READ CAREFULLY. WORK ON ALL QUESTIONS. Justify your answers. Cross out what is not meant to be part of your final answer.

1. [10pts] Suppose that the interest in an account is compounded continuously with a rate \( r(s) \), which is changing with the time \( s (s \geq 0) \). Assume that \( r(0) = r_0 \), a fixed nonrandom number.

(a) [2pts] Define the yield curve \( \bar{r}(t), t \geq 0 \), associated with the rate \( r(s) \). Find the limit of \( \bar{r}(t) \) as \( t \) approaches 0 from above (from positive values), i.e., find \( \bar{r}(0) \).

(b) [2pts] Suppose that \( \bar{r}(t) \) is Brownian motion with drift parameter \( \mu \) and variance parameter \( \sigma^2 \). Find the probability \( P(1/2 > \bar{r}(t) > 1/3) \). Note: Provide an answer in terms of the distribution \( \Phi(z) \) for appropriate value(s) of \( z \).

(c) Under the assumptions of part 1(b) about \( \bar{r}(t) \), let \( D(t) \) be the total amount in the account at time \( t \) if the given, fixed nonrandom amount \( D_0 \) is deposited at time 0.

   (c.i) [3pts] Find \( \text{Var}(D(t)) \).

   (c.ii) [3pts] Determine \( P(D(t) > A) \), for a given fixed \( A > D_0 \), in terms of distribution \( \Phi(z) \) for appropriate value(s) of \( z \).

2. [10pts] Let \( P \) be the cost of a European put option that gives one the right to sell a stock at strike price \( K \) at time \( t > 0 \), in the future. Assume that the interest rate \( r \) is compounded continuously. Let \( S \) be the price of the stock today \( (t = 0) \). Show that \( S + P \geq Ke^{-rt} \) if there is no arbitrage opportunity. Note: Give a proof that does NOT use the put-call parity formula. No credit will be given to students who use the put-call parity formula in their proofs.

3. [10pts] For an initial amount \( A \), an investment yields returns \( X_i \) at the end of period \( i \) for \( i = 1, 2, \ldots, n \) where \( n \) is some integer, \( n \geq 1 \). Suppose that the amounts \( X_1, X_2, \ldots, X_n \) are independent, normal random variables, each of which has mean \( \mu \) and variance \( \sigma^2 \). You may assume that \( X_i > 0 \) for \( i = 1, 2, \ldots, n \) with probability nearly equal to one (i.e., \( \mu > 0 \) with \( \mu \gg \sigma > 0 \)).

   Define the rate of return, \( r^* \), for this investment. What is the probability that this \( r^* \) is larger than or equal to a given number \( r \) \((0 < r < 1) \) in the limit \( n \to \infty \)? Beware: You do not need to use the Central Limit Theorem for this problem. Note: Assume that the interest is compounded periodically. Express your answer in terms of the distribution \( \Phi(z) \) where \( z \) should depend on \( A, r, \mu, \) and \( \sigma \). In order that you receive full credit, your final answer must be given in terms of simple enough expressions (and should not involve sums etc). Simplify your answer as much as possible.