Problem 4 Solution

Question:
Find the particular solution of the following differential equation:
\[ \frac{dy}{dx} + \sin x = y \sin x, \]
with the condition that \( y(0) = 1. \)

Solution:
First recognize that the differential equation is a first order linear and put it into standard form
\[ \frac{dy}{dx} - y \sin x = \sin x. \]  
(3 pts)

Then find the integrating factor, which is
\[ e^{\int -\sin x \, dx} = e^{\cos x}. \]  
(5 pts)

Multiple through by the integrating factor and simplify the left hand side.
\[ e^{\cos x} \frac{dy}{dx} - ye^{\cos x} \sin x = -e^{\cos x} \sin x \]
\[ \frac{d}{dx} \left( ye^{\cos x} \right) = -e^{\cos x} \sin x \]  
(7 pts)

Integrate both sides
\[ ye^{\cos x} = - \int e^{\cos x} \sin x \, dx \]
and use a u-substitution on the right hand side
\[ u = \cos x \]
\[ dx = -\sin x \, dx, \]
to get
\[ ye^{\cos x} = e^{\cos x} + C. \]  
(8 pts)

Then use the initial condition \( x = 0, \ y = 1 \) to find the value \( C = 0, \) so the final answer is
\[ y = 1. \]  
(2 pts)