6.2 Multiperiod Binomial Model

The purpose of this lecture is to introduce a discrete model for the application of the Arbitrage Theorem to stocks and call options.

Consider the price $S(i)$ of a stock in the end of period $i$; $i = 1, 2, \ldots, n$. The initial price $S(0)$ is non-random and fixed. The stock price $S(i)$ is random, going up or down by probabilistic rules.

The evolution of the stock price can be considered to be an experiment. An outcome of this experiment is the set of values

$$\{S(1) = s_1, S(2) = s_2, \ldots, S(i) = s_i, \ldots, S(n) = s_n\}$$

for specific values $s_1, s_2, \ldots, s_n$.

To express the random "up's and down's" of $S(i)$, consider the RV:

$$X_i = \begin{cases} 1, & \text{if } S(i) \text{ goes up} \\ 0, & \text{if } S(i) \text{ goes down} \end{cases}$$

An outcome of the above experiment is the set

$$\{X_1 = x_1, X_2 = x_2, \ldots, X_i = x_i, \ldots, X_n = x_n\}$$

for specific values of $x_1, x_2, \ldots, x_n$ which are 0's and 1's.
Suppose one wants to purchase a call option \((K,t=n)\) at \(t=0\)
in order to buy stock after \(n\) periods; the interest rate
is \(r\) per period, compounded periodically.

Questions:

1) What is \(P(X_1=I_1, \ldots, X_n=I_n)\) so that all "bets" are fair?

2) What should the cost of the call option be so that there is
no arbitrage?

We are going to answer 1) and 2) via defining a "bet" for period \(i\).

Type of bet for fixed \(i\) : Pick an \(i\); \(i=1,2,\ldots,n\).

- 0's or 1's

Pick a set of values \(\{I_1, I_2, \ldots, I_{i-1}\}\); and observe the actual
prices of stock \(\{S(1), S(2), \ldots, S(i-1)\}\) up to period \(i\).

Bet (wager) associated with \(i\) : If \(X_k = I_k\) for each \(k=1,2,\ldots,i-1\),
then: Buy stock at \(t=i-1\); sell stock at \(t=i\).

(Otherwise: Do nothing)

Define
\[
p = P(X_i = 1 / X_1 = I_1, X_2 = I_2, \ldots, X_{i-1} = I_{i-1}) \quad \text{conditional probability}
\]

We want to find \(p\) (for fixed \(i\)) so that all bets are fair.

Let \(\alpha = P(X_1 = I_1, X_2 = I_2, \ldots, X_{i-1} = I_{i-1})\) for given \(I_k\) \((k=1,\ldots,i-1)\).
Expected profit (gain) at \( t = i-1 \):

\[
G(i-1) = \alpha \cdot \left[ \text{Expected gain if one applies bet at } t = i-1 \right] - \text{cost at } t = i-1
\]

\[
= \alpha \cdot \left[ \frac{\text{Present value at } t = i-1 \text{ of purchase}}{S(i) \text{ if up}} - \frac{S(i-1)}{S(i) \text{ if down}} \right]
\]

\[
= \alpha \cdot \left[ p \frac{u S(i-1)}{1+r} + (1-p) \frac{d S(i-1)}{1+r} - S(i-1) \right]
\]

\[
= \alpha \left[ \frac{pu}{1+r} + \frac{(1-p)d}{1+r} - 1 \right] S(i-1)
\]

The bet is fair if

\[
G(i-1) = 0
\]

\[
\Rightarrow \frac{pu}{1+r} + \frac{(1-p)d}{1+r} - 1 = 0 \quad \Rightarrow \quad p = \frac{1+r-d}{u-d} \quad (u>d)
\]

Notice: \( 0 < p < 1 \iff d < 1+r < u \).

This conditional probability is independent of \( i \) and the values \( \xi_k \).

Thus, the important conclusion is reached that:

\( X_k \) must all be independent RV's.

Recall that \( X_i \) is Bernoulli RV, with parameter \( p \).

Answer to question (2): All bets are fair if \( p = \frac{1+r-d}{u-d} \); \( X_i \)'s: indep.
Now it is time to address question (1):

What is the cost (at $t=0$) of a call option $(K, t=n)$ so that there is no arbitrage?

For no arbitrage, we must assume that all bets are fair: $p = \frac{1+r-d}{u-d}$, and $X_i$'s are independent Bernoulli RV's with parameter $p$.

Next, we need to determine the statistics of $S(t)$.

Notice that $S(i) = u^{X_i} \, d^{1-X_i} \, S(i-1)$ 

[ $X_i$ = 1 if "up"; $X_i = 0$ if "down" ]

Thus:

$S(1) = u^{X_1} \, d^{1-X_1} \, S(0)$

$S(2) = u^{X_2} \, d^{1-X_2} \, S(1)$

Thus: $S(n) = u^{X_n} \, d^{1-X_n} \, S(n-1)$

Let $Y = \sum_{i=1}^{n} X_i$; this is a binomial RV with parameters $(n, p)$

Thus: $S(n) = u^Y \, d^{n-Y} \, S(0)$

where $Y$ is a known (binomial) RV.

The cost of the call option $(K, t=n)$ at $t=0$ must be:

$$C = \mathbb{E} \left[ \text{PV (Payoff of call at } t=n) \right]$$

$$\Rightarrow C = \mathbb{E} \left[ (1+r)^{-n} (S(n)-K)^+ \right] = (1+r)^{-n} \mathbb{E} \left[ (u^Y d^{n-Y} S(0)-K)^+ \right]$$

For no arbitrage on average