Math 463, Jeffrey Adams

Solutions to selected problems in Homework set 7

Assignment 7, due Monday, November 3 page 188, #1,3,6,7,10; page 198, #2,7 I graded everything, 5 points each, for a total of 35.

1. page 188, #6. Once you have \( f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{4n+2} \) you should use this to show \( f^{(m)}(0) = 0 \) for some \( m \). That is \( f(z) = \sum_{m=0}^{\infty} \frac{f^{(m)}(0)}{m!} z^m \), so
\[
\sum_{m=0}^{\infty} \frac{f^{(m)}(0)}{m!} z^m = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{4n+2}
\]
and we conclude \( f^{(m)}(0) = (-1)^n/(2n+1)! \) if \( m = 2n+2 \), and 0 otherwise. That is \( f^{(m)}(0) = 0 \) unless \( m \) is of the form \( 4n+2 \), i.e. \( m = 2,6,10,\ldots \).

2. #10. Since \( \tanh(z) \) has a pole at \( \pm \frac{\pi}{2} i \) the radius of convergence is \( \frac{\pi}{2} \).
Although \( \tanh(z) = \frac{\sinh(z)}{\cosh(z)} \) you cannot use the Taylor series of sinh and cosh to get one for \( \tanh(z) \). If \( \sinh(z) = \sum a_n z^n \) and \( \cosh(z) = \sum b_n z^n \) then \( \tanh(z) = \sum a_n b_n z^n \) but this doesn’t tell you anything about how to write \( \tanh(z) = c_n z^n \). It certainly isn’t the case that \( c_n = \frac{a_n}{b_n} \) (since when is \( \frac{1}{a+b} = \frac{1}{a} + \frac{1}{b} \)?)

You just have to compute derivatives of \( f(z) = \tanh(z) \); \( f'(z) = \sech^2(z) \), \( f''(z) = 2\sech^2(z) \tanh(z) \), \( f'''(z) = -4\sech(z) \tanh^2(z) - 2\sech^4(z) \). At 0 this gives \( f(0) = 0, f'(0) = 1, f''(0) = 0, f'''(0) = -2 \). Therefore \( \tanh(z) = z - \frac{2}{3} z^3 + \cdots = z - \frac{1}{3} z^3 + \cdots \).

3. page 198, #2 You need to find the Taylor expansion of \( e^z \) about \(-1\), i.e. as a power series in \( z+1 \). There is a cute trick here: \( e^z = \sum \frac{1}{n!} z^n \), so \( e^{z+1} = \sum \frac{1}{n!} (z+1)^n \). For most functions this wouldn’t tell you anything about \( f(0) \), but \( e^{z+1} = e e^z \), so \( e e^z = \sum \frac{1}{n!} z^n \), and \( e^z = \frac{1}{e} \sum \frac{1}{n!} z^n \).
Therefore \( \frac{e^z}{(z+1)^2} = \frac{1}{e} \sum \frac{1}{n!} (z+1)^{n-2} \).

4. #6 These are both Laurent series about \( z_0 = 0 \). The \( \frac{1}{z} \) term is fine as it is. You have to expand \( \frac{1}{1+z^2} \) as a Taylor Laurent series about 0.
If \( |z| < 1 \):
\[
\frac{1}{1+z^2} = \frac{1}{1-(-z^2)} = \sum_{n=0}^{\infty} (-z^2)^n = \sum_{n=0}^{\infty} (-1)^n z^{2n}.
\]
If \( |z| > 1 \):
\[
\frac{1}{1+z^2} = \frac{1}{z^2} \frac{1}{1-\frac{1}{z^2}} = \frac{1}{z^2} \sum_{n=0}^{\infty} (-\frac{1}{z^2})^n = \frac{1}{z^2} \sum_{n=0}^{\infty} (-1)^n z^{-2n}.
\]
The rest of the problem follows easily from this.