Answers to HW 10

1. Let \( g \) be in the center of \( G \). Then \( \rho(g) \) commutes with all the matrices in \( \rho(G) \).
Since \( \rho \) is irreducible, Schur’s Lemma implies that \( \rho(g) \) is a scalar matrix \( \lambda_g I \).
The map \( \phi : g \mapsto \lambda_g \) is a homomorphism from the center of \( G \) to \( \mathbb{C}^\times \).
Since \( \rho \) is injective, so is \( \phi \). Therefore, \( G \) is isomorphic to \( \rho(G) \), which is a finite multiplicative subgroup of \( \mathbb{C}^\times \), hence cyclic.

2. (a) Let \( g \in G \). Then \( gg_i = g_j a \) for some \( a \in A \).
Therefore \( \rho(g)\rho(g_i)W = \rho(g_j)\rho(a)W = \rho(g_j)W \), since \( A \) is invarinat under \( \rho(A) \).
Therefore, \( \rho(g) \) maps each \( g_i W \) into \( V_1 \), so \( \rho(g)V_1 \subseteq V_1 \).
(b) This follows immediately from (a) and the definition of irreducibility.
(c) If \( A \) is abelian, then \( W \) is one-dimensional.
Therefore, \( V_1 \) has dimension at most \( d \).
(d) \( D_n \) contains a cyclic subgroup of index 2.
Take \( A \) to be this cyclic subgroup and \( d = 2 \).

3. The commutator subgroup of \( Q_8 \) is \( \{\pm 1\} \).
The quotient \( Q_8/\{\pm 1\} \) is the Klein 4-group, so the four one-dimensional representations of the Klein 4-group give us the 4 one-dimensional representation of \( Q_8 \).
Since \( 8 = \sum n_i^2 \), there can be only one 2-dimesnional representation. The character table is

<table>
<thead>
<tr>
<th></th>
<th>{1}</th>
<th>{-1}</th>
<th>{\pm i}</th>
<th>{\pm j}</th>
<th>{\pm k}</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \chi_1 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \chi_2 )</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>( \chi_3 )</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>( \chi_4 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>( \chi_5 )</td>
<td>2</td>
<td>-2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

The entries for \( \chi_5 \) are determined from the orthogonality of the columns.

4. (a) If \( M \) is fixed by \( \tilde{\rho} \), then \( M \) commutes with \( \rho(G) \).
By Schur’s Lemma, \( M \) is scalar.
Conversely, the scalar matrices are clearly fixed by \( \tilde{\rho} \).
The trivial representation is given exactly by the set of fixed elements in \( V \).
Since the scalar matrices are a one-dimensional subspace, the trivial representation occurs once.
(b) We have \( \mathbb{C}^2 = V_1 \oplus V_2 \), where each \( V_i \) is a one-dimensional representation of \( G \).
Use a basis for \( \mathbb{C}^2 \) formed from a basis of \( V_1 \) and a basis of \( V_2 \).
Then all of the matrices in \( \rho(G) \) are diagonal. If \( \rho_1 \neq \rho_2 \), then there exists some matrix \( \rho(g) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \) with \( \alpha \neq \beta \).
Suppose \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) is fixed by \( \tilde{\rho}(g) \).
Then
\[
\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}
\]
This yields \( b\alpha = b\beta \) and \( c\alpha = c\beta \). Since \( \alpha \neq \beta \), we have \( b = c = 0 \), so \( M \) is diagonal. The diagonal matrices are clearly fixed by conjugation by the diagonal matrices in \( \rho(G) \), so the fixed elements are exactly the two dimensional space of diagonal matrices in \( V \). Therefore, the trivial representation occurs twice. Explicitly, let \( W_1 \) be the matrices that have 0’s everywhere except possibly in the upper left corner. Let \( V_2 \) be the matrices that have 0’s everywhere except possibly in the lower right corner. Then \( W_1 \oplus W_2 \) is the sum of two trivial representations occurring in \( V \).

(c) If \( \rho \) is the sum of two equal one-dimensional representations, then the reasoning in (b) shows that each matrix \( \rho(g) \) is scalar. Therefore, \( \tilde{\rho}(g) \) acts trivially on \( V \), so \( V \) is the sum of only trivial representations. There must be 4 of them since \( V \) has dimension 4.

5. (a) The cosets \( g_1H, \ldots, g_nH \) give a basis for \( \mathbb{C}[G/H] \) as a vector space. If \( \sigma \in G \), then multiplication by \( \sigma \) permutes these basis elements. The trace of \( T_\sigma \) is the number of basis elements that are mapped to themselves by \( \sigma \). We have \( \sigma gH = gH \) if and only if \( \sigma H g H = g H \), which happens if and only if \( \sigma H = H \). Therefore, if \( \sigma \in H \), then \( T_\sigma \) is the identity and \( \chi(\sigma) = [G : H] \), which is the dimension of the space. If \( \sigma \notin H \), then the matrix for \( T_\sigma \) has only 0’s on the diagonal, so \( \chi(\sigma) = 0 \).

(b) \( \chi, \chi = |G|^{-1} \sum_{g \in G} \chi(g)\overline{\chi(g)} = |G|^{-1} \sum_{\sigma \in H} [G : H]^2 = |G|/|H| \). Therefore, \( \chi \) is irreducible \( \iff \chi, \chi = 1 \iff |G| = |H| \).

6. (a) If \( \rho \) is not irreducible, it is the sum of two one-dimensional representations. But a one-dimensional representation of \( G \) has \( G^c \) in its kernel. Since \( G \) is nonabelian and simple, \( G^c = G \), so the trivial representation is the only one-dimensional representation. Therefore, if \( \rho \) is not irreducible, it is the sum of two trivial representations, which means \( \rho(g) = I \) for all \( g \in G \).

(b) Consider the image of \( G \) in \( \text{GL}_2(\mathbb{C})/\mathbb{C}^* \). Since \( G \) has no nontrivial normal subgroups, the image is either trivial or isomorphic to \( G \). If the image is trivial, then \( \rho(G) \) is contained in the scalar matrices, which means that \( \rho \) is the direct sum of two one-dimensional representations, contradicting the assumption that \( \rho \) is irreducible. Therefore, the image is isomorphic to \( G \). Among the groups \( D_n, A_5, S_4 \), only \( A_5 \) has a simple nonabelian subgroup, namely \( A_5 \). Therefore, \( G = A_5 \).