1. Let \((X,x_0)\) be a based space (always assumed compactly generated and Hausdorff). Prove that 
\(\Omega X\), equipped with the multiplication \(\Omega X \times \Omega X \xrightarrow{\mu} \Omega X\) and inversion \(\Omega X \xrightarrow{\iota} \Omega X\),

\[
\mu(\alpha, \beta)(s) = \begin{cases} 
\alpha(2s), & 0 \leq s \leq \frac{1}{2}, \\
\beta(2s-1), & \frac{1}{2} \leq s \leq 1,
\end{cases}
\]

\(\iota(\alpha)(s) = \alpha(1-s),\)

is an \(H\)-group. (Thus for any based space \(Y\), \([Y, \Omega X]\) is a group.)

2. Let \((X,x_0)\) be a based space. Recall that the (based) suspension \(\Sigma X\) is the smash product 
\(X \wedge S^1\), i.e., the quotient space of the product \(X \times S^1\) in which \(x_0 \times S^1\) and \(X \times \ast\) (\(\ast\) the basepoint of \(S^1\)) are collapsed to a common point which serves as the basepoint of \(\Sigma X\). In turn one can represent \(S^1\) as the quotient group \(\mathbb{R}/\mathbb{Z}\) or the quotient space \([0, 1]/(0 \sim 1)\). Represent points in \(\Sigma X\) by equivalence classes \([x, t]\) of pairs \((x, t)\), with \(x \in X, t \in S^1\). Show that \(\Sigma X\), equipped with the comultiplication \(\nu: \Sigma X \to \Sigma X \vee \Sigma X\) and inversion \(\Sigma X \xrightarrow{\iota} \Sigma X\),

\[
\nu([x, t]) = \begin{cases} 
[x, 2t], & 0 \leq t \leq \frac{1}{2}, \\
[x, 2t - 1], & \frac{1}{2} \leq t \leq 1,
\end{cases}
\]

where the subscripts indicate which copy of \(\Sigma X\),

\(\iota([x, t]) = [x, 1-t],\)

is an \(H\)-cogroup. (Thus for any based space \(Y\), \([\Sigma X, Y]\) is a group.) **Caution:** You need to check that \(\nu\) and \(\iota\) are well defined on equivalence classes and send basepoint to basepoint.

3. (Cf. May, page 56, top.) Use the fact that in the category of compactly generated spaces, 
\(\text{Maps}(S^1 \times X, Y) \cong \text{Maps}(X, \text{Maps}(S^1, Y))\), to prove the adjunction formula \([\Sigma X, Y] \cong [X, \Omega Y]\), and show that this identification is a group isomorphism with respect to the group structures defined in \#1 and \#2.

4. There are several (\textit{a priori} different) group laws defined on \([\Sigma^2 X, Y] \cong [\Sigma X, \Omega Y] \cong [X, \Omega^2 Y]\) via \#1 and \#2. Prove that all these group structures coincide and are abelian. (See for instance Bredon, pp. 442 ff., or Spanier, pp. 43–44.)