Abstract

In this note we establish some properties of exponentiable motivic measures. As a first application, we show that the rationality of Kapranov’s zeta function is stable under products. As a second application, we give an elementary proof of a special case of a result of Totaro.

Keywords: Grothendieck ring of varieties, motivic measure, Kapranov’s zeta function, Witt vectors, λ-ring, Kimura-finiteness, pure and mixed motives, G-varieties.

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1 Motivic measures

Let $k$ be an arbitrary base field and $\text{Var}(k)$ the category of varieties, i.e. reduced separated $k$-schemes of finite type. The Grothendieck ring of varieties $K_0\text{Var}(k)$ is defined as the quotient of the free abelian group on the set of isomorphism classes of varieties $[X]$ by the relations $[X] = [Y] + [X \setminus Y]$, where $Y$ is a closed subvariety of $X$. The multiplication is induced by the product of varieties. When $k$ is of positive characteristic, one needs also to impose the relation $[X] = [Y]$ for every surjective radicial morphism $X \to Y$; see Mustaţă [19, Page 78]. Let $L := [A^1]$.

The structure of the Grothendieck ring of varieties is quite mysterious; see Poonen [21] for instance. In order to capture some of its flavor several motivic measures, i.e. ring homomorphisms $\mu : K_0\text{Var}(k) \to R$, have been built. Here are some classical examples:

(i) When $k$ is finite, the assignment $X \mapsto \# X(k)$ gives rise to the counting measure $\mu_{\#} : \text{Var}(k) \to \mathbb{Z}$; see [19, Ex. 7.7].

(ii) When $k = \mathbb{C}$, the assignment $X \mapsto \chi_c(X) := \sum_i (-1)^i \dim_{\mathbb{Q}} H^i_c(X^{an}, \mathbb{Q})$ gives rise to the Euler characteristic measure $\chi_c : \text{Var}(k) \to \mathbb{Z}$; see [19, Ex. 7.8].

(iii) When $k$ is of characteristic zero, the assignment $X \mapsto H_X(u, v) := \sum_{p, q \geq 0} h^{p,q}(X) u^p v^q$, with $X$ smooth projective, gives rise to the Hodge characteristic measure $\mu_H : \text{Var}(k) \to \mathbb{Z}[u, v]$; see [14, §4.1].
(iv) When \( k \) is of characteristic zero, the assignment \( X \mapsto P_X(u) := \sum_i \dim_k H^i_{dR}(X)u^i \), with \( X \) smooth projective, gives rise to the Poincaré characteristic measure \( \mu_P : \text{Var}(k) \to \mathbb{Z}[u] \); see [14, §4.1].

Other motivic measures include the Larsen-Lunts “exotic” measure \( \mu_{LL} \) (see [13]); the Albanese measure \( \mu_{Alb} \) with values in the semigroup ring of isogeny classes of abelian varieties (see [19, Thm. 7.21]); the Gillet-Soulé measure \( \mu_{GS} \) with values in the Grothendieck ring \( K_0(\text{Chow}(k)_{\mathbb{Q}}) \) of Chow motives (see [6]); and the measure \( \mu_{NC} \) with values in the Grothendieck ring of noncommutative motives (see [23]). There exist several relations between the above motivic measures. For example, \( \chi_c, \mu_H, \mu_P, \mu_{NC} \) factor through \( \mu_{GS} \).

2 Kapranov’s zeta function

As explained in [19, Prop. 7.27], in the construction of the Grothendieck ring of varieties we can restrict ourselves to quasi-projective varieties. Given a motivic measure \( \mu \), Kapranov introduced in [11] the associated zeta function
\[
\zeta_\mu(X; t) := \sum_{n=0}^{\infty} \mu([S^n(X)]) t^n \in (1 + t\mathbb{R}[t]),
\]
where \( S^n(X) \) stands for the \( n^{th} \) symmetric product of the quasi-projective variety \( X \). In the particular case of the counting measure, (1) agrees with the classical Weil zeta function. Here are some other computations (with \( X \) smooth projective)
\[
\zeta_{\chi_c}(X; t) = (1 - t)^{-\chi_c(X)} \quad \zeta_P(X; t) = \prod_{r \geq 0} \left( \frac{1}{1 - \frac{1}{1-at} t^r} \right) (-1)^b_r \quad \zeta_{Alb}(X; t) = \frac{[\text{Alb}(X)]t}{1 - t},
\]
where \( b_r := \dim \mathbb{C} H^r_{dR}(X) \) and \( \text{Alb}(X) \) is the Albanese variety of \( X \); see [22, §3].

3 Big Witt ring

Given a commutative ring \( R \), recall from Bloch [2, Page 192] the construction of the big Witt ring \( W(R) \). As an additive group, \( W(R) \) is equal to \( (1 + tR[[t]], \times) \). Let us write \( +_W \) for the addition in \( W(R) \) and \( 1 = 1 + 0t + \ldots \) for the zero element. The multiplication \( * \) in \( W(R) \) is uniquely determined by the following requirements:

(i) The equality \((1 - at)^{-1} * (1 - bt)^{-1} = (1 - abt)^{-1}\) holds for every \( a, b \in R \);

(ii) The assignment \( R \mapsto W(R) \) is an endofunctor of commutative rings.

The unit element is \((1 - t)^{-1}\). We have also a (multiplicative) Teichmüller map
\[
R \rightarrow W(R) \quad a \mapsto [a] := (1 - at)^{-1}
\]
such that \( g(t) * [a] = g(at) \) for every \( a \in R \) and \( g(t) \in W(R) \); see [2, Page 193].
Definition 3.1. Elements of the form \( p(t) - W q(t) \in W(R) \), with \( p(t), q(t) \in R[t] \) and \( p(0) = q(0) = 1 \in R \), are called rational functions.

Let \( W_{\text{rat}}(R) \) be the subset of rational elements. As proved by Naumann in [20, Prop. 6], \( W_{\text{rat}}(R) \) is a subring of \( W(R) \). Moreover, \( R \mapsto W_{\text{rat}}(R) \) is an endofunctor of commutative rings. Recall also the construction of the commutative ring \( \Lambda(R) \). As an additive group, \( \Lambda(R) \) is equal to \( W(R) \). The multiplication is uniquely determined by the requirement that the involution group isomorphism \( \iota : \Lambda(R) \to W(R), g(t) \mapsto g(-t)^{-1} \), is a ring isomorphism. The unit element is \( 1 + t \).

4 Exponentiation

Let \( \mu \) be a motivic measure. As explained by Mustaţă in [19, Prop. 7.28], the assignment \( X \mapsto \zeta_{\mu}(X; t) \) gives rise to a group homomorphism

\[
\zeta_{\mu}(-; t) : K_0 \text{Var}(k) \to W(R).
\]

(2)

Definition 4.1. ([22, §3]) A motivic measure \( \mu \) is (uniquely) exponentiable\(^1\) if the above group homomorphism (2) is a ring homomorphism.

Corollary 4.2. Given an exponentiable measure, the following holds:

(i) The ring homomorphism (2) is a new motivic measure;

(ii) Any motivic measure which factors through \( \mu \) is also exponentiable.

This class of motivic measures is well-behaved with respect with rationality:

Proposition 4.3. Let \( \mu \) be an exponentiable motivic measure. If \( \zeta_{\mu}(X; t) \) and \( \zeta_{\mu}(Y; t) \) are rational functions, then \( \zeta_{\mu}(X \times Y; t) \) is also a rational function.

Proof. It follows automatically from the fact that \( W_{\text{rat}}(R) \) is a subring of \( W(R) \).

As proved by Naumann in [20, Prop. 8] (see also [22, Thm. 2.1]), the counting measure \( \mu_{\#} \) is exponentiable. On the other hand, Larsen-Lunts “exotic” measure \( \mu_{\text{LL}} \) is not exponentiable! This would imply, in particular, that

\[
\zeta_{\mu_{\text{LL}}}(C_1 \times C_2; t) = \zeta_{\mu_{\text{LL}}}(C_1; t) \ast \zeta_{\mu_{\text{LL}}}(C_2; t)
\]

(3)

for any two smooth projective curves \( C_1 \) and \( C_2 \). As proved by Kapranov in [11] (see also [19, Thm. 7.33]), \( \zeta_{\mu}(C; t) \) is a rational function for every smooth projective curve \( C \)

\(^1\)Note that Kapranov’s zeta function is similar to the exponential function \( e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \). The product \( X^n \) corresponds to \( x^n \) and the symmetric product \( S^n(X) \) corresponds to \( \frac{x^n}{n!} \) since \( n! \) is the size of the symmetric group on \( n \) letters.
and motivic measure $\mu$. Using Proposition 4.3, this hence implies that the right-hand side of (3) is also a rational function. On the other hand, as proved by Larsen-Lunts in [13, Thm. 7.6], the left-hand side of (3) is not a rational function whenever $C_1$ and $C_2$ have positive genus. We hence obtain a contradiction.

At this point, it is natural to ask which motivic measures are exponentiable? We now provide a general answer to this question using the notion of $\lambda$-ring. Recall that a $\lambda$-ring $R$ consists of a commutative ring equipped with a sequence of maps $\lambda^a : R \to R$, $a \geq 0$, such that $\lambda^0(a) = 1$, $\lambda^1(a) = a$, and $\lambda^n(a + b) = \sum_{i+j=n} \lambda^i(a) \lambda^j(b)$ for every $a, b \in R$. In other words, the map

$$\lambda_t : R \longrightarrow \Lambda(R) \quad a \mapsto \lambda_t(a) := \sum_n \lambda^n(a)t^n$$

is a group homomorphism. Equivalently, the composed map

$$\sigma_t : R \overset{\lambda}{\longrightarrow} \Lambda(R) \overset{\iota}{\longrightarrow} W(R) \quad a \mapsto \sigma_t(a) = \sum_n \sigma^n(a)t^n := \lambda_{-t}(a)^{-1} \quad (4)$$

is a group homomorphism. This homomorphism is called the opposite $\lambda$-structure.

**Proposition 4.4.** Let $\mu$ be a motivic measure and $R$ a $\lambda$-ring such that:

(i) The above group homomorphism (4) is a ring homomorphism;

(ii) We have $\mu([S^n(X)]) = \sigma^n(\mu([X]))$ for every quasi-projective variety $X$.

Under these conditions, the motivic measure $\mu$ is exponentiable.

**Proof.** Consider the following composed ring homomorphism

$$K_0 \text{Var}(k) \overset{\mu}{\longrightarrow} R \overset{\sigma_t}{\longrightarrow} W(R). \quad (5)$$

The equalities $\mu([S^n(X)]) = \sigma^n(\mu([X]))$ allow us to conclude that (5) agrees with the group homomorphism $\zeta_\mu(-; t)$. This achieves the proof.

**Remark 4.5.** Let $C$ be a $\mathbb{Q}$-linear additive idempotent complete symmetric monoidal category. As proved by Heinloth in [9, Lem. 4.1], the exterior powers give rise to a special $\lambda$-structure on the Grothendieck ring $K_0(C)$, with opposite $\lambda$-structure given by the symmetric powers $\text{Sym}^n$. In this case, (4) is a ring homomorphism.

**Remark 4.6.** Let $T'$ be a $\mathbb{Q}$-linear thick triangulated monoidal subcategory of compact objects in the homotopy category $T = \text{Ho}(C)$ of a simplicial symmetric monoidal model category $C$. As proved by Guletskii in [8, Thm. 1], the exterior powers give rise to a special $\lambda$-structure on $K_0(T')$, with opposite $\lambda$-structure given by the symmetric powers $\text{Sym}^n$. In this case, (4) is a ring homomorphism.
Remark 4.7. Assume that $k$ is of characteristic zero. Thanks to Heinloth’s presentation of the Grothendieck ring of varieties (see [10, Thm. 3.1]), it suffices to verify the equality $\mu([S^n(X)]) = \sigma^n(\mu([X]))$ for every smooth projective variety $X$.

As an application of the above Proposition 4.4, we obtain the following result:

**Proposition 4.8.** The Gillet-Soulé motivic measure $\mu_{GS}$ is exponentiable.

**Proof.** Recall from [6] that $\mu_{GS}$ is induced by the symmetric monoidal functor

$$\mathfrak{h}: \text{SmProj}(k) \longrightarrow \text{Chow}(k)_\mathbb{Q}$$

from the category of smooth projective varieties to the category of Chow motives. Since the latter category is $\mathbb{Q}$-linear, additive, idempotent complete, and symmetric monoidal, Remark 4.5 implies that the Grothendieck ring $K_0(\text{Chow}(k)_\mathbb{Q})$ satisfies condition (i) of Proposition 4.4. As proved by del Baño-Aznar in [4, Cor. 2.4], we have $\mathfrak{h}(S^n(X)) \simeq \text{Sym}^n\mathfrak{h}(X)$ for every smooth projective variety $X$. Using Remark 4.7, this hence implies that condition (ii) of Proposition 4.4 is also satisfied.

**Remark 4.9.** Thanks to Corollary 4.2(ii), all the motivic measures which factor through $\mu_{GS}$ (e.g. $\chi_c, \mu_H, \mu_P, \mu_{NC}$) are also exponentiable.

5 Application I: rationality of zeta functions

By combining Propositions 4.3 and 4.8, we obtain the following result:

**Corollary 5.1.** Let $X, Y$ be two varieties. If $\zeta_{\mu_{GS}}(X; t)$ and $\zeta_{\mu_{GS}}(Y; t)$ are rational functions, then $\zeta_{\mu_{GS}}(X \times Y; t)$ is also a rational function.

**Remark 5.2.** Corollary 5.1 was independently obtained by Heinloth [9, Prop. 6.1] in the particular case of smooth projective varieties and under the extra assumption that $\zeta_{\mu_{GS}}(X; t)$ and $\zeta_{\mu_{GS}}(Y; t)$ satisfy a certain functional equation.

**Example 5.3.** Let $X, Y$ be smooth projective varieties (e.g. abelian varieties) for which $\mathfrak{h}(X), \mathfrak{h}(Y)$ are Kimura-finite; see [12, §3]. Consider the ring homomorphism

$$\sigma_t: K_0(\text{Chow}(k)_\mathbb{Q}) \longrightarrow W(K_0(\text{Chow}(k)_\mathbb{Q})).$$

As proved by Andrè in [1, Prop. 4.6], $\sigma_t([\mathfrak{h}(X)])$ and $\sigma_t([\mathfrak{h}(Y)])$ are rational functions. Since $\zeta_{\mu_{GS}}(-; t)$ agrees with the composition of $\mu_{GS}$ with (7), these latter functions are equal to $\zeta_{\mu_{GS}}(X; t)$ and $\zeta_{\mu_{GS}}(Y; t)$, respectively. Using Corollary 5.1, we hence conclude that $\zeta_{\mu_{GS}}(X \times Y; t)$ is also a rational function.
When \( k \) is of characteristic zero, Voevodsky constructed in [24, §2.2] a functor
\[
M^c : \text{Var}(k)^{op} \rightarrow \text{DM}_{\text{gm}}(k)_\mathbb{Q}
\]
from the category of varieties and proper morphisms to the triangulated category of geometric motives. As proved in [24, Prop. 4.1.7], the functor (8) is symmetric monoidal. Moreover, given a variety \( X \) and a closed subvariety \( Y \subset X \), we have a triangle
\[
M^c(Y) \rightarrow M^c(X) \rightarrow M^c(X \setminus Y) \rightarrow M^c(Y)[1]
\]
in \( \text{DM}_{\text{gm}}(k)_\mathbb{Q} \); see [24, Prop. 4.1.5]. Consequently, we obtain the motivic measure:
\[
K_0 \text{Var}(k) \rightarrow K_0(\text{DM}_{\text{gm}}(k)_\mathbb{Q}) \quad [X] \mapsto [M^c(X)].
\]

**Proposition 5.4.** The above motivic measure (9) agrees with \( \mu_{\text{GS}} \).

**Proof.** As proved by Voevodsky in [24, Prop. 2.1.4], there exists a \( \mathbb{Q} \)-linear additive fully-faithful symmetric monoidal functor
\[
\text{Chow}(k)_\mathbb{Q} \rightarrow \text{DM}_{\text{gm}}(k)_\mathbb{Q}
\]
such that \( (10) \circ h(X) \simeq M^c(X) \) for every smooth projective variety. Thanks to the work of Bondarko [3, Cor. 6.4.3 and Rk. 6.4.4], the above functor (10) induces a ring isomorphism \( K_0(\text{Chow}(k)_\mathbb{Q}) \simeq K_0(\text{DM}_{\text{gm}}(k)_\mathbb{Q}) \). Therefore, the proof follows from Heinloth’s presentation of the Grothendieck ring of varieties in terms of smooth projective varieties; see [10, Thm. 3.1].

Thanks to Proposition 5.4, Example 5.3 admits the following generalization:

**Example 5.5.** Let \( X,Y \) be varieties for which \( M^c(X), M^c(Y) \) are Kimura-finite. Similarly to Example 5.3, \( \zeta_{\mu_{\text{GS}}}(X \times Y; t) \) is then a rational function.

In the above Examples 5.3 and 5.5, the rationality of \( \zeta_{\mu_{\text{GS}}}(X \times Y; t) \) can alternatively be deduced from the stability of Kimura-finiteness under tensor products; see [12, §5]. Thanks to the work of O’Sullivan-Mazza [18, §5.1] and Guletskii [8], the above Corollary 5.1 can also be applied to non Kimura-finite situations.

**Proposition 5.6.** Let \( X_0 \) be a connected smooth projective surface, over an algebraically closed field \( k_0 \), with geometric genus \( p_g > 0 \) and irregularity \( q = 0 \). Let \( k := k_0(X_0) \) the function field of \( X_0 \), \( x_0 \) a \( k_0 \)-point of \( X_0 \), \( z \) the zero-cycle which is the pull-back of the cycle \( \Delta(X_0) - (x_0 \times X) \) along \( X_0 \times k \to X_0 \times X_0 \), \( Z \) the support of \( z \), and finally \( U \) the complement of \( Z \) in \( X = X_0 \times k \). Under these notations, the following holds:

(i) The geometric motive \( M^c(U) \) is not Kimura-finite;

(ii) Kapranov’s zeta function \( \zeta_{\mu_{\text{GS}}}(U; t) \) is rational.
Proof. As proved by O’Sullivan-Mazza in [18, Thm. 5.18], \( M(U) \) is not Kimura-finite. Since the surface \( \hat{U} \) is smooth, we have \( M^c(U) \simeq M(U)^{\vee} (2)[4] \) where \((-)^\vee\) stands for the dual; see [24, Thm. 4.3.7]. Using the fact that \(- (2)[4]\) is an auto-equivalence and that \( M(U)^{\vee} \) is Kimura-finite if and only if \( M(U) \) is Kimura-finite (see Deligne [5, Prop. 1.18]), we conclude that \( M^c(U) \) also is not Kimura-finite.

We now prove item (ii). As proved by Guletskii in [8, §3], the category \( \text{DM}_{\text{gm}}(k)_\mathbb{Q} \) satisfies the conditions of Remark 4.6. Consequently, we have a ring homomorphism

\[
\sigma_t : K_0(\text{DM}_{\text{gm}}(k)_{\mathbb{Q}}) \to W(K_0(\text{DM}_{\text{gm}}(k)_{\mathbb{Q}})).
\]  

(11)

As explained by Guletskii in [8, Ex. 5], \( \sigma_t([M(U)]) \) is a rational function. Equivalently, \( \sigma_t([M(U)])^{\vee} \) (obtained from \( \sigma_t([M(U)]) \) by applying \((-)^{\vee}\) to each term) is a rational function. Thanks to Lemma 5.7 below, we hence conclude that \( \sigma_t([M^c(U)]) \) is also a rational function. The proof follows now from the fact that \( \zeta_{\text{mgS}}(-; t) \) agrees with the composition of the ring homomorphisms (9) and (11). \[\square\]

Lemma 5.7. Given a smooth variety \( X \) of dimension \( d \), we have the equality

\[
\sigma_t([M^c(X)]) = \sigma_{\mu_{\text{mgS}}(L)^d t}([M(X)])^{\vee}.
\]

Proof. The proof is given by the following identifications

\[
\sigma_t([M^c(X)]) = \sigma_t([M(X)^{\vee}(d)[2d]]) = \sigma_t([M(X)^{\vee}]_{\mu_{\text{mgS}}(L^d)}) = \sigma_t([M(X)^{\vee}] \ast \zeta_{\text{mgS}}(L^d; t)) = \sigma_t([M(X)])^{\vee} \ast \zeta_{\text{mgS}}(L^d; t) = \sigma_{\mu_{\text{mgS}}(L)^d t}([M(X)])^{\vee},
\]

(12) where (12) follows from [24, Thm. 4.3.7], (13) from [5, Prop. 1.18], and (14) from Remark 6.2 below with \( \mu := \mu_{\text{mgS}} \) and \( g(t) := \sigma_t([M(X)])^{\vee} \). \[\square\]

Example 5.8. Let \( U_1, U_2 \) be two surfaces as in Proposition 5.6. Thanks to the above Corollary 5.1, we hence conclude that \( \zeta_{\text{mgS}}(U_1 \times U_2; t) \) is a rational function. Note that the geometric motive \( M^c(U_1 \times U_2) \) is not Kimura-finite! Choose a rational point \( x_1 \) of \( U_1 \) and consider the associated morphism \( x_1 \times \text{id} : U_2 \to U_1 \times U_2 \). Using the projection \( U_1 \times U_2 \to U_2 \) we observe that \( M(U_2) \) is a direct summand of \( M(U_1 \times U_2) \). As explained in the proof of Proposition 5.6, \( M^c(U_2) \) (resp. \( M^c(U_1 \times U_2) \)) is Kimura-finite if and only if \( M(U_2) \) (resp. \( M(U_1 \times U_2) \)) is Kimura-finite. Consequently, if \( M^c(U_1 \times U_2) \) were Kimura-finite, \( M^c(U_2) \) also would be Kimura-finite. This contradicts Proposition 5.6. Finally, note that self-products \( U_1 \times \cdots \times U_1 \) are examples of arbitrarily high dimension.

Remark 5.9. Thanks to Corollary 4.2(ii), the above Examples 5.3, 5.5, and 5.8, hold mutatis mutandis for any motivic measure which factors through \( \mu_{\text{mgS}} \).
6 Application II: Totaro’s result

The following result plays a central role in the study of the zeta functions.

**Proposition 6.1** (Totaro). The equality \( \zeta_\mu(X \times \mathbb{A}^n; t) = \zeta_\mu(X; \mu(L)^n t) \) holds for every variety \( X \) and motivic measure \( \mu \).

Its proof (see [7, Lem. 4.4] [19, Prop. 7.32]) is non-trivial and based on a stratification of the symmetric products of \( X \times \mathbb{A}^n \). In all the cases where the motivic measure \( \mu \) is exponentiable, this result admits the following elementary proof:

**Proof.** Since \( [X \times \mathbb{A}^n] = [X][\mathbb{A}^n] \) in the Grothendieck ring of varieties and the motivic measure \( \mu \) is exponentiable, the proof is given by the identifications

\[
\zeta_\mu(X \times \mathbb{A}^n; t) = \zeta_\mu(X; t) \ast \zeta_\mu(L^n; t) = \zeta_\mu(X; t) \ast (1 + \mu(L)t + \mu(L)t^2 + \cdots)^n \tag{15}
\]

where (15) follows from [19, Ex. 7.23] and \([\mu(L)]\) stands for the image of \( \mu(L) \in R \) under the multiplicative Teichmüller map \( R \to W(R) \).

**Remark 6.2.** The above proof shows more generally that \( g(t) \ast \zeta_\mu(L^n; t) = g(\mu(L)^n t) \) for every \( g(t) \in W(R) \) and exponentiable motivic measure \( \mu \).

**Remark 6.3.** (Fiber bundles) Given a fiber bundle \( E \to X \) of rank \( n \), we have \([E] = [X][\mathbb{A}^n]\) in the Grothendieck ring of varieties; see [19, Prop. 7.4]. Therefore, the above proof, with \( X \) replaced by \( E \), shows that \( \zeta_\mu(E; t) = \zeta_\mu(X; \mu(L)^n t) \).

**Remark 6.4.** (\( \mathbb{P}^n \)-bundles) Given a \( \mathbb{P}^n \)-bundle \( E \to X \), we have \([E] = [X][\mathbb{P}^n]\) in the Grothendieck ring of varieties; see [19, Ex. 7.5]. Therefore, by combining the equality \([\mathbb{P}^n] = 1 + L + \cdots + L^n\) with the above proof, we conclude that

\[
\zeta_\mu(E; t) = \zeta_\mu(X; t) + W \zeta_\mu(X; \mu(L)t) + W \cdots + W \zeta_\mu(X; \mu(L)^n t).
\]

7 \( G \)-varieties

Let \( G \) be a finite group and \( \text{Var}^G(k) \) the category of \( G \)-varieties, i.e. varieties \( X \) equipped with a \( G \)-action \( \lambda : G \times X \to X \) such that every orbit is contained in an
affine open set. The Grothendieck ring of $G$-varieties $K_0 \text{Var}^G(k)$ is defined as the quotient of the free abelian group on the set of isomorphism classes of $G$-varieties $[X,\lambda]$ by the relations $[X,\lambda] = [Y,\tau] + [X \setminus Y,\lambda]$, where $(Y,\tau)$ is a closed $G$-invariant subvariety of $(X,\lambda)$. The multiplication is induced by the product of varieties. A motivic measure is a ring homomorphism $\mu^G : K_0 \text{Var}^G(k) \to R$. As mentioned in [15, §5], the above measures $\chi_c, \mu_H, \mu_P$ admit $G$-extensions $\chi_c^G, \mu_H^G, \mu_P^G$.

**Notation 7.1.** Let $\text{Chow}^G(k)_Q$ be the category of functors from the group $G$ (considered as a category with a single object) to the category $\text{Chow}(k)_Q$.

Note that $\text{Chow}^G(k)_Q$ is still a $Q$-linear additive idempotent complete symmetric monoidal category and that (6) extends to a symmetric monoidal functor

$$h^G : \text{SmProj}^G(k) \to \text{Chow}^G(k)_Q.$$  \hfill (16)

Note also that the $n$th symmetric product of a $G$-variety is still a $G$-variety. Therefore, the notion of exponentiation makes sense in this generality. Gillet-Soulé’s motivic measure $\mu_{GS}$ admits the following $G$-extension:

**Proposition 7.2.** The above functor (16) gives rise to an exponentiable motivic measure:

$$\mu_{GS}^G : K_0 \text{Var}^G(k) \to K_0(\text{Chow}^G(k)_Q).$$

**Proof.** Given a smooth projective variety $X$ and a closed subvariety $Y$, let us denote by $\text{Bl}_Y(X)$ the blow-up of $X$ along $Y$ and by $E$ the associated exceptional divisor. As proved by Manin in [16, §9], we have a natural isomorphism $h(\text{Bl}_Y(X)) \otimes h(Y) \simeq h(X) \otimes h(E)$ in $\text{Chow}(k)_Q$. Since this isomorphism is natural, it also holds in $\text{Chow}^G(k)_Q$ when $X$ is replaced by a smooth projective $G$-variety $(X,\lambda)$ and $Y$ by a closed $G$-invariant subvariety $(Y,\tau)$. Therefore, thanks to Heinloth’s presentation of the Grothendieck ring of $G$-varieties in terms of smooth projective $G$-varieties (see [10, Lem. 7.1]), the assignment $X \mapsto h^G(X)$ gives rise to a (unique) motivic measure $\mu_{GS}^G$. The proof of Proposition 4.8, with (6) replaced by (16), shows that this motivic measure $\mu_{GS}^G$ is exponentiable.

**Remark 7.3.** Similarly to Remark 4.9, all the motivic measures which factor through $\mu_{GS}^G$ (e.g. $\chi_c^G, \mu_H^G, \mu_P^G$) are also exponentiable.

**Proposition 4.3** admits the following $G$-extension:

**Proposition 7.4.** Let $\mu^G$ be an exponentiable motivic measure and $(X,\lambda), (Y,\tau)$ two $G$-varieties. If $\zeta_{\mu^G}((X,\lambda);t)$ and $\zeta_{\mu^G}((Y,\tau);t)$ are rational functions, then $\zeta_{\mu^G}(((X \times Y,\lambda \times \tau);t)$ is also a rational function.

**Example 7.5.** Assume that the group $G$ (of order $r$) is abelian and that the base field $k$ is algebraically closed of characteristic zero or of positive characteristic $p$ with $p \nmid r$. Under these assumptions, Mazur proved in [17, Thm. 1.1] that $\zeta_{\mu^G}((C,\lambda);t)$ is a rational function for every smooth projective $G$-curve $(C,\lambda)$ and motivic measure $\mu^G$. Thanks
to Proposition 7.4, we hence conclude that \( \zeta_{\mu^G}((C_1 \times C_2, \lambda_1 \times \lambda_2); t) \) is still a rational function for every exponentiable motivic measure \( \mu^G \) and for any two smooth projective \( G \)-curves \((C_1, \lambda_1)\) and \((C_2, \lambda_2)\).

Finally, Totaro’s result admits the following \( G \)-extension:

**Proposition 7.6.** Let \( \mu^G \) be an exponentiable motivic measure and \((X, \lambda), (\mathbb{A}^n, \tau)\) two \( G \)-varieties. When \( G \) (of order \( r \)) is abelian and \( k \) is algebraically closed, Kapranov’s zeta function \( \zeta_{\mu^G}((X \times \mathbb{A}^n, \lambda \times \tau); t) \) agrees with

\[
\zeta_{\mu^G}((X, \lambda); \mu^G(S^r(\mathbb{A}^n, \tau)); t) + W \zeta_{\mu^G}((X, \lambda); t) = \left( \sum_{l=0}^{r-1} \prod_{i=1}^{n} \mu^G([\mathbb{A}^1, \tau_i] \cdots [\mathbb{A}^1, \tau_i^l]) t^l \right),
\]

where \([\mathbb{A}^n, \tau] = [\mathbb{A}^1, \tau_1] \cdots [\mathbb{A}^1, \tau_n]\).

**Proof.** Since \([X \times \mathbb{A}^n, \lambda \times \tau] = [X, \lambda][\mathbb{A}^n, \tau]\) in the Grothendieck ring of \( G \)-varieties and the motivic measure \( \mu^G \) is exponentiable, we have the equality

\[
\zeta_{\mu^G}((X \times \mathbb{A}^n, \lambda \times \tau); t) = \zeta_{\mu^G}((X, \lambda); t) * \zeta_{\mu^G}((\mathbb{A}^n, \tau); t).
\]

Moreover, as explained in [17, Page 1338], we have the following computation

\[
\zeta_{\mu^G}((\mathbb{A}^n, \tau); t) = \frac{1}{1 - \mu^G(S^r(\mathbb{A}^n, \tau)) t} \left( \sum_{l=0}^{r-1} \prod_{i=1}^{n} \mu^G([\mathbb{A}^1, \tau_i] \cdots [\mathbb{A}^1, \tau_i^l]) t^l \right).
\]

Therefore, since \((1 - \mu^G(S^r(\mathbb{A}^n, \tau)) t)^{-1}\) is the Teichmüller class \( [\mu^G(S^r(\mathbb{A}^n, \tau))] \), the proof follows from the combination of the above equalities. \( \square \)

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**References**


