(1)
(a) \(1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = 32\)
(b) \((-1 - 0) \vec{i} - (-1 - 1) \vec{j} + (0 - 1) \vec{k} = (-1, 2, -1)\)
(c) \(|c\vec{v}| = |c||\vec{v}| \) so \(|-2\vec{v}| = |-2||\vec{v}| = 2 \cdot 5 = 10\).
(d) \(\vec{n} = \vec{v}/|\vec{v}| = (-1, 2, -3)/\sqrt{15}\)

(2)
(a) \(\vec{v} = \vec{v}_1 \times \vec{v}_2 = (2, -1, -1)\), which is easily seen to be perpendicular to \(\vec{v}_1\) and \(\vec{v}_2\).
(b) The plane is \((x, y, z) \cdot \vec{v} = 0\), i.e. \(2x - y - z = 0\).
(c) The translated plane has the same normal vector, but the equation is not homogeneous. That is \(P'\) is given by \((x, y, z) \cdot \vec{v} = C\) for some \(C\). What is \(C\)? Well, \((2, 3, 5)\) has to be a solution, so \((2, 3, 5) \cdot (2, -1, -1) = C\), i.e. \(C = -4\). So \(P'\) is given by \(2x - y - z = -4\).

(3) The determinant is \((2-a)*3\), which can be seen by any of the formulas, for example expanding via the first row gives \(2*3 - a*3 = (2-a)*3\). The matrix is invertible if and only if the determinant is nonzero. The determinant is zero if and only if \(a = 2\).

Take \(a = 2\): \(
\begin{pmatrix}
1 & 1 & 0 \\
2 & 2 & 0 \\
0 & 0 & 3
\end{pmatrix}
\)
The reduced form is \(M = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\), which has null space \((x, y, z)\) with \(x + y = 0\) and \(z = 0\). This is one-dimensional, with basis \((1, -1, 0)\) (or any nonzero multiple of this).

(4)
\(2x - y = -1\)
\(x - y + 2z = 0\)

(a) The matrix equation is
\[
\begin{pmatrix}
2 & -1 & 0 \\
1 & -1 & 2
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
= \begin{pmatrix} -1 \\ 0 \end{pmatrix}
\]

To get reduced form add \(-2\) times the second row to the first:
\[
\begin{pmatrix}
0 & 1 & -4 \\
1 & -1 & 2
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
= \begin{pmatrix} -1 \\ 0 \end{pmatrix}
\]

Now add the first row to the second:
\[
\begin{pmatrix}
0 & 1 & -4 \\
1 & 0 & -2
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
= \begin{pmatrix} -1 \\ -1 \end{pmatrix}
\]
Now the corresponding homogenous equation is
\[
\begin{pmatrix}
0 & 1 & -4 \\
1 & 0 & -2
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\] or \( y - 4z = 0, x - 2z = 0 \). So \( z \) is a free variable, and the general solution is \((2z, 4z, z)\), or all multiples of \((2, 4, 1)\).

(b) The nonhomogeneous equation is \( y - 4z = -1 \) and \( x - 2z = -1 \). Take \( z = 0 \) so that \( x = y = -1 \). The particular solution is \((-1, -1, 0)\).

(c) The general solution is \( \{(−1,−1,0) + t(2,4,1)\} \) with \( t \in \mathbb{R} \).

(5) The eigenvalues are given by \( \det \begin{pmatrix}
2 & -\lambda \\
0 & 2 - \lambda
\end{pmatrix} = 0 \). The determinant is \( (2 - \lambda)^2 \), so \( (2 - \lambda)^2 = 0 \) has the unique solution 2. The only eigenvalue is 2.

Plug in 2, to give \( M = \begin{pmatrix}
2 & -2 \\
0 & 2 - 2
\end{pmatrix} = \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix} \). The null space of this matrix is \( 0x + y = 0 \) and \( 0 = 0 \), i.e. \( y = 0 \), i.e. \( \{(x,0)\} \). This is one-dimensional, all multiples of \((1,0)\). These are the only eigenvectors. (In this case \( \mathbb{R}^2 \) is not spanned by eigenvectors for the matrix.)

(6)
(1) What does it mean for \( S \) to be linearly independent? Suppose \( a_1 f(\vec{v}_1) + a_2 f(\vec{v}_2) + \ldots + a_n f(\vec{v}_n) = 0 \). To show that \( S \) is linearly independent we have to show this only happens if \( a_1 = a_2 = \cdots = a_n = 0 \). Let’s see, applying linearity of \( f \) twice gives:
\[
0 = a_1 f(\vec{v}_1) + a_2 f(\vec{v}_2) + \cdots + a_n f(\vec{v}_n)
= f(a_1 \vec{v}_1) + f(a_2 \vec{v}_2) + \cdots + f(a_n \vec{v}_n)
= f(a_1 \vec{v}_1 + a_2 \vec{v}_2 + \cdots + a_n \vec{v}_n).
\]
Now \( f \) is injective, so \( f(\vec{v}) = 0 \) implies \( \vec{v} = 0 \). So \( a_1 \vec{v}_1 + a_2 \vec{v}_2 + \cdots + a_n \vec{v}_n = 0 \). But \( \{\vec{v}_1, \ldots, \vec{v}_n\} \) is linearly independent. This implies \( a_1 = a_2 = \cdots = a_n = 0 \) as required.

(2) Since \( \{\vec{v}_1, \ldots, \vec{v}_n\} \) is a basis of \( V \), \( \dim(V) = n \). But \( S \) is a basis of \( W \), and has \( n \) elements, so \( \dim(W) = n \) also.

(3) As in (2) \( |S| = n = \dim(V) < \dim(W) \). So \( S \) can’t be a basis of \( W \).