Computer Computations in Representation Theory III:
Unitary Representations of real Lie groups
With an appendix on mathematical software

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Recap

In Lecture I we showed that $SL(2, \mathbb{R})$ has interesting infinite dimensional unitary representations. That is we found representations $\pi_\nu$ on a Hilbert space $L^2(\mathbb{R})$ for which the operators $\pi_\nu$ are unitary, i.e. $(\pi_\nu(g)v, \phi_\nu(g)w) = (v, w)$ for all $v, w \in L^2(\mathbb{R})$ and $g \in SL(2, \mathbb{R})$. Here $\nu \in i\mathbb{R}$ (with $(,) \,$ the usual inner product) or $-1 \leq \nu \leq 1$ (with a different inner product).

We are interested in similar representations of other Lie groups such as $GL(n, \mathbb{R}), Sp(2n, \mathbb{R})$. 
and $SO(n, \mathbb{R})$. We are also interested in the exceptional groups $E_6, E_7, E_8, F_4$ and $G_2$.

Each of the groups under discussion contains a subgroup $T$ isomorphic to $\mathbb{R}^* \mathbb{R}$, which gives rise to the root system $R \subset \mathbb{R}^n$ and the Weyl group $W$. Recall $R$ is a finite set of roots satisfying certain conditions. In particular if $\alpha, \beta \in R$ then

$$< \alpha, \beta > = \frac{2(\alpha, \beta)}{(\beta, \beta)} \in \mathbb{Z}$$

and the corresponding reflection

$$\sigma_\alpha(v) = v - < v, \alpha > \alpha$$
takes \( R \) to itself. The Weyl group is the group generated by the reflections \( \sigma_\alpha (\alpha \in R) \). This acts on \( \mathbb{R}^n \).

Recall a set of simple roots \( \Delta \subset R \) is a basis of \( V \) satisfying certain other condition.

We say \( \nu \) is dominant if \( < \nu, \alpha > \geq 0 \) for all \( \alpha \in \Delta \).

**Lemma 6.1** Given \( \nu \) there exists \( w \in W \) such that \( w\nu \) is dominant.
7 Representations

Let $G$ be one of our groups. The diagonal subgroup $T$ is isomorphic to $\mathbb{R}^* \times \cdots \times \mathbb{R}^*$. I’ll assume the number of factors is $n$, i.e. $T \cong \mathbb{R}^n$.

For example take $G$ to be $GL(n, \mathbb{R})$, $Sp(2n, \mathbb{R})$, $SO(2n, \mathbb{R})$ or $SO(2n + 1, \mathbb{R})$.

Note that if $\nu = (z_1, \ldots, z_n) \in \mathbb{C}^n$ then $\nu$ gives a one–dimensional representation $\nu$ of $T \cong \mathbb{R}^n$ by

$$\nu(x_1, \ldots, x_n) = |x_1|^{z_1} \cdots |x_n|^{z_n}$$

Lemma 7.1 For every $\nu \in \mathbb{C}^n$ there is an
irreducible representation, denoted \( \pi_\nu \), of \( G \).

It is unitary if (but not only if!) \( \nu \in i\mathbb{R}^n \).

Finally, \( \pi_\nu \simeq \pi_{w\nu} \) for all \( w \in W \).

By the last statement we may as well assume \( \nu \) is dominant.

The representation \( \pi_\nu \) is on a space of functions \( \mathcal{F} \) on \( G \). It is an example of an induced representation. In the case of \( SL(2, \mathbb{R}) \) these are the representations of Lecture I.

Recall in the case of \( SL(2, \mathbb{R}) \) \( \pi_\nu \) is unitary if \( \nu \in i\mathbb{R} \) or \( -1 \leq \nu \leq 1 \). The interesting and
hard case is the latter one.

From now on we assume $\nu$ is real, i.e. $\nu \in \mathbb{R}^n$.

In our setting we are interested in the following problem:

**Problem:** For which $\nu \in \mathbb{R}^n$ is $\pi_\nu$ unitary?

Recall the Weyl group $W$ acts on $\mathbb{R}^n$, and $w_0$ is the longest element of $W$.

**Lemma 7.2** If $\pi_\nu$ is unitary then $w_0 \nu = -\nu$.

In some cases $w_0 = -1$, in which case this is no condition.
8 A calculation in $W$

We now give a formal construction in the Weyl group which will answer the question posed in the previous section.

Let $V = \mathbb{R}^n$, with the standard inner product $(v, w) = v \cdot w$. Suppose $R \subset \mathbb{R}^n$ is a root system, and $W$ is the Weyl group of $R$. Pick a set $\Delta$ of simple roots. Write

$$w_0 = s_N s_{N-1} \ldots s_1$$

where $s_i = s_\alpha$ for some $\alpha \in \Delta$. Here $N = \frac{1}{2}|N|$.

Fix a representation $\tau$ of $W$ of dimension $m$. 
Definition 8.1 Suppose $\nu \in V$.

(1) For $\alpha \in \Delta$ define

$$A_{\pi,\alpha}(\nu) = \pi(1) + \langle \nu, \alpha \rangle \pi(s_\alpha)$$

(2) Define

$$A_{\pi}(\nu) = A_{\alpha_N}(s_{\alpha_{N-1}} \ldots s_{\alpha_1} \nu) \ldots A_{\alpha_2}(s_{\alpha_1} \nu) A_{\alpha_1}(\nu)$$

Thus $A(\nu)$ is an $m \times m$ matrix. Note that $\pi(s_\alpha) = diag(\pm 1, \ldots, \pm 1)$ in some basis, so

$$A_{\pi,\alpha} = \begin{pmatrix} 1 \pm \langle \nu, \alpha \rangle \\ \vdots \\ 1 \pm \langle \nu, \alpha \rangle \end{pmatrix}$$

In particular $A_{\pi,\alpha}$ is invertible if $\langle \nu, \alpha \rangle \neq \pm 1$. 
Example: $Sp(4, \mathbb{R})$

We’ll compute $A_\pi(\nu)$ where $\pi$ is the reflection representation.

This group has type $C_2$, and the root system looks like this ($n = 2$):
The simple roots are $\alpha, \beta$. Write $\nu = (x, y)$.

We have $s_\alpha$ is reflection in the $x$ coordinate, and $s_\beta$ is reflection in the line $x = y$:

$$s_\alpha(x, y) = (-x, y) \quad s_\beta(x, y) = (y, x)$$

That is, the reflection representation $\pi$ is given by matrices

$$\pi(s_\alpha) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\pi(s_\beta) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
Also note that

\[< \nu, \alpha > = 2(2x)/4 = x\]
\[< \nu, \beta > = 2(x - y)/2 = x - y\]

As noted in Lecture II, \(w_0 = s_\beta s_\alpha s_\beta s_\alpha\). Therefore \(A_\pi(\nu)\) is the following product:

\[A_\pi, s_\beta (s_\alpha s_\beta s_\alpha \nu) A_\pi, s_\alpha (s_\beta s_\alpha \nu) A_\pi, s_\beta (s_\alpha \nu) A_\pi, s_\alpha (\nu)\]

Thus

\[A_\pi, s_\alpha (\nu) = \pi(1) + < \nu, \alpha > \pi(s_\alpha)\]

\[= \pi(1) + x\pi(s_\alpha)\]

\[= \begin{pmatrix} 1 - x & 0 \\ 0 & 1 + x \end{pmatrix}\]

Next \(< s_\alpha \nu, \beta > = <(-x, y), (1, -1) > -x -\)
\[ A_{\pi,s_\beta}(s_\alpha \nu) = \pi(1) + \langle s_\alpha \nu, \beta \rangle \pi(s_\beta) \]
\[ = \pi(1) + (-x - y)\pi(s_\beta) \]
\[ = \begin{pmatrix} 1 & -x - y \\ -x - y & 1 \end{pmatrix} \]

For the next step

\[ \langle s_\beta s_\alpha \nu, \alpha \rangle = \langle (y, -x), (2, 0) \rangle = y \]

and

\[ A_{\pi,s_\alpha}(s_\beta s_\alpha \nu) = \pi(1) + \langle s_\beta s_\alpha \nu, \alpha \rangle \pi(s_\alpha) \]
\[ = \pi(1) + y\pi(s_\alpha) \]
\[ = \begin{pmatrix} 1 - y & 0 \\ 0 & 1 + y \end{pmatrix} \]

Finally \[ \langle s_\alpha s_\beta s_\alpha \nu, \beta \rangle = \langle (-y, -x), (1, -1) \rangle \]
\(y - x\), and
\[
A_{\pi, s_\beta}(s_\alpha \nu) = \pi(1) + \langle s_\alpha s_\beta s_\alpha \nu, \beta \rangle > \pi(s_\beta)
\]
\[
= \pi(1) + (-x - y)\pi(s_\beta)
\]
\[
= \begin{pmatrix} 1 & y - x \\ y - x & 1 \end{pmatrix}
\]
This gives
\[
A_\pi(\nu) = \begin{pmatrix} 1 & y - x \\ y - x & 1 \end{pmatrix} \begin{pmatrix} 1 - y & 0 \\ 0 & 1 + y \end{pmatrix}
\]
\[
\begin{pmatrix} 1 & -x - y \\ -x - y & 1 \end{pmatrix} \begin{pmatrix} 1 - x & 0 \\ 0 & 1 + x \end{pmatrix}
\]
or
\[
\begin{pmatrix}
(1 + y)(1 + x) + \\
(1 + y)(1 - x)(x - y)(x + y) \\
2x(y^2 - 1)
\end{pmatrix}
\begin{pmatrix}
2x(y^2 - 1) \\
(1 - y)(1 - x) + \\
(1 - y)(1 + x)(x - y)(x + y)
\end{pmatrix}
\]
Definition 8.2 A real symmetric matrix is positive semi–definite if its eigenvalues are all $\geq 0$.

Remark: The eigenvalues of a real symmetric matrix are real.

Example:

$$X := \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 7 \end{pmatrix}$$

According to Mathematica the eigenvalues are:

$$\frac{11}{3} + \frac{\frac{235^2}{3 (241 + 9 i \sqrt{34})^{\frac{1}{3}}}}{3} + \frac{\frac{5 (241 + 9 i \sqrt{34})^\frac{1}{3}}{3}}{3}$$

$$\frac{11}{3} - \frac{\frac{235^\frac{2}{3} (1 + i \sqrt{3})}{6 (241 + 9 i \sqrt{34})^{\frac{1}{3}}}}{6} - \frac{\frac{(1 - i \sqrt{3}) (5 (241 + 9 i \sqrt{34})^\frac{1}{3})}{6}}{6}$$

Numerically:

$$10.79 + 0. i$$
$$-0.34 + 4.44 \times 10^{-16} i$$
$$0.54 - 4.44 \times 10^{-16} i$$

Moral: Don’t trust what computers tell you.
Theorem 8.3  1. $A_\pi(\nu)$ is well-defined

  2. $A_\pi(\nu)$ is a polynomial function in $\nu$

  3. $A_\pi(\nu)$ is invertible unless $<\nu, \alpha> = \pm1$ for some root $\alpha$

  4. $A_\pi(\nu)$ is symmetric if $w_0\nu = -\nu$

Conjecture 8.4  Assume $\nu \in \mathbb{R}^n$ and $w_0\nu = -\nu$. Then $\pi_{\nu}$ is unitary if and only if $A_\pi(\nu)$ is positive semi-definite for all irreducible representations $\pi$ of $W$.

This is almost certainly true. It is true for classical groups. If it isn’t true it would be good to know why.

Theorem 8.5  The conjecture is true if you replace $\mathbb{R}$ with a “p-adic” field.

Problem: Compute the set of $\nu$ for which $A_\pi(\nu)$ is positive definite for all $\pi$. Given the conjecture this answers the Problem stated earlier.

We may assume $\nu$ is dominant. The eigenvalues of $A_\pi(\nu)$ are continuous functions of $\nu$; they can only change sign at where $<w\nu, \alpha> = 1$ for some $\alpha \in \Delta$ and $w \in W$. 
We therefore divide the dominant region into a finite number of facets: those cut out by the planes $\langle v, \alpha \rangle = 1$ ($\alpha \in \Delta$).

**Example:** $Sp(4, \mathbb{R})$
Here are the regions for which \( A_\pi(\nu) \) is positive semidefinite where \( \pi \) is the reflection representation:

![Diagram of regions]

The eigenvalues of \( A_\pi(\nu) \) are:

\[
\begin{align*}
&[1 + x^2 + xy - x^3y - y^2 + xy^3] \pm \\
&[5x^2 - 2x^4 + x^6 + 2xy - 2x^5y + y^2 - 4x^2y^2 - \\
&x^4y^2 + 4x^3y^3 - 2y^4 + 3x^2y^4 - 2xy^5 + y^6]\frac{1}{2}
\end{align*}
\]

In the picture the (dominant) points where both eigenvalues are \( \geq 0 \) are colored black (and made Weyl group invariant).
In all representations of $W$:

This is the picture of the unitary representations of $Sp(4, \mathbb{R})$. 
Example: $G_2$

The facets for $G_2$

It suffices to use a reducible four-dimensional representation. The first eigenvalue is:
\begin{align*}
(54 - 108x^2 + 54x^4 + 144\sqrt{3}xy + 72\sqrt{3}x^3y + 36y^2 - \\
36x^2y^2 - 192\sqrt{3}xy^3 - 240\sqrt{3}x^3y^3 - 90y^4 + 72\sqrt{3}xy^5 - \\
[(54 - 108x^2 + 54x^4 + 144\sqrt{3}xy + 72\sqrt{3}x^3y + 36y^2 - 36x^2y^2 - \\
192\sqrt{3}xy^3 - 240\sqrt{3}x^3y^3 - 90y^4 + 72\sqrt{3}xy^5)^2 + 108(-27 + 216x^2 - \\
594x^4 + 756x^6 - 459x^8 + \\
108x^{10} + 216y^2 - 1188x^2y^2 + 1332x^4y^2 - 108x^6y^2 - 108x^8y^2 - \\
144x^{10}y^2 - 594y^4 + 2892x^2y^4 - 2178x^4y^4 + 216x^6y^4 + 960x^8y^4 + \\
652y^6 - 2796x^2y^6 + 1224x^4y^6 - 1888x^6y^6 - 267y^8 + 828x^2y^8 + \\
960x^4y^8 + 36y^{10} - 144x^2y^{10})^{1/2})/54
\end{align*}

The \((1, 1)\) entry of \(A_\pi(\nu)\) is

\begin{align*}
&(1 - x + 3(x + y) - 3(-2x - y + 3(x + y)))(1 + 3x + y - 6(x + y) + \\
3(-2x - y + 3(x + y)))((1 + x - 3(x + y))(1 - 2x - y + 3(x + y))(3x(x + y) + \\
(1 - x)(1 + x + y)) + 3(-2x - y + 3(x + y))(x(1 - x - y)(1 - x + 3(x + y)) + \\
(-x + 3(x + y))(3x(x + y) + (1 - x)(1 + x + y)))) + 3(3x + y - 6(x + y)
+ 3(-2x - y + 3(x + y))((1 + 2x + y - 3(x + y))(1 + x - 3(x + y) + \\
3(-2x - y + 3(x + y))(x(1 - x - y)(1 - x + 3(x + y)) + (-x + 3(x + y))
(3x(x + y) + (1 - x)(1 + x + y)) + (x - 3x + y) + 3(-2x - y + 3(x + y)))
((1 + x - 3(x + y))(1 - 2x - y + 3(x + y))(3x(x + y) + (1 - x)(1 + x + y)) + \\
3(-2x - y + 3(x + y))(x(1 - x - y)(1 - x + 3(x + y)) + \\
(-x + 3(x + y))(3x(x + y) + (1 - x)(1 + x + y))))
\end{align*}
The matrix $A_\pi(1/4, 1/6)$ is

\[
\begin{pmatrix}
\frac{207393-81191\sqrt{3}}{373248} & \frac{425}{1728} & 0 & 0 \\
\frac{425}{1728} & \frac{207393+81191\sqrt{3}}{124416} & 0 & 0 \\
0 & 0 & \frac{167025+39913\sqrt{3}}{124416} & \frac{13}{54} \\
0 & 0 & \frac{13}{54} & \frac{167025-39913\sqrt{3}}{373248}
\end{pmatrix}
\]

And the final picture is:

This is the picture of the spherical unitary representations of $G_2(\mathbb{R})$ and also $G_2(F)$ for $F$ a $p$–adic field.
Example: $F_4$, $\pi = \text{the reflection representation, } \nu = (7/12, 7/24, 7/24, 0)$

\[ A : \{ \{179084422223504289318018271/27262293279626489757696, 2848741016732394535581473/1009714565912092213248, 43933519598549036670413563/27262293279626489757696, 20482625885253000005047721/3029143697736276639744\}, \{2848741016732394535581473/1009714565912092213248, 1588707472675380417781223/1009714565912092213248, 2107274845436801783022007/3029143697736276639744, 805420891824960442145471/3029143697736276639744\}, \{43933519598549036670413563/27262293279626489757696, 2107274845436801783022007/3029143697736276639744, 10953541816452899251348189/27262293279626489757696, 169635969006910190915221/1009714565912092213248\}, \{2048262588525300005047721/3029143697736276639744, 805420891824960442145471/3029143697736276639744, 169635969006910190915221/1009714565912092213248, 218511037306943908817039/3029143697736276639744\} \} \]
9 Computations

Here are the ingredients for computing unitarity of spherical representations.

**Note:** All computations must be done with rational numbers. The reason is we need to know whether a number (for example an eigenvalue) is \( > 0 \). If you compute with a fixed precision you need to be very careful you don’t make a mistake.

**Models of representations of Weyl groups** Find a model for every irreducible representation of an exceptional Weyl group. The character tables are known.

- \( W(G_2) \) has 6 irreducible representations, of dimension 2, 2, 1, 1, 1, 1.
- \( W(F_4) \) has 25 representations, the largest of dimension 16.
- \( W(E_6) \) has 25 representation, the largest of dimension 81.
- \( W(E_7) \) has 50 representation, the largest of dimension 512.
- \( W(E_8) \) has 112 representation, the largest of dimension 5600.

**The facets** Given a Weyl group, list the facets given by the hyperplanes \( < \nu, \alpha >= 1 \). Give a sample point on each facet. This is a problem in linear programming.
There are 712 facets for $E_6$.

**Computation of $A_{\pi\nu}$** This is straightforward, but slow for large representations.

**Compute whether $A_{\pi}(\nu)$ is positive semi–definite**
Algorithms are known, but this may be difficult for large matrices. You certainly can’t compute eigenvalues. I don’t know if it is possible to test a matrix of size 5600. It would be interesting to find out.

**Compute the unitary set** Compute the facets for which $A_{\pi,\nu}$ is positive semi–definite for all $\nu$.

**Compute a minimal set of $\pi$** We should only need a small set of $\pi$ to give the set of unitary $\nu$. Find a small set of $\pi$ which works.
10 Software

Here is some of the mathematical software we have available, with an emphasis on things useful for this project or similar one.

- Mathematica, Matlab, Maple: general purpose mathematical software, including symbolic computations
- Magma: Powerful algebra package, including finite groups of Lie type, Weyl groups, and character tables. Command: magma2.9 (new version of magma).
- Gap: Group Algebra Package, with an emphasis on group theory. Includes the character tables from the Atlas of Finite Groups
- LiE: Computations in Lie theory, including finite dimensional representations of complex Lie groups and algebras, Kazhdan–Lusztig polynomials
- Cocoa: Commutative Algebra package (Grobner bases, ideals, polynomials, ...)
- Magnus: Combinatorial Group Theory, with a graphical interface. Handles infinite groups
- Pari: Advanced programmable calculator: symbolic computations, number theoretic functions (elliptic curves, class field theory...). Command: gp.
Perl: Nice software for gluing things together, for example for taking the output of LiE and preparing it for input to Magma