tag{11.26}$$

where $M = \left(\begin{smallmatrix} m & 0 \\ 0 & m^{-1} \end{smallmatrix}\right)$ and $L = \left(\begin{smallmatrix} l & 0 \\ 0 & l^{-1} \end{smallmatrix}\right)$. Using this, we obtain

$$\begin{aligned} a &= \begin{pmatrix} \frac{-(m+l)x}{m^4-m^2} & \frac{l^2}{m^2} \\ \frac{-1}{ml} & \frac{-lx}{m^2} \end{pmatrix}, \qquad b = \begin{pmatrix} \frac{1}{m^2} & \frac{l^2x}{m^3} \\ \frac{-x}{m^2l} & \frac{m^2+ml}{m^2-1} \end{pmatrix}, \qquad c = \begin{pmatrix} \frac{-(ml+1)x}{m^4-m^2} & \frac{l^2}{m} \\ \frac{m+l}{l^2(m^2-1)} & \frac{-lx}{m} \end{pmatrix} \\ d &= \begin{pmatrix} m^{-1} & 0 \\ \frac{-x}{ml} & m \end{pmatrix}, \qquad \mu = \begin{pmatrix} m & \frac{-l^2x}{m^2} \\ 0 & m^{-1} \end{pmatrix}, \qquad \lambda = \begin{pmatrix} l & \frac{(-l^3+l)x}{m^3-m} \\ 0 & l^{-1} \end{pmatrix}, \end{aligned} \tag{11.27}$$

where x is given by (11.20).

Remark 11.1. As mentioned in Remark 1.12 there is also an enhanced Ptolemy variety for representations in $\mathrm{PSL}(2,\mathbb{C})$ that are not necessarily boundary-unipotent. Using this, one can show that the representation (11.24) deforms into a curve of representations with image in an infinite dihedral group. None of these representations lift to $\mathrm{SL}(2,\mathbb{C})$, so the variety $\mathcal{E}\overline{P}(\mathcal{T})$ does not detect these.

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