Quiz 6 Solutions, Math 246, Professor David Levermore
Wednesday, 26 March 2008

(1) [5] Find a general solution of
\[ \frac{d^2x}{dt^2} + 2 \frac{dx}{dt} - 3x = e^t - 3t \]

**Solution.** The characteristic polynomial is \( P(z) = z^2 + 2z - 3 = (z - 1)(z + 3) \) and has roots 1 and -3. The solution of the associated homogeneous problem is
\[ Y_H(t) = c_1 e^t + c_2 e^{-3t} \]

The forcing term \( e^t \) has degree 0 and characteristic 1 which is a root of \( P(z) \) of multiplicity 1. The forcing term \(-3t\) has degree 1 and characteristic 0 which is a root of \( P(z) \) of multiplicity 0. We therefore need the KEY identity and its first derivative with respect to \( z \):
\[
L(e^{zt}) = (z^2 + 2z - 3)e^{zt}, \\
L(t e^{zt}) = (z^2 + 2z - 3)t e^{zt} + (2z + 2)e^{zt}.
\]

Evaluating these at \( z = 1 \) gives \( L(e^t) = 0 \) and \( L(t e^t) = 4e^t \), whereby \( L(t^2 e^t) = e^t \).

Evaluating these at \( z = 0 \) gives \( L(1) = -3 \) and \( L(t) = -3t + 2 \), which implies \( L(t + \frac{2}{3}) = -3t \). Hence, \( Y_P(t) = \frac{1}{4} t e^t + t + \frac{2}{3} \). A general solution is
\[ y = c_1 e^t + c_2 e^{-3t} + \frac{1}{4} t e^t + t + \frac{2}{3} \].

Alternatively, to get the forcing term \( e^t \) you set \( Y_P(t) = At e^t \), plug it into the equation, and solve for \( A \); to get the forcing term \(-3t\) you set \( Y_P(t) = A_0 t + A_1 \), plug it into the equation, and solve for \( A_0 \) and \( A_1 \). You get the same general solution.

(2) [5] The functions \( 1 + t \) and \( e^t \) are solutions of the equation
\[ t \frac{d^2y}{dt^2} - (1 + t) \frac{dy}{dt} + y = 0. \]

(You do not have to check that this is true.) Find a general solution of
\[ t \frac{d^2y}{dt^2} - (1 + t) \frac{dy}{dt} + y = t^2 e^t. \]

**Solution.** Divide the equation by \( t \) to bring it into normal form. A general solution of the associated homogeneous problem is
\[ Y_H(t) = c_1 (1 + t) + c_2 e^t. \]

Use variation of parameters to find \( Y_P(t) \) in the form
\[ Y_P(t) = u_1(t)(1 + t) + u_2(t)e^t, \]
where
\[ u_1'(t)(1 + t) + u_2'(t)e^t = 0, \\
u_1'(t)1 + u_2'(t)e^t = t e^t. \]

The solution of this linear algebraic system is \( u_1'(t) = -e^t \) and \( u_2'(t) = 1 + t \), whereby \( u_1(t) = c_1 - e^t \) and \( u_2(t) = c_2 + t + \frac{1}{2} t^2 \). A general solution is
\[ y = c_1 (1 + t) + c_2 e^t - e^t (1 + t) + (t + \frac{1}{2} t^2) e^t. \]