Characters of Non-Linear Groups

Notes available at www.math.umd.edu/~jda/preprints.

General setup:

\( \mathbb{F} \): a local field of characteristic 0

\( \mathbb{G} \): an algebraic group over \( \mathbb{F} \), often simply connected, \( G = \mathbb{G}(\mathbb{F}) \),

\( \tilde{G} \) a central extension of \( G \):

\[ 1 \longrightarrow A \longrightarrow \tilde{G} \xrightarrow{\rho} G \longrightarrow 1 \]

(\(|A| < \infty\)).

When \( \mathbb{G} \) is simply connected \( \tilde{G} \) is a non-linear group.

Relevant semisimple orbits

Definition: An irreducible representation \( \pi \) of \( \tilde{G} \) is genuine if the central character \( \chi_\pi \) is an injection when restricted to \( A \).

Fix a Cartan subgroup \( T \) of \( G \), let \( \tilde{T} = p^{-1}(T) \), \( Z(\tilde{T}) \) the center of \( \tilde{T} \).

Definition: If \( g \in \tilde{T} \) is regular,

\( g \) is relevant if \( g \in Z(\tilde{T}) \)

The next Proposition originates in [Flicker]:

Proposition: Suppose \( \pi \) is an irreducible genuine representation, with character \( \Theta_\pi \). If \( g \) is not relevant then

\[ \Theta_\pi(g) = 0 \]

Proof: Suppose \( g \) is not relevant. Then

\[ g \sim_{\tilde{T}} g' \]

for some \( g' \neq g \). Projecting both sides to \( G \) gives \( p(g) \sim p(g') \Rightarrow p(g) = p(g') \Rightarrow g' = ag \) for some \( a \in A, a \neq 1 \). Therefore

\[ g \sim g' = ga \]

Since \( \Theta_\pi \) is conjugation invariant this gives

\[ \Theta_\pi(g) = \Theta_\pi(ga) = \chi_\pi(a) \Theta_\pi(g) \]

Since \( \pi \) is genuine, \( \chi_\pi(a) \neq 1 \) proving the result.

Typeset by \LaTeX
Remark: Rebecca Herb noticed an error in one version of Harish-Chandra’s character formula for the discrete series. In [Acta 1965] formula (pg. 304) is correct, but in (Lemma 57) the terms are rearranged, and written:

$$\Theta_\pi(g) = \frac{1}{\Delta(g)} \sum_{t \in W \setminus W_0} c(t) \lambda^t(h_1) \sum_{s \in W(A^+)} c(s) c_\lambda(s : t : A^+) \exp(st(\lambda)^\beta(H_2))$$

The first sum is not well defined for $g \not\in Z(T)$. However by the preceding discussion this may be corrected as follows:

$$\Theta_\pi(g) = \begin{cases} \frac{1}{\Delta(g)} \sum_{t \in W \setminus W_0} c(t) \lambda^t(h_1) \\ \sum_{s \in W(A^+)} c(s) c_\lambda(s : t : A^+) \exp(st(\lambda)^\beta(H_2)) & g \in Z(T) \\ 0 & g \not\in Z(T) \end{cases}$$

Remark: A similar phenomenon holds for $\bar{G}$ a linear cover of $G$. Recall $g$ is regular if $\text{Cent}_G(g)^0$ is a torus, and strongly regular if $\text{Cent}_G(g)$ is a torus. Suppose $g \in \bar{G}$ and $p(g)$ is regular, but not strongly regular. Then $\Theta_\pi(g) = 0$ for any genuine representation of $\bar{G}$.

Example: $G = SO(2n), \bar{G} = \text{Spin}(2n)$. Suppose $g \in SO(2n)$ is regular. Then $g$ is not strongly regular if and only both $\pm 1$ are eigenvalues of $g$.

$$|\Theta_{\text{Spin}^+} \pm \Theta_{\text{Spin}^-}(g)| = |\det(1 \pm p(g))|^{\frac{1}{2}}$$

It follows that $\Theta_{\text{Spin}^+}(g) = 0$ if and only if $p(g)$ is not strongly regular.

**Spin-Oscillator Duality**

Let $G = Sp(2n, \mathbb{R}), \bar{G} = SO(n + 1, n)$. For $g, g'$ semisimple elements, write $g \leftrightarrow g'$ if $g, g'$ have the same eigenvalues (This is the stabilized orbit correspondence.)

Let $\text{osc} = \text{osc}^+ \oplus \text{osc}^-$ be the oscillator representation of $\tilde{Sp}(2n, \mathbb{R})$. Then (Howe)

$$|\Theta_{\text{osc}}(g)| = |\det(1 - p(g))|^{-\frac{1}{2}}$$

**Proposition:** The correspondence $g \to g'$ may be lifted uniquely to a correspondence $\tilde{g} \to \tilde{g}'$ between $Sp(2n, \mathbb{R})$ and $\text{Spin}(n + 1, n)$ with the following property:

$$\Theta_{\text{osc}^+ \oplus \text{osc}^-}(\tilde{g}) \Theta_{\text{Spin}}(\tilde{g}') = 1$$

Remark: In absolute value this follows from

$$|\Theta_{\text{Spin}}(g)| = |\det(1 + p(g))|^\frac{1}{4}$$

$$|\Theta_{\text{osc}^+ \pm \text{osc}^-}(g)| = |\det(1 \mp p(g))|^{-\frac{1}{2}}$$
Flicker/Kazhdan/Patterson Lifting

Let \( G = GL(n, \mathbb{F}) \), \( \hat{G} \) an \( N \)-fold central extension.

Flicker (for \( GL(2) \)) and Kazhdan/Patterson (for \( GL(n) \)) defined an operation \( t_\ast \), taking a virtual character of \( G \) to a genuine virtual character of \( \hat{G} \). If \( \sigma \) is irreducible and tempered, then \( t_\ast(\sigma) \) is zero or \( \pm \) an irreducible tempered representation. Over \( \mathbb{C} \) (Tadic) and \( \mathbb{R} \) (Adams-Huang), for \( \sigma \) irreducible unitary, \( t_\ast(\sigma) \) is zero, or \( \pm \) an irreducible unitary representation.

Inversion

Flicker/Kazhdan/Patterson lifting completely describe the irreducible characters of covers of \( GL(n) \). What about \( SL(n) \)?

Consider a non-trivial \( N \)-fold cover \( \overline{SL(n)} \) of \( SL(n) \), and assume \( N \) divides \( n \), for example the 2-fold cover of \( SL(2) \). This extends to a cover \( \overline{GL(n)} \) of \( GL(n) \) \((c = 0 \) in Kazhdan/Patterson’s notation).

Let \( \pi \) be an irreducible genuine representation of \( \overline{GL(n)} \), and write \( \sum_i \pi_i \) for the restriction of \( \pi \) to \( SL(n) \).

Let
\[
GL(n)_+ = SL(n)D = \{ g \in GL(n) \mid \det(g) \in \mathbb{F}^* \}
\]

The key point is:
\[
Z(\overline{GL(n)}_+)/Z(\overline{GL(n)}) \simeq \mathbb{F}^*/\mathbb{F}^N
\]

Write \( \chi_i \) for the central character of \( \pi_i \) considered as a representation of \( \overline{GL(n)}_+ \). Choose representatives \( x_i \) of \( \mathbb{F}^*/\mathbb{F}^N \), and \( z_i \in Z(\overline{GL(n)}_+) \), \( p(z_i) = x_i I \).

Proposition:
\[
\theta_{\pi_i}(g) = \frac{1}{m} \sum_j \chi_i(z_j)^{-1} \theta_{\pi}(z_j g)
\]

Remark: This is very different from the case of linear groups, for which the character of a representation of \( SL(n) \) is not obtained directly from the corresponding characters of \( GL(n) \). In fact for \( SL(2) \) this is the first example of endoscopy, in which an endoscopic group \( T \) (an elliptic torus) plays a role.

Support of Distributions

It is interesting to consider the restriction to \( \overline{SL(n)} \) of invariant distributions of \( SL(n) \). By lifting a virtual character of \( GL(n) \) which vanishes on \( SL(n) \), we obtain
a virtual character of $SL(n)$ with small support. To obtain such a virtual character of $GL(n)$, simply take $\pi - \pi \otimes \beta$ for some character $\beta$.

For simplicity we consider the case $GL(2)$, $N = 2$. (The general situation is similar, up to complications arising from the center of $GL(n)$). Suppose $\sigma$ is a virtual representation of $GL(2)$, and let $\pi = t_\sigma(\sigma)$. By [Flicker],

$$\Theta_{\pi}(g) = \sum_{h \in p(g)} \Phi(h) \Theta_{\sigma}(h)$$

Here $\Phi(h)$ is a (transfer) factor which need not concern us here. Note that if $p(g) = h^2$ is elliptic, the sum is over $\pm h$.

Now let $\sigma$ be an irreducible representation of $GL(2)$, with $t_\sigma(\sigma) \neq 0$. Fix a character $\beta_0$ of $\mathbb{F}^*$ with $\beta_0(-1) = -1$, and let $\beta(g) = \beta_0(\det(g))$. Let $\pi = t_\sigma(\sigma - \sigma \otimes \beta)$, restricted to $SL(2)$.

Suppose $g$ is elliptic. By the preceding formula, $\Theta_{\pi}(g) = 0$ if

1. $p(g) = h^2$ with $h \in SL(2)$,
2. $p(g) \neq h^2$ for any $h \in GL(2)$

Therefore: the support of $\Theta_{\pi}$, restricted to the regular elliptic set in $\overline{SL}(2)$, is

$$\{g \mid p(g) = h^2, \det(h) = -1\}$$

This is a fairly small but non-empty set. Suppose $T$ is an elliptic torus in $GL(2)$, isomorphic to the multiplicative group $\mathbb{F}^*$ of a quadratic extension $\mathbb{E} = \mathbb{F}(\sqrt{\Delta})$. The determinant of an element $z \in \mathbb{E}^*$ is the norm $N(z)$, the torus in $SL(2)$ is the norm one elements, and the set we are considering is

$$\{z^2 \mid N(z) = -1\}$$

This is non-empty if and only if $(-1, \Delta)_F = 1$. For example if $-1$ is not a square in $\mathbb{F}$ and the residual characteristic is odd, this holds if and only if $\Delta = -1$. In this case $\Theta_{\pi}$ is supported on:

$$\{g \in T - T^2 \mid T \text{ unramified}\}.$$ 

Note that $T - T^2 = -T^2$ is a coset of index 2 in $T$.

Finally we note that $\pi$ may be zero. By considering the hyperbolic set, we see $\pi \neq 0$ if $\Theta_{\sigma}(g) \neq 0$ for some $g = \begin{pmatrix} x & \frac{1}{x} \\ -\frac{1}{x} & \frac{1}{x} \end{pmatrix}$.

For example, taking $\sigma$ to be the trivial representation gives $\pi$ is the sum of the oscillator representations of $SL(2)$ (there are $|\mathbb{F}^*/\mathbb{F}^*|^2$ such representations).