Math 340, Jeffrey Adams
Test I, November 19, 2010
For complete credit you must show all work

Question 1 [15 points] Let \( f(x, y) = \left( \frac{2xy}{x^2 - y^2} \right) \).

(a) \( f'(x, y) = \left( \frac{2x}{2x} \frac{2y}{-2y} \right) \)

(b) The determinant is \(-4y^2 - 4x^2 = -4(x^2 + y^2)\). This equals 0 only if \( x = 0, y = 0 \), so the only point is \((0, 0)\).

Question 2 [10]

(a) The gradient is \((2x, 2y, -4z)\), which at \((1, 1, 1)\) gives \( \vec{v} = (2, 2, -4) \).

The gradient points in the direction of maximum increase; when you have a level set, this is perpendicular to the surface. If you take something perpendicular to \( \vec{v} \) you are getting a tangent vector.

(b) The equation is \(((x, y, z) - (1, 1, 1)) \cdot \vec{v} = 0\), which is \(2(x - 1) + 2(y - 1) - 4(z - 1) = 0\), which simplifies to \(x + y - 2z = 0\).

Question 3 [20] The region is under a parabola, and over the \( x \)-axis. To find extremal points in the interior interior, \( \nabla f(x, y) = (4x, 1 + 2y) \), which equals 0 only at \((0, -\frac{1}{2})\) which is not in the region. So the only max/min are on the boundary.

Now do two Lagrange interpolation problems, one for each component of the boundary.

Let \( G(x, y) = x^2 + y - 1 \). Then \( \nabla f - \lambda \nabla G = (4x + 2\lambda x, 1 + 2y + \lambda) \). Setting this equal to 0 gives \( x = 0 \), or \( \lambda = -2 \). If \( x = 0 \) then \( y = 1 \). Otherwise \( \lambda = -2 \) gives \( y = \frac{1}{2} \), and then \( x = \pm \frac{\sqrt{2}}{2} \).

If \( y = 0 \), just look at \( f(x, y) = f(x, 0) = 2x^2 \), which has a minimum at \( x = 0, y = 0 \).

You should consider the ”corners” separately, i.e. \((\pm 1, 0)\). So the points to check are: \((0, 1), (\pm 1, 0), (\pm \frac{\sqrt{2}}{2}, \frac{1}{2})\).

Checking these points give \( f(0, 1) = f(\pm 1, 0) = 2 \) is a max, and \( f(0, 0) = 0 \) is a min.

Question 4 [15] This is an ellipse. \( f'(t) = (-2\sin(t), 3\cos(t)) \), and \(|f'(t)| = \sqrt{4\sin^2(t) + 9\cos^2(t)} \), so the integral is \( \int_0^{2\pi} \sqrt{4\sin^2(t) + 9\cos^2(t)} \). (By the way the corresponding indefinite integral is an elliptic integral which cannot be evaluated in terms of elementary functions.)
Question 5

The matrix equation is:

\[
\begin{pmatrix}
x_s & x_t \\
y_s & y_t \\
\end{pmatrix} = - \begin{pmatrix}
s^3 & - \sin(\pi(y + t))
\end{pmatrix}^{-1} \begin{pmatrix}
3x_s^2 & - \sin(\pi(y + t))
\end{pmatrix}
\]

where you need to first plug in \((x, y, s, t) = (2, 1, 1, 1)\) in the matrices on the right, i.e.

\[
\begin{pmatrix}
x_s & x_t \\
y_s & y_t \\
\end{pmatrix} = - \begin{pmatrix}
1 & 0 \\
1 & 2 \\
\end{pmatrix}^{-1} \begin{pmatrix}
6 & 0 \\
2 & 2 \\
\end{pmatrix}
\]

which gives

\[
\begin{pmatrix}
x_s & x_t \\
y_s & y_t \\
\end{pmatrix} = - \frac{1}{2} \begin{pmatrix}
2 & 0 \\
-1 & 1 \\
\end{pmatrix} \begin{pmatrix}
6 & 0 \\
2 & 2 \\
\end{pmatrix} = \begin{pmatrix}
0 & 6 \\
2 & 1 \\
\end{pmatrix}
\]

So \(x_s = -6, y_s = 2, x_t = 0, y_t = -1\).

Question 6

(a) closed
(b) neither
(c) closed
(d) open