1. [15pts] An experiment has three possible outcomes (namely, outcomes $j = 1, 2, 3$) with nonzero unknown probabilities $p_j$, $0 < p_j < 1$.
   (a) [7pts] Suppose there are two different wagers ($i = 1, 2$) with returns (per unit bet) $r_i(j)$:
   $$
   r_1(1) = 4, \quad r_1(2) = 6, \quad r_1(3) = c, \\
   r_2(1) = 6, \quad r_2(2) = 9, \quad r_2(3) = -12.
   $$
   What is the value of $c$ if there is no arbitrage? Are the risk-neutral probabilities $p_j$ determined uniquely or not? Explain from scratch. **Hint:** You are asked to write down all relations satisfied by $p_1, p_2$ and $p_3$; and figure out if you can find one value only for each $p_j$.
   (b) [8pts] Now suppose there are three different wagers ($i = 1, 2, 3$) with returns (per unit bet)
   $$
   r_1(1) = 1, \quad r_1(2) = 2, \quad r_1(3) = -3, \\
   r_2(1) = 3, \quad r_2(2) = -1, \quad r_2(3) = -1, \\
   r_3(1) = -2, \quad r_3(2) = -4, \quad r_3(3) = 6.
   $$
   Show that no arbitrage can occur. What are the risk-neutral probabilities $p_j$? Explain in detail.

2. [15pts] In all parts of this problem, $C_{BS}(s,t,K,\sigma,r)$ denotes the Black-Scholes formula.
   (a) [7pts] Show that $C_{BS}(se^{-ft},t,K,\sigma,r) = e^{-ft}C_{BS}(s,t,K,\sigma,r-f)$, where $f < r$.
   (b) [8pts] Determine the limit of $C_{BS}(s,t,K,\sigma,r)$ as $t \to \infty$. Explain by doing the math.

3. [20pts] The price $S(t)$ of a security follows a geometric Brownian motion, where $S(0) = s$. Consider a call option to buy the security with strike price $K = s$ at time $T > 0$. Assume that the geometric Brownian motion is risk neutral. The volatility parameter of $S(t)$ is $\sigma$ and the interest rate (compounded continuously) is $r = \sigma^2/2$.
   (a) [5pts] What is the probability that the above call option will be exercised (at time $t = T$)?
   (b) [5pts] What is the expected no-arbitrage present cost $C$ (at $t = 0$) of the call option?
   (c) [4pts] Compute $\partial C/\partial s$ for fixed $\sigma$ and $T$. (Note that $C$ is defined in 3(b).)
   (d) [6pts] Compute $\partial C/\partial \sigma$ for fixed $s$ and $T$. (Note that $C$ is defined in 3(b).)

**Note:** In all parts [3(a),(b),(c),(d)] of this problem, your answers should involve value(s) of $\Phi(z)$ and may depend on $s$, $T$ and $\sigma$. If you use the Black-Scholes formula, you may use it without proving it. However, any subsequent result, e.g., derivatives of the no-arbitrage cost, must be derived in great detail. In all calculations or derivations, show all your steps (for full credit).