MATH 601, SAMPLE PROBLEMS

1. Let \( L = \mathbb{Q}(i, 2^{1/4}) \). (a) Prove that \( L/\mathbb{Q} \) is Galois with \([L : \mathbb{Q}] = 8\).
   (b) Determine generators and relations for \( \text{Gal}(L/\mathbb{Q}) \).
   (c) Find all fields \( F \) with \( \mathbb{Q}(i) \subseteq F \subseteq L \).

2. (a) Let \( L/K \) be a finite Galois extension of fields with \( \text{Gal}(L/K) \) abelian. Let \( F \) be a field with \( K \subseteq F \subseteq L \). Show that \( F/K \) is a Galois extension.
   (b) Let \( f(X) \in K[X] \) be an irreducible polynomial whose splitting field has abelian Galois group. Let \( \alpha \) be a root of \( f(X) \). Show that \( K(\alpha) \) contains all roots of \( f(X) \).

3. (a) Let \( L \) and \( L' \) be subfields of \( \mathbb{Q} \) such that \( L' \not\subseteq L \). Show that there is an automorphism \( \tau \) of \( \mathbb{Q} \) such that \( \tau|_L = \text{id} \) but \( \tau|_{L'} \neq \text{id} \).
   (b) Suppose \( L/\mathbb{Q} \) is a normal extension. Let \( \sigma \) be an automorphism of \( \mathbb{Q} \) and let \( \tau \) be an automorphism of \( \mathbb{Q} \) such that \( \tau|_L = \text{id} \). Show that \( (\sigma^{-1}\tau\sigma)|_L = \text{id} \).
   (c) Suppose that \( L/\mathbb{Q} \) is not normal. Show that there exist automorphisms \( \tau \) and \( \sigma \) of \( \mathbb{Q} \) such that \( \tau|_L = \text{id} \) but \( (\sigma^{-1}\tau\sigma)|_L \neq \text{id} \).

4. Let \( f(x) \) be an irreducible polynomial of degree \( n \) over a field \( F \). Let \( g(x) \) be any polynomial in \( F[x] \). Prove that every irreducible factor of the polynomial \( f(g(x)) \) has degree divisible by \( n \).

5. Determine the splitting field \( F \) over \( \mathbb{Q} \) of \( X^4 + X^2 + 1 \).
   (b) Describe \( \text{Gal}(F/\mathbb{Q}) \).

6. Let \( L/\mathbb{Q} \) be a finite Galois extension of odd degree. Show that \( L \subseteq \mathbb{R} \) (more precisely, any embedding of \( L \) into \( \mathbb{C} \) has image in \( \mathbb{R} \)).

7. Find all irreducible polynomials of degrees 1, 2, 4 over \( \mathbb{F}_2 \) and prove that their product is \( X^{16} - X \).