READ CAREFULLY. WORK ON ALL QUESTIONS. Justify your answers. Cross out what is not meant to be part of your final answer.

1. [10pts] An experiment has three possible outcomes (namely, outcomes $j = 1, 2, 3$) with nonzero unknown probabilities $p_j$, where $0 < p_j < 1$.

Suppose that there are three different wagers ($i = 1, 2, 3$) with returns (per unit bet) $r_i(j)$:

- $r_1(1) = 1, \quad r_1(2) = 2, \quad r_1(3) = -3,$
- $r_2(1) = \alpha, \quad r_2(2) = -1, \quad r_2(3) = 1,$
- $r_3(1) = -8, \quad r_3(2) = -16, \quad r_3(3) = 24.$

where $\alpha > 0$.

Show that no arbitrage can occur. What are the risk-neutral probabilities $p_j$? (Note: Each $p_j$ should be determined as a function of $\alpha$). What happens to each $p_j$ as $\alpha$ becomes larger and larger?

2. [10pts] In all parts of this problem, suppose that the non-arbitrage cost of a call option $(K, t)$ is given by the Black-Scholes formula, $C_{BS}(s, t, K, \sigma, r)$. By doing the algebra carefully, give detailed answers to the following questions.

What should be the non-arbitrage cost of the call option if:

(a) [3pts] The strike price approaches zero?

(b) [3pts] The exercise time becomes larger and larger, i.e., it approaches infinity?

(c) [4pts] The volatility becomes larger and larger?

3. [10pts] The price $S(t)$ of a traded security follows a geometric Brownian motion with volatility $\sigma$. A brokerage firm offers, at cost $C$ (at $t = 0$), an investment that will pay amount $A$ at $t = 1$ if the security price at $t = 1$ increases by at least 100% percent in comparison to its initial value. Assume that the compounded continuously interest is $r = \sigma^2$. Suppose that this investment does not give rise to arbitrage. Note: The drift of $S(t)$ should be risk-neutral.

(a) [6pts] Find a relation among $y$, $C$, $A$ and $\sigma$.

(b) [4pts] By using the result of part 3(a) above, derive a formula for the derivative $\partial C/\partial y$ for fixed $A$ and $\sigma$. Does $C$ increase or decrease with $y$? Explain.