MATH 141H SOLUTIONS TO SAMPLE EXAM

1. (a) converges for $r < 1$ (Do the integral explicitly)
   (b) converges for $r > 1$ (Do the integral explicitly)

2. Take logarithm, then exponentiate:
   \[
   \lim_{n\to\infty} \frac{\ln(n/(n+1))}{1/(n-1)} = 0
   \]
   so l’Hospital applies. The numerator is $\ln n - \ln(n+1)$, and its derivative is $\frac{1}{n} - \frac{1}{n+1}$. The derivative of the denominator is $-1/(n-1)^2$. Some algebra shows that we now have $\frac{-1(n-1)^2}{n(n+1)} = -1$. Exponentiating yields the answer $e^{-1}$.

3. It’s probably easiest to use the shell method. Use $f(x) = \sqrt{R^2 - x^2}$. The part of the graph with $-3 \leq y \leq 3$ is revolved around the $y$-axis. When $f(x) = y = 3$ we have $x = \sqrt{R^2 - 3^2}$. The largest $x$ can be is $R$. Therefore, the volume above the $x$-axis is
   \[
   V = 2\pi \int_{\sqrt{R^2 - 9}}^{R} x\sqrt{R^2 - x^2} \, dx = \pi \left[ \frac{-2}{3} (R^2 - x^2)^{3/2} \right]_{\sqrt{R^2 - 9}}^{R} = 18\pi
   \]
   Double to include the lower half. The final answer is $36\pi$.

4. Partial fractions yields
   \[
   \frac{2x^2 + 3}{x(x-1)^2} = \frac{3}{x} + \frac{-1}{x-1} + \frac{5}{(x-1)^2}.
   \]
   Integrating yields
   \[
   3\ln|x| - \ln|x-1| - 5(x-1)^{-1} + C.
   \]

5. Partial fractions, with a lot of calculation, yields
   \[
   \frac{2x}{(x+1)(x^2+1)^2} = \frac{-1}{2} \frac{1}{x+1} + \frac{1}{2} \frac{x-1}{x^2+1} + \frac{x+1}{(x^2+1)^2}.
   \]
   Integrating these yields
   \[
   -\frac{1}{2} \ln|x+1| + \frac{1}{4} \ln|x^2+1| + \frac{1}{2} \frac{x-1}{x^2+1} + C
   \]
\[ S = 2\pi \int_0^2 f(x)\sqrt{1 + f'(x)^2} \, dx = 2\pi \int_0^2 x^3\sqrt{1 + 9x^2} \, dx \]
Use the substitution \( u = 9x^4 \). The final answer is \((\pi/27)(145^{3/2} - 1)\).

7. \[ V = \int_0^{30} A(x) \, dx = \int_0^{30} (30 - x)^2 \, dx = 9000. \]

8. The ball goes down 6, then up \((3/4)^6\) and down \((3/4)^6\), then up \((3/4)^2\), and down the same amount, etc. The total distance is
\[
6 + 2\left(\frac{3}{4}6 + \left(\frac{3}{4}\right)^26 + \cdots\right) = 6 + 12\left(\frac{3/4}{1 - (3/4)}\right) = 42.
\]
(the infinite sum is a geometric series)

9. (a) Converges: Limit comparison test with \(\sum 1/n^{3/2}\).
(b) Diverges. The terms are larger than the terms of \(\sum 1/n\), which diverges. The comparison test implies the answer.
(c) Converges: use the integral test:
\[
\int_2^\infty \frac{dx}{x(\ln x)^2} = -\left. \frac{1}{\ln x}\right|_2^\infty = \frac{1}{\ln 2} < \infty,
\]
so the integral converges. Therefore, the sum converges.

10. Draw the boxes under the curve \( y = 1/x^2 \), starting at \( x = 1000 \). The sum is less than
\[
\int_{1000}^\infty \frac{dx}{x^2} = .001
\]