Problem (25pts): Determine whether the following series converge (absolutely/conditionally). If they converge, find their value.

1. \[ \sum_{k=2}^{\infty} \frac{(-1)^k 5^{k-1}}{3^{2k}} \]
2. \[ \sum_{n=2}^{\infty} \frac{1}{n \cdot \ln n} \]

Solution:

Part 1. 15 points.

(2pts): We have

\[ \sum_{k=2}^{\infty} \frac{(-1)^k 5^{k-1}}{3^{2k}} = \frac{1}{5} \sum_{k=2}^{\infty} \left( \frac{-5}{9} \right)^k \]

(6pts): This is a geometric series with ratio \( r = \frac{-5}{9} \) and first term \( a = \frac{5}{81} \). [3 points given for each item.]

(3pts): Thus, since \( |r| < 1 \), the series converges absolutely.

(4pts): In particular, it converges to

\[ \frac{a}{1 - r} = \frac{\frac{5}{81}}{1 - \left( -\frac{5}{9} \right)} = \frac{5}{126} \]

(Note: Students may alternatively use the root or ratio tests to determine convergence of the series. Only doing this, without finding the value of the series, will merit 8 points.)

Part 2. 10 points.

(2pts): We use the Integral Test to determine the behavior of the series.

(2pts): Let \( f(x) = \frac{1}{x \cdot \ln(x)} \). Then, \( f \) is decreasing, integrable, and agrees with the sequence that we are summing. Thus, the original sum converges iff \( \int_{x=2}^{\infty} \frac{dx}{x \cdot \ln(x)} \) converges.
(5pts): Now,

\[
\int_{x=2}^{\infty} \frac{dx}{x \cdot \ln(x)} = \lim_{b \to \infty} \left( \int_{x=2}^{b} \frac{dx}{x \cdot \ln(x)} \right)
\]

\[
= \lim_{b \to \infty} (\ln \ln b - \ln \ln 2)
\]

\[
= +\infty, \quad \text{i.e. diverges.}
\]

(1pts): Thus, the original sum diverges.