First In-Class Exam Solutions
Math 246, Spring 2009, Professor David Levermore

(1) [12] Suppose you have used a numerical method to approximate the solution of an initial-value problem over the time interval [0, 5] with 1000 uniform time steps. About how many uniform time steps do you need to reduce the global error of your approximation by a factor of 81 if the method you had used was each of the following?

(a) Runge-Kutta method
**Solution:** This method is fourth order, so its global error scales like $h^4$. To reduce the error by a factor of 81, you must reduce $h$ by a factor of $81^{\frac{1}{4}} = 3$. You must therefore increase the number of time steps by a factor of 3, which means you need 3000 uniform time steps.

(b) Heun-midpoint method
**Solution:** This method is second order, so its global error scales like $h^2$. To reduce the error by a factor of 81, you must reduce $h$ by a factor of $81^{\frac{1}{2}} = 9$. You must therefore increase the number of time steps by a factor of 9, which means you need 9000 uniform time steps.

(c) Heun-trapezoidal method
**Solution:** This method is second order, so its global error scales like $h^2$. To reduce the error by a factor of 81, you must reduce $h$ by a factor of $81^{\frac{1}{2}} = 9$. You must therefore increase the number of time steps by a factor of 9, which means you need 9000 uniform time steps.

(d) Euler method
**Solution:** This method is first order, so its global error scales like $h$. To reduce the error by a factor of 81, you must reduce $h$ by a factor of 81. You must therefore increase the number of time steps by a factor of 81, which means you need 81000 uniform time steps.

(2) [20] Find the explicit solution for each of the following initial-value problems and identify its interval of existence (interval of definition).

(a) $\frac{dy}{dx} = \frac{e^x}{1 + y}, \quad y(0) = -2.$

**Solution:** This equation is separable. Its separated differential form is

$$(y + 1) \, dy = e^x \, dx, \quad \Rightarrow \quad \frac{1}{2} (y + 1)^2 = e^x + c.$$  

The initial condition $y(0) = -2$ implies that $c = \frac{1}{2} (-2 + 1)^2 - e^0 = \frac{1}{2} - 1 = -\frac{1}{2}$. Therefore $(y + 1)^2 = 2e^x - 1$, which can be solved as

$$z = -1 - \sqrt{2e^x - 1}, \quad \text{with interval of existence} \ x > \log\left(\frac{1}{2}\right).$$

The negative square root is needed to satisfy the initial condition.

(b) $\frac{du}{dt} = \frac{t^3 - u}{1 + t}, \quad u(2) = 3.$
**Solution:** This equation is linear. Its linear normal form is
\[
\frac{du}{dt} + \frac{1}{1+t} u = \frac{t^3}{1+t}.
\]

An integrating factor is \( \exp \left( \int_0^t \frac{1}{1+s} \, ds \right) = \exp(\log(1+t)) = 1+t \), so that
\[
\frac{d}{dt} \left( (1+t)u \right) = (1+t) \cdot \frac{t^3}{1+t} = t^3, \quad \implies (1+t)u = \frac{1}{4}t^4 + c.
\]
The initial condition \( u(2) = 3 \) implies that \( c = (1+2) \cdot 3 - \frac{1}{4}2^4 = 9 - 4 = 5 \). Therefore
\[
u = \frac{1}{4}t^4 + 5, \quad \text{with interval of existence } t > -1.
\]

(3) [16] Consider the differential equation \( \frac{dx}{dt} = x(2-x)(4-x)^2 \).

(a) Sketch its phase-line. Indicate all of the stationary (equilibrium) solutions and classify each as being either stable, unstable, or semistable.

(b) If \( x(0) = 6 \), how does the solution \( x(t) \) behave as \( t \to \infty \)?

(c) If \( x(0) = 3 \), how does the solution \( x(t) \) behave as \( t \to \infty \)?

(d) If \( x(0) = 1 \), how does the solution \( x(t) \) behave as \( t \to \infty \)?

(e) If \( x(0) = -2 \), how does the solution \( x(t) \) behave as \( t \to \infty \)?

**Solution (a):** The stationary solutions are \( x = 0 \), \( x = 2 \), and \( x = 4 \). A sign analysis of \( x(2-x)(4-x)^2 \) shows that the phase-line for this equation is therefore

\[
\begin{array}{ccccccc}
& & & & & & \\
- & & & & & & \\
\vdots & \longrightarrow & \longrightarrow & \longrightarrow & \longrightarrow & \longrightarrow & y \\
& & & & & & \\
0 & & & & & & \\
& & & & & & \\
\vdots & \longrightarrow & \longrightarrow & \longrightarrow & \longrightarrow & \longrightarrow & \\
& & & & & & \\
2 & & & & & & \\
& & & & & & \\
\vdots & \longrightarrow & \longrightarrow & \longrightarrow & \longrightarrow & \longrightarrow & \\
& & & & & & \\
4 & & & & & & \\
& & & & & & \\
\end{array}
\]

unstable stable semistable

(b): The phase-line shows that if \( x(0) = 6 \) then \( x(t) \to 4 \) as \( t \to \infty \).

(c): The phase-line shows that if \( x(0) = 3 \) then \( x(t) \to 2 \) as \( t \to \infty \).

(d): The phase-line shows that if \( x(0) = 1 \) then \( x(t) \to 2 \) as \( t \to \infty \).

(e): The phase-line shows that if \( x(0) = -2 \) then \( x(t) \to -\infty \) as \( t \to \infty \).

(4) [16] Consider the following MATLAB function M-file.

```matlab
function [t,y] = solveit(ti, yi, tf, n)

h = (tf - ti)/n;

end
```
(a) What is the initial-value problem being approximated numerically?
(b) What is the numerical method being used?
(c) What are the output values of $t(2)$ and $y(2)$ that you would expect for input values of $ti = 1, yi = 1, tf = 5, n = 20$?

**Solution (a):** The initial-value problem being approximated numerically is

$$\frac{dy}{dt} = 2y - y^2, \ \ y(ti) = yi.$$

(b): It is being approximated by the Heun-midpoint method.
(c): When $ti = 1, yi = 1, tf = 5, n = 20$ one has $h = (tf - ti)/n = (5 - 1)/20 = .2$, $t(1) = ti = 1$, and $y(1) = yi = 1$.

Setting $k = 1$ inside the “for” loop then yields

$$y_{half} = y(1) + (h/2) (2 y(1) - y(1)^2) = 1 + .1 (2 \cdot 1 - 1) = 1.1,$$

$$t(2) = t(1) + h = 1 + .2 = 1.2,$$

$$y(2) = y(1) + h (2 y_{half} - y_{half}^2) = 1 + .2 (2 \cdot 1.1 - (1.1)^2).$$

The above answer got full credit, but $y(2) = 1.198$ if you worked out the arithmetic.

(5) [16] A student borrows $6000 at an interest rate of 10% per year compounded continuously. Assume that the student makes payments continuously at a constant rate of $k$ dollars per year. Let $B(t)$ denote the balance of the loan at $t$ years.

(a) Write down an initial-value problem that governs $B(t)$ at any positive time for which the balance is still positive.
(b) Determine the value of $k$ required to pay off the loan in five years.

**Solution (a):** The balance $B(t)$ satisfies the initial-value problem

$$\frac{dB}{dt} = .1B - k, \quad B(0) = 6000.$$

(b): The equation is linear and can be put into the integrating factor form

$$\frac{d}{dt}(e^{-.1t}B) = -ke^{-.1t}, \quad \implies \quad e^{-.1t}B(t) = 10ke^{-1t} + c, \quad \implies \quad B(t) = 10k + ce^{.1t}.$$

The initial condition $B(0) = 6000$ implies that $c = 6000 - 10k$. Hence,

$$B(t) = 10k(1 - e^{-.1t}) + 6000e^{.1t}.$$

Paying off the loan in five years means that $B(5) = 0$. Therefore $k$ must satisfy

$$0 = 6000e^5 - 10k(e^5 - 1), \quad \implies \quad k = \frac{600e^5}{e^5 - 1}.$$

(6) [20] Give an implicit general solution to each of the following differential equations.

(a) $2xy \, dx + (2x^2 + e^y) \, dy = 0$.

**Solution:** This differential form is *not exact* because

$$\partial_y(2xy) = 2x \neq \partial_x(2x^2 + e^y) = 4x.$$

You therefore seek an *integrating factor* $\mu$ such that

$$\partial_y[2xy\mu] = \partial_x[(2x^2 + e^y)\mu].$$
Expanding the partial derivatives yields
\[2xy\partial_y\mu + 2x\mu = (2x^2 + e^y)\partial_x\mu + 4x\mu.\]
If you set \(\partial_x\mu = 0\) then this becomes
\[2xy\partial_y\mu + 2x\mu = 4x\mu,
\]
which reduces to \(y\partial_y\mu = \mu\). This has the normal form
\[\partial_y\mu - \frac{1}{y} \mu, \quad \Rightarrow \quad \partial_y\left(\frac{\mu}{y}\right) = 0,
\]
which yields the integrating factor \(\mu = y\).

Because \(y\) is an integrating factor, the differential form
\[2xy^2 \, dx + (x^2y + ye^y) \, dy = 0\]
is exact.

You can therefore find \(H(x, y)\) such that
\[\partial_x H(x, y) = 2xy^2, \quad \partial_y H(x, y) = x^2y + ye^y.
\]
Integrating the first equation with respect to \(x\) yields
\[H(x, y) = \int 2xy^2 \, dx = x^2y^2 + h(y).
\]
Plugging this expression for \(H(x, y)\) into the second equation gives
\[2x^2y + h'(y) = \partial_y H(x, y) = 2x^2y + ye^y,
\]
which yields \(h'(y) = ye^y\). One integration by parts then yields
\[h(y) = \int ye^y \, dy = ye^y - \int e^y \, dy = ye^y - e^y - c.
\]
Taking \(h(y) = (y - 1)e^y\), a general solution is therefore given implicitly by
\[x^2y^2 + (y - 1)e^y = c.
\]

(b) \((3x^2y^2 + 5x^4) \, dx + (2x^3y + 4y^3) \, dy = 0\).

**Solution:** This differential form is exact because
\[\partial_y(3x^2y^2 + 5x^4) = 6x^2y = \partial_x(2x^3y + 4y^3) = 6x^2y.
\]
We can therefore find \(H(x, y)\) such that
\[\partial_x H(x, y) = 3x^2y^2 + 5x^4, \quad \partial_y H(x, y) = 2x^3y + 4y^3.
\]
Integrating the second equation with respect to \(y\) yields
\[H(x, y) = \int (2x^3y + 4y^3) \, dy = x^3y^2 + y^4 + h(x).
\]
Plugging this expression for \(H(x, y)\) into the first equation gives
\[3x^2y^2 + h'(x) = \partial_x H(x, y) = 3x^2y^2 + 5x^4,
\]
which yields \(h'(x) = 5x^4\). Taking \(h(x) = x^5\), a general solution is therefore given implicitly by
\[x^3y^2 + y^4 + x^5 = c.
\]