Homework #10

1. Let $G$ be a finite group and let $\rho : G \to GL_n(\mathbb{C})$ be an irreducible representation. Suppose $\rho$ is injective. Show that the center of $G$ is cyclic. (Hint: What do you know about finite subgroups of the multiplicative group of a field?)

2. Let $G$ be a group and suppose there exists a subgroup $A \subseteq G$ with $[G : A] = d$. Let $\rho : G \to GL(V)$ be a representation of $G$. Restrict $\rho$ to $A$ and decompose this representation of $A$ into irreducible representations. Let $\phi : A \to GL(W)$ be one of these irreducible representations of $A$.

(a) Let $g_1, \ldots, g_d$ be left coset representatives for $A$ in $G$, so $G = \cup g_iA$. Let

$$V_1 = g_1W + g_2W + \cdots + g_dW \subseteq V.$$ 

Show that $\rho(G)$ maps $V_1$ into itself, so $V_1$ gives a subrepresentation $V$.

(b) Suppose $\rho$ is irreducible. Show that $V_1 = V$.

(c) Suppose $A$ is abelian. Show that every irreducible representation of $G$ has dimension less than or equal to $d$.

(d) Let $D$ be a dihedral group $D_n$. Show that every irreducible representation of $D$ has dimension at most 2.

3. Find the character table of the quaternion group $Q_8$.

4. Let $\rho : G \to GL_2(\mathbb{C})$ be a two-dimensional complex representation of the finite group $G$. Let $V$ be the 4-dimensional vector space of $2 \times 2$ complex matrices, and let $G$ act on $V$ by

$$\tilde{\rho}(g)(M) = \rho(g)M\rho(g)^{-1}$$

for $M \in V$.

(a) Show that if $\rho$ is irreducible then $\tilde{\rho}$ contains the trivial representation exactly once.

(b) Show that if $\rho$ is the sum of two distinct one-dimensional representations, then $\tilde{\rho}$ contains the trivial representation exactly twice.

(c) Show that if $\rho$ is the sum of two equal one-dimensional representations, then $\tilde{\rho}$ equals the sum of four copies of the trivial representation.

5. Let $G$ be a finite group and let $H$ be a normal subgroup. Let $R = \mathbb{C}[G/H]$ be the group ring of $G/H$ with complex coefficients. Then $R$ is a complex vector space. For $\sigma \in G$, let $T_\sigma$ be the linear transformation of $R$ given by multiplication on the left by $\sigma$ (so $T_\sigma(gH) = \sigma gH$). Define a representation $\rho$ of $G$ by $\rho(\sigma) = T_\sigma$.

(a) Let $\chi$ be the character of $\rho$. Show that $\chi(\sigma) = |G|/|H|$ if $\sigma \in H$ and $\chi(\sigma) = 0$ if $\sigma \notin H$.

(b) Show that $\rho$ is irreducible if and only if $H = G$.

6. (a) Let $G$ be a finite nonabelian simple group (that is, $G$ has no nontrivial normal subgroups). Let $\rho : G \to GL_2(\mathbb{C})$ be a two-dimensional representation of $G$. Show that $\rho$ is either irreducible or trivial.

(b) It is known that the only finite subgroups of $GL_2(\mathbb{C})/\mathbb{C}^*$ (where $\mathbb{C}^*$ denotes the multiplicative group of nonzero scalar multiples of the identity) are isomorphic to subgroups of one of the following: $D_n$, $A_5$, $S_4$ (where $D_n$ is the $n$th dihedral group, $S_n$ is the group of permutations of $n$ objects, and $A_n$ is the subgroup of even permutations). Use this fact to prove that if $G$ is a nonabelian finite simple group with a nontrivial two-dimensional representation over $\mathbb{C}$, then $G = A_5$. (Note: You do not need to prove that $A_5$ has such a representation.)