Exam 4 Problem 2:

In order to find the interval of convergence of
\[ \sum_{n=1}^{\infty} (-1)^n \frac{\ln(n)x^n}{n} , \]
first the radius of convergence is calculated using either the generalized root or ratio test.

Using the ratio test, the power series converges for values of \( x \) such that
\[ \lim_{n \to \infty} \left| \frac{\ln(n+1)x^{n+1}}{\ln(n)x^n} \cdot \frac{n}{n+1} \right| |x| < 1. \]

We consider the limits of sequences separately,
\[ \lim_{n \to \infty} \frac{n}{n+1} = \lim_{n \to \infty} \frac{1}{1+1/n} = 1. \]
Then let \( f(x) = \frac{\ln(x+1)}{\ln(x)} \) and use L’Hopital’s rule and the continuity of \( f \) to conclude that
\[ \lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{1/(x+1)}{1/x} = \lim_{x \to \infty} \frac{x}{x+1} = 1. \]
Thus we get that the radius of convergence is 1. A similar argument using the root test gives the same answer by calculating \( \lim_{n \to \infty} \ln(n)^{1/n} \) by taking the limit of the equivalent real-variable function.

This means that there’s absolute convergence on the interval \(-1 < x < 1\). When \( x = 1 \):
\[ \sum_{n=1}^{\infty} (-1)^n \frac{\ln(n)(1)^n}{n} = \sum_{n=1}^{\infty} (-1)^n \frac{\ln(n)}{n} \]
is an alternating series. \( \frac{\ln(n)}{n} \) is a decreasing sequence since its equivalent real-variable function \( \frac{\ln(x)}{x} \) has a negative derivative \( \frac{1-\ln(x)}{x^2} \) is negative for \( x > e \). Using a comparison, we obtain
\[ 0 \leq \lim_{n \to \infty} \frac{\ln(n)}{n} \leq \lim_{n \to \infty} \frac{2\sqrt{n}}{n} \Rightarrow \lim_{n \to \infty} \frac{\ln(n)}{n} = 0. \]
So by the alternating series test, the series converges conditionally at \( x = 1 \).

When \( x = -1 \):
\[ \sum_{n=1}^{\infty} (-1)^n \frac{\ln(n)(-1)^n}{n} = \sum_{n=1}^{\infty} (-1)^{2n} \frac{\ln(n)}{n} = \sum_{n=1}^{\infty} \frac{\ln(n)}{n} \geq \sum_{n=1}^{\infty} \frac{1}{n} = \infty \]
by the p-series test, hence the series diverges at \( x = 1 \). This also means that the series does not converge absolutely at \( x = -1 \).

Therefore the interval of convergence is \((-1, 1]\).

- 10 points for finding the radius of convergence
  - 2 points for using ratio or root test
  - 2 points for using absolute values where necessary
  - 4 points for correctly evaluating limits in ratio or root test computation, only 2 points without correct justification
  - 2 points for getting that the radius is 1
- 7 points for analysis of series at \( x = 1 \)
  - 4 for correct use of alternating series test: 2 points for showing the sequence decreases, 2 points for showing the sequence converges to 0
  - 3 points for concluding that the convergence is conditional, only 1 point if type of convergence is not specified
• 7 points for analysis of series at $x = -1$
  – 5 points for making a valid comparison to a divergent series or correctly using the integral test
  – 2 points for stating that the series diverges
• 1 point for writing the interval $(-1, 1]$