Midterm 3, Problem 3

November 13, 2013

Problem (25pts): Determine whether \( \int_{0}^{\pi} \cos(x) \frac{dx}{\sqrt{\sin(x)}} \) converges. If so, evaluate.

Solution:

(6pts): Since the integrand is defined on \((0, \pi)\), and

\[
\lim_{x \to 0^+} \left( \cos(x) \frac{1}{\sqrt{\sin(x)}} \right) = +\infty
\]

and

\[
\lim_{x \to \pi^-} \left( \cos(x) \frac{1}{\sqrt{\sin(x)}} \right) = -\infty,
\]

one must break the integral into two improper integrals.

(2pts): So, pick a value \(0 < d < \pi\). Then,

\[
\int_{0}^{\pi} \cos(x) \frac{dx}{\sqrt{\sin(x)}} = \int_{0}^{d} \cos(x) \frac{dx}{\sqrt{\sin(x)}} + \int_{d}^{\pi} \cos(x) \frac{dx}{\sqrt{\sin(x)}},
\]

provided that both of the improper integrals on the right-hand side exist.

Note: I will pick \(d = \frac{\pi}{2}\). Also OK to leave it as a variable.

(8pts): Then,

\[
\int_{0}^{d} \cos(x) \frac{dx}{\sqrt{\sin(x)}} = \lim_{a \to 0^+} \int_{a}^{d} \cos(x) \frac{dx}{\sqrt{\sin(x)}}
\]

\[
= \lim_{a \to 0^+} \left( 2\sqrt{\sin(x)} \right|_{a}^{d}
\]

\[
= 2\sqrt{\sin(d)}
\]

\[
= 2.
\]

(Note: 4 points given for the substitution \(u = \sin(x)\), etc, while evaluating the integral; 4 points given for the remainder of the work.)
Similarly, 
\[
\int_{\pi}^{b} \frac{\cos(x)}{\sqrt{\sin(x)}} \, dx = \lim_{b \to \pi^-} \int_{d}^{b} \frac{\cos(x)}{\sqrt{\sin(x)}} \, dx \\
= \lim_{b \to \pi^-} \left( 2\sqrt{\sin(x)} \right)_{d}^{b} \\
= -2\sqrt{\sin(d)} \\
= -2.
\]
(Note: Points given identically as in the other integral.)

Thus, 
\[
\int_{0}^{\pi} \frac{\cos(x)}{\sqrt{\sin(x)}} \, dx = 2\sqrt{\sin(d)} + \left( -2\sqrt{\sin(d)} \right) = 0.
\]