1. (14 points)
Let \( f \) be the function \( f(x) = x^2 - 8 \ln x \) with domain \([1, 10] \).

(a) (4 pts) What properties of \( f \) and its domain guarantee that \( f \) will assume maximum and minimum values?

Solution.
\( f \) is continuous and the domain is a finite closed interval.

(b) (10 pts) What are the maximum and minimum values assumed by \( f \) on its domain?

Solution.
\( f'(x) = 2x - \frac{8}{x} \). So, \( f'(x) = 0 \) at \( x = 2 \).

Because \( f \) is differentiable, the max and min values can only be assumed at inputs from \( \{1, 2, 10\} \).

Minimum value is \( f(2) = 4 - 8 \ln(2) \) (by the first derivative test, or by comparing values).

Maximum value is \( f(10) = 100 - 8 \ln(10) \) (this number is larger than \( f(1) = 1 \)).
2. (10 points)
Find the equation of the tangent line to the curve $4e^{2x} - y^2 = 0$ at the point $(0, 2)$.

**Solution.**
Use implicit differentiation. Differentiating with respect to $x$:

$$4e^{2x} - y^2 = 0$$
$$8e^{2x} - 2yy' = 0$$
$$2yy' = 8e^{2x}$$
$$y' = \frac{4e^{2x}}{y}.$$ 

For $(x, y) = (0, 2)$, we have $y' = 4e^0/2 = 2$, and an equation for that tangent line is

$$y - 2 = 2x.$$
3. (15 points)
Let $f$ be the function with domain $[0, 2]$ defined by $f(x) = \sqrt{2x + 1}$.

(a) (7 pts) Compute the left endpoint Riemann sum estimate $\sum_{i=1}^{4} f(x_{i-1}) \Delta x$ of $\int_{x=0}^{2} f(x) \, dx$ when $n = 4$. (Do not simplify the expression you obtain from the definition.)

Solution.
\[ \sqrt{2(0) + 1 \cdot (1/2)} + \sqrt{2(1/2) + 1 \cdot (1/2)} + \sqrt{2(1) + 1 \cdot (1/2)} + \sqrt{2(3/2) + 1 \cdot (1/2)}. \]

(b) (5 pts) Draw the graph of $f$ and the rectangles corresponding to this Riemann sum.

Solution.
Not included for technical reasons.

(c) (3 pts) Is this Riemann sum greater or smaller than $\int_{x=0}^{2} f(x) \, dx$?

Solution.
Smaller.
4. (14 points)

Let \( f \) be the function on \([0, 4]\) defined by \( f(x) = (2x + 1)^{1/4} \). Let \( R \) be the “region under the curve”, i.e. the set of points \((x, y)\) such that \(0 \leq x \leq 4\) and \(0 \leq y \leq f(x)\). Let \( S \) be the solid of revolution obtained by rotating \( R \) about the \( x \)-axis.

What is the volume of \( S \)?

Solution.

\[
\text{volume}(S) = \int_{x=0}^{4} \pi [f(x)]^2 \\
= \int_{x=0}^{4} \pi (2x + 1)^{1/2} \\
= \pi \left[ \frac{1}{3} (2x + 1)^{3/2} \right]_{x=0}^{4} \\
= \pi \left( \frac{1}{3} (9^{3/2} - (1/3)(1)) \right) \\
= \frac{\pi}{3} (27 - 1) \\
= \frac{26\pi}{3} .
\]
5. (18 points)
   (a) (8 pts) Compute the average value of the function $f(x) = \sec^2(x)$ over the interval $[0, \pi/4]$.
   Solution.
   This average value $\bar{f}$ is
   \[
   \bar{f} = \frac{1}{(\pi/4)} \int_{x=0}^{\pi/4} \sec^2(x) \, dx \\
   = \frac{4}{\pi} \left[ \tan(x) \right]_{x=0}^{\pi/4} \\
   = \frac{4}{\pi} \left( \tan(\pi/4) - \tan(0) \right) \\
   = \frac{4}{\pi} (1 - 0) = \frac{4}{\pi}.
   \]

   (b) (10 pts) Evaluate the definite integral
   \[
   \int_{x=\pi/4}^{\pi/2} \sqrt{\sin x} \cos x \, dx
   \]
   Solution.
   We use a substitution $u(x) = u = \sin(x)$. Then $du/dx = \cos x$, and
   \[
   \int_{x=\pi/4}^{\pi/2} \sqrt{\sin x} \cos x \, dx = \int_{u=u(\pi/4)}^{u(\pi/2)} \sqrt{u} \, du \\
   = \int_{u=1/\sqrt{2}}^{1} \sqrt{u} \, du = \left[ \frac{2}{3} u^{3/2} \right]_{u=1/\sqrt{2}}^{1} \\
   = \frac{2}{3} - \frac{2}{3} (1/\sqrt{2})^{3/2} \\
   = \frac{2}{3} (1 - 2^{-3/4}).
   \]
6. (14 points)
Let $s(t)$ be the position of a certain object at time $t$. Suppose its velocity at time $t$ is $e^{2t}$, and suppose $s(0) = 1$.
What is the position of the object at time $t = 3$?
Solution.

$$s(3) - s(0) = \int_{t=0}^{3} e^{2t} \, dt = \left[ \frac{1}{2} e^{2t} \right]_{t=0}^{3}$$
$$= \frac{1}{2} e^{6} - \frac{1}{2} e^{0}$$
$$= (e^{6} - 1)/2 .$$

Therefore

$$s(3) = s(0) + (e^{6} - 1)/2$$
$$= 1 + (e^{6} - 1)/2$$
$$= \frac{e^{6} + 1}{2} .$$
7. (15 points) According to Poiseuille’s laws, the velocity \( v \) of blood in a blood vessel is given by \( v(r) = k(R^2 - r^2) \), where \( R \) is the (constant) radius of the blood vessel, \( r \) is the distance of the flowing blood from the center of the blood vessel, and \( k \) is a positive constant.

Given \( R \), let \( Q(R) \) be the total blood flow (in milliliter per minute) in the vessel. For \( n \) a positive integer, \( Q(R) \) is approximated by a sum

\[
\sum_{i=1}^{n} v(r_i)2\pi r_i \Delta r
\]

in which \( \Delta R = R/n \) and \( r_i = i\Delta r \). As \( n \) goes to \( \infty \), the sum converges to \( Q(R) \).

(a) (5 pts) Write a definite integral which equals \( Q(R) \).

Solution.

\[
\int_{r=0}^{R} v(r)2\pi r \, dr,
\]

which equals \( \int_{r=0}^{R} k(R^2 - r^2)2\pi r \, dr \).

(b) (10 pts) Compute the definite integral.

Solution.

\[
\int_{r=0}^{R} k(R^2 - r^2)2\pi r \, dr = 2\pi k \int_{r=0}^{R} (R^2 - r^2)r \, dr
\]

\[
= 2\pi k \int_{r=0}^{R} R^2r - r^3 \, dr
\]

\[
= 2\pi k \left[ (1/2)R^2r^2 - (1/4)r^4 \right]_{r=0}^{R}
\]

\[
= 2\pi k \left( (1/2)R^4 - (1/4)R^4 \right)
\]

\[
= 2\pi k (1/4) R^4 = \pi k (1/2) R^4
\]

\[
= \frac{\pi k R^4}{2}.
\]