Least squares solutions

You can find the least squares solution to $Ax = b$ using the methods in Lay, solving the normal equation $A^T Ax = A^T b$. But Matlab also finds a least squares solution for you automatically. If $A$ is not a square matrix then $A \backslash b$ always finds a least squares solution to $Ax = b$. Matlab may complain if the columns of $A$ are not linearly independent, but generally seems to come up with a good answer nevertheless. However, some care must be taken if $A$ is square and not invertible since then $A \backslash b$ will not determine the least squares solution. You can get around this by adding a last row of zeroes to $A$ and $b$ to fool Matlab. For example if $A$ is $6 \times 6$ you could do $x = [A; \text{zeros}(1,6)] \backslash [b; 0]$. Then $x$ will be a least squares solution to $Ax = b$.

Singular value decomposition

The command $[U S V] = \text{svd}(A)$ finds the singular value decomposition of $A$. So if $A$ is $m \times n$ then $U$ and $V$ are unitary matrices (which means orthogonal in case $A$ is real), $A = USV^*$, and $S$ is a diagonal $m \times n$ matrix (that is, if $A$ has rank $r$ then $S$ has an $r \times r$ diagonal matrix in its upper left corner and the rest of $S$ is zero). The diagonal entries of $S$ are real and nonnegative, in decreasing order.

Problems due May 10

For all the following problems let $A$ be a random $4 \times 4$ complex matrix with rank 2 and nonzero imaginary part. Check that its rank is in fact 2 before proceeding.

**Problem 1:** Generate a random vector $b$ in $\mathbb{C}^4$. Find a least squares solution $\hat{x}$ to $Ax = b$. You could use the trick given above if you wanted or any other way you prefer. Compute the error vector $b - A\hat{x}$. Check that this error vector is perpendicular to the column space of $A$. The error is the length $||b - A\hat{x}||$. Check that the error is minimized for the least squares solution by computing the quantity $||b - Ax||$ for several random vectors $x$ and seeing that it is larger than than the error.

**Problem 2:** Find the singular value decomposition of $A$. Also compute $\text{orth}(A)$ and $\text{null}(A)$. Based on your results, guess how Matlab computes $\text{orth}(A)$ and $\text{null}(A)$.

**Problem 3:** Find the pseudoinverse $A^+$ of $A$ (see page 480). Calculate $A^+b$ (using the same $b$ as in problem 1). Exercise 13 on page 492 says that $A^+b$ should be the least squares solution to $Ax = b$ of smallest length. Check that $A^+b$ is a least squares solution by seeing whether its image is the same as $A(\hat{x})$. Compare the length of $A^+b$ with the length of your solution $\hat{x}$ from problem 1. Determine whether or not the length is smaller.