Answer any four of the five problems. Clearly indicate which four you want graded.
To save writing, you may use $J_{k,c}$ to represent a $k \times k$ Jordan block with $c$ on the diagonal. Give sufficient reasons for your answers.

1. (25) Let $A = \begin{bmatrix} 0 & 2 & 3 & 7 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 4-i \\ 0 & 0 & 0 & 3+i \end{bmatrix}$.
   a) Find the characteristic polynomial of $A$.
   Answer: $x^3(x - 3 - i)$
   b) Find the minimal polynomial of $A$.
   Answer: Since the minimal polynomial divides the characteristic polynomial and has the same factors, it is either $x(x - 3 - i)$ or $x^2(x - 3 - i)$ or $x^3(x - 3 - i)$. Calculate $A(A - (3 + i)I)$ and see if it is nonzero, then calculate $A^2(A - (3 + i)I)$ which is 0. So the minimal polynomial is $x^2(x - 3 - i)$.
   c) Find the Jordan form of $A$.
   Answer: Since the minimal polynomial is $x^2(x - 3 - i)$ one of the Jordan blocks is $J_{2,0}$ and another is $J_{1,3+i}$. There is only room for one more $1 \times 1$ block $J_{1,0}$. So the Jordan form of $A$ has three Jordan blocks, $J_{2,0}$, $J_{1,0}$, and $J_{1,3+i}$. In other words, it is all zeros except for a 1 in row 2 column 1 and a 3 + i in the fourth row fourth column. If you wanted to find a basis $B = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ so that $[A]_B$ is in this Jordan form, you would take $\alpha_1$ to be any vector in $NS(A^2 - NS(A)$, for example $\alpha_1 = \epsilon_2$. Then let $\alpha_2 = A\alpha_1 = 2\epsilon_1$. Then let $\alpha_3$ be any vector in $NS(A)$ which is not a multiple of $\alpha_2$, for example $\alpha_3 = 3\epsilon_2 - 2\epsilon_3$. Finally $\alpha_4$ is any nonzero vector in $NS(A - (3 + i)I)$ for example $\alpha_4 = 3 + 4\epsilon_1$. Then if $P$ is the matrix $P = [\alpha_1 \alpha_2 \alpha_3 \alpha_4]$ we have $P^{-1}AP$ in Jordan form.

2. (25) Let $N$ be a nilpotent $3 \times 3$ real matrix.
   a) Show that $(I + N)^{-1} = 1 - N + N^2$.
   Answer: $(I + N)(I - N + N^2) = I - N + N^2 + N - N^2 + N^3 = I + N^3$. But $N^3 = 0$ by the Cayley-Hamilton theorem since $N$ is nilpotent. So $(I + N)(I - N + N^2) = I$ which implies $(I + N)^{-1} = 1 - N + N^2$.
   b) Suppose $N^3 + 3N^2 + N \neq 0$ and $N^7 - 5N^6 + 4N^3 - N^2 = 0$. What is the minimal polynomial of $N$?
   Answer: Since $N^3 = 0$ we see that $3N^2 + N \neq 0$ and $-N^2 = 0$. So $N^2 = 0$ but $N \neq 0$ which means the minimal polynomial of $N$ is $x^2$.

3. (25) Let $V \subset \mathbb{R}[x]$ be the polynomials of degree 2 or less, let $T: V \to V$ be the linear operator $T(p(x)) = (x + 1)p'(x)$, so $T(x^2) = 2x(x + 1)$, $T(x) = x + 1$, and $T(1) = 0$.
   a) Find the minimal polynomial of $T$.
   Answer: The matrix of $T$ with respect to the standard basis $\{1, x, x^2\}$ is $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$. So the characteristic polynomial of $T$ is $x(x - 1)(x - 2)$. Since the minimal polynomial
always has the same roots and not larger exponents, the minimal polynomial must also be \( x(x - 1)(x - 2) \).

b) Does \( T \) have a Jordan form? If so, find it. If not, explain why not.
Answer: Yes, since the minimal polynomial is a product of linear factors. In fact, since all exponents are 1 we know \( T \) is diagonalizable so the Jordan form is diagonal with 0, 1, 2 on the diagonal. A basis diagonalizing \( T \) is \( \{1, x + 1, (x + 1)^2\} \).

4. (25) Answer four of the following six short questions by either finding the requested matrices or subspaces, or showing they do not exist.

a) Find a \( 4 \times 4 \) complex matrix which is not diagonalizable, and write down its minimal and characteristic polynomials.
Answer: For example \( J_{4,0} \) with characteristic and minimal polynomials both \( x^4 \). What you need is that some linear factor in the minimal polynomial have exponent > 1.

b) Find a \( 4 \times 4 \) real matrix which is not triangulable, and write down its minimal and characteristic polynomials.
Answer: The minimal and characteristic polynomials cannot be products of linear factors so they need to be divisible by a polynomial without a real root, for example \( x^2 + 1 \). So for example if \( B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \) is rotation by 90 degrees, then \( B^2 \) is rotation by 180 degrees, so \( B^2 + I = 0 \). So we could take the block matrix \( A = \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix} \). You can calculate the characteristic polynomial as \( (x^2 + 1)x^2 \). Since \( (A^2 + I)A = 0 \) the minimal polynomial is \( (x^2 + 1)x \).

c) Find a \( 4 \times 4 \) complex matrix which is not triangulable, and write down its minimal and characteristic polynomials.
Answer: Impossible, every complex matrix is triangulable since the minimal polynomial can be written as a product of linear factors.

d) Find a \( 3 \times 3 \) matrix \( A \) and a subspace \( W \subset \mathbb{R}^3 \) and an \( \alpha \in \mathbb{R}^3 \) so that \( S_A(\alpha; W) \) is not an ideal. (Recall \( S_A(\alpha; W) = \{p \in \mathbb{R}[x] \mid p(A)\alpha = 0\} \).)
Answer: We know \( W \) cannot be invariant under \( A \). So let's just try a random example and see if it works. Let \( W \) be the span of \( \epsilon_1 \) and take any \( A \) so that \( Ae_1 \) is not a multiple of \( \epsilon_1 \), say \( Ae_1 = \epsilon_2 \), \( Ae_2 = Ae_3 = 0 \). Note \( A^2 = 0 \) so computations will be easy. Then if \( p(x) = d + ex + \cdots \) is any polynomial, \( p(A)(a, b, c) = d(a, b, c) + e(0, a, 0) \) so \( p(A)(a, b, c) \in W \) if and only if \( cd = 0 \) and \( bd + ae = 0 \). So if we let \( \alpha = (1, 0, 0) \) for example then \( S_A(\alpha; W) \) is the set of all polynomials whose \( x \) coefficient is 0. This is not an ideal since for example \( 1 \in S_A(\alpha; W) \) but \( x \cdot 1 \not\in S_A(\alpha; W) \).

e) Find an upper triangular real matrix which is not similar to a real matrix in Jordan form.
Answer: Impossible, the characteristic polynomial is a product of linear factors, so the matrix is similar to a matrix in Jordan form.

f) Let \( A = J_{3,0} \). Find all subspaces \( W \subset \mathbb{R}^3 \) invariant under \( A \).
Answer: There are very few invariant subspaces. If \( (a, b, c) \in W \) then \( A(a, b, c) = (0, a, b) \in W \) and \( A^2(a, b, c) = (0, 0, a) \in W \). So if \( a \neq 0 \) then \( W = \mathbb{R}^3 \). So suppose \( W \neq \mathbb{R}^3 \). Then \( a = 0 \) so \( W \) must be contained in the span of \( \epsilon_2, \epsilon_3 \). If \( b \neq 0 \) then \( (0, b, c) \in W \) and \( (0, 0, b) \in W \) so \( W \) is the span of \( \epsilon_2, \epsilon_3 \). So suppose \( W \) is neither \( \mathbb{R}^3 \) nor the span of \( \epsilon_2, \epsilon_3 \).
Then $a = b = 0$ and if $c \neq 0$ then $W$ is the span of $\epsilon_1$, otherwise $W = 0$. So there are only 4 subspaces invariant under $A$, namely $\mathbb{R}^3$, 0, the span of $\epsilon_2, \epsilon_3$ and the span of $\epsilon_3$.

5. (25) Let $p(x) = x^2(x^2 + 4)(x - 1)(x + 3)^3$, $q(x) = (x + 5)^3(x + 1)^4(x + 3)^4$, and $r(x) = x^2(x + 1)(x + 3)^2$. Suppose $A$ is a $5 \times 5$ complex matrix and $p$ and $q$ are both in the annihilating ideal of $A$, so $p(A) = 0$ and $q(A) = 0$. Suppose also that $r(A) \neq 0$.

a) What are all possible characteristic polynomials of $A$?
Answer: The minimal polynomial of $A$ must divide both $p$ and $q$ so it must be some $(x + 3)^k$ with $k \leq 3$. Since the characteristic polynomial has the same roots and has degree 5 it must be $(x + 3)^5$.

b) What are all possible minimal polynomials of $A$?
Answer: The minimal polynomial $(x + 3)^k$ does not divide $r$ so $k = 3$. So the minimal polynomial is $(x + 3)^3$.

c) What are all possible Jordan forms of $A$?
Answer: There must be a $3 \times 3$ block $J_{3,-3}$ so the only possibilities are:
- One $J_{3,-3}$ and one $J_{2,-3}$
- One $J_{3,-3}$ and two $J_{1,-3}$