Math 463, Jeffrey Adams
Solutions to selected problems in Homework set 3

1. page 53, #9. Note that \(|\sin(x)| \leq 1\), and \(\lim_{x \to 0} \sin(x)\) does not exist. So: \(|g(z)| \leq M\) does not imply \(\lim_{z \to z_0} g(z)\) exists. You **cannot** apply Theorem 2, page 47; the assumption that both limits exist doesn’t hold.

So: given \(\epsilon\), there exists \(\delta\) such that \(|z - z_0| < \delta\) implies \(|f(z) - f(z_0)| < \epsilon \frac{1}{M}\). Then \(|z - z_0| < \delta\) implies \(|f(z)g(z) - 0| = |f(z)g(z)| = |f(z)||g(z)| \leq \epsilon \frac{1}{M} M = \epsilon\). This proves the limit is 0.

2. page 59, #7. It is not enough to take the formula \(\frac{d}{dz} z^n = n z^{n-1}\) for \(n > 0\) and let \(m = -n\). You are trying to prove that this formal procedure is legitimate. You need to use the quotient rule. Suppose \(n > 0\):

\[
\frac{d}{dz} z^n = \frac{0 z^n - n z^{n-1}}{z^{2n}} = -n z^{n-2} = -n z^{n+1}.
\]

Therefore \(\frac{d}{dz} z^{-n} = (-n) z^{(-n)+1} = (-n) z^{-(n-1)}\). Letting \(m = -n\), so \(m < 0\) gives \(\frac{d}{dz} z^m = m z^{m-1}\).

3. #9. The derivative is easily seen to be \(\lim_{z \to 0} \frac{z^2}{z^2} = 1\). Take the limit along the ray \(z = re^{i\theta}\) with \(\theta\) fixed. This becomes \(\lim_{r \to 0} r e^{-2i\theta} r e^{i\theta} = \lim_{r \to 0} e^{-i\theta}\). This can take on any value of absolute value 1, depending on \(\theta\), so the limit doesn’t exist.

4. page 68, #3.

(a) (a) Note that \(
\frac{1}{z} = \frac{1}{r} \frac{r}{z} = \frac{\overline{z}}{|z|^2} = \frac{x}{x^2+y^2} - i \frac{y}{x^2+y^2},
\)
so \(u(x, y) = \frac{x}{x^2+y^2}\) and \(v(x, y) = -\frac{y}{x^2+y^2}\).

This is easier in polar coordinates: \(\frac{1}{z} = \frac{1}{r} e^{-i\theta} = \frac{1}{r} \cos(\theta) - i \frac{1}{r} \sin(\theta)\).

Therefore \(u(r, \theta) = \frac{1}{r} \cos(\theta)\) and \(v(r, \theta) = -\frac{1}{r} \sin(\theta)\).

(b) (b) This is differential if and only if \(x = y\).

(c) (c) Note that \(z - \overline{z} = 2i Im(z)\) so \(Im(z) = -\frac{1}{2}(z - \overline{z})\) and \(z Im(z) = -\frac{i}{2} z \overline{z} + \frac{i}{2} z \overline{z} = -\frac{i}{2} z \overline{z} + \frac{i}{2} |z|^2\). Since \(z\overline{z}\) is analytic for all \(z\) the given function is differential exactly where \(|z|^2\) is, i.e. 0.

5. #6. Use polar coordinates (even though the exercise asks about \(u_x\), etc., this is equivalent). That is \(f(re^{i\theta}) = \frac{r^2}{r^2} e^{-i\theta} = re^{-3i\theta} = r \cos(3\theta) - ir \sin(3\theta)\). Therefore \(u(r, \theta) = r \cos(3\theta)\) and \(v(r, \theta) = -r \sin(3\theta)\). The Cauchy–Riemann equations give \(r \cos(3\theta) = -3r \cos(3\theta)\) and \(-3r \sin(3\theta) = r \sin(3\theta)\). These hold at 0.