The t tests

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1 Introduction

In this lecture we will derive the formulas for the two-sided t-test and the upper-tailed t-test for the mean in a normal distribution when the variance \( \sigma^2 \) is unknown. Let \( x_1, x_2, \ldots, x_n \) be a sample from a normal distribution with mean \( \mu \) and variance \( \sigma^2 \). Recall that \( \bar{X} \) is the sample mean (the point estimator for the populations mean \( \mu \)).

2 The two-sided t-test

We wish to give a test to decide between:

\[
H_0 : \mu = \mu_0 \\
H_a : \mu \neq \mu_0
\]

The two-sided t-test is the decision rule:
reject \( H_0 \) if either \( \bar{x} \leq \mu_0 - t_{\alpha/2,n-1}(\frac{s}{\sqrt{n}}) \) or \( \bar{x} \geq \mu_0 + t_{\alpha/2,n-1}(\frac{s}{\sqrt{n}}) \).

We will now prove that the two-sided t-test has significance level (i.e. Type I error probability) equal to \( \alpha \). We will need the following theorem from Probability Theory.

Theorem 1. \( T = (\bar{X} - \mu)/(\frac{s}{\sqrt{n}}) \) has standard normal distribution.

We now prove

Theorem 2. The two-sided t-test has significance level \( \alpha \).
Proof.

\[ P(\text{Type I error}) = P(\text{Reject } H_0 \text{ when } H_0 \text{ is correct}) \]
\[ = P(\bar{X} \leq \mu_0 - t_{\alpha/2,n-1} \frac{S}{\sqrt{n}} \text{ or } \bar{X} \geq \mu_0 + t_{\alpha/2,n-1} \frac{S}{\sqrt{n}} \text{ when } \mu = \mu_0) \]
\[ = P(\bar{X} - \mu_0 \leq -t_{\alpha/2,n-1} \frac{S}{\sqrt{n}} \text{ or } \bar{X} - \mu_0 \geq t_{\alpha/2,n-1} \frac{S}{\sqrt{n}} \text{ when } \mu = \mu_0) \]
\[ = P((\bar{X} - \mu_0)/(\frac{S}{\sqrt{n}}) \leq -t_{\alpha/2,n-1} \text{ or } (\bar{X} - \mu_0)/(\frac{S}{\sqrt{n}}) \geq t_{\alpha/2,n-1} \text{ when } \mu = \mu_0). \]

Now we use the assumption that \( \mu = \mu_0 \) to replace \( \mu_0 \) by \( \mu \) in the ratio \((\bar{X} - \mu_0)/(\frac{S}{\sqrt{n}})\).

Then we apply Theorem 1 above to deduce that the rewritten ratio \( T = (\bar{X} - \mu)/(\frac{S}{\sqrt{n}}) \) has t-distribution with \( n - 1 \) degrees of freedom. Thus we obtain the new equation

\[ P(\text{Type I error}) = P((T \leq -t_{\alpha/2,n-1} \text{ or } T \geq t_{\alpha/2,n-1}) = P(T \leq -t_{\alpha/2,n-1}) + P(T \geq t_{\alpha/2,n-1}). \]

Each of the two probabilities in the last term are equal to \( \alpha/2 \). To prove this draw a picture, the second is equal to \( \alpha/2 \) by definition, the second by symmetry.

\[ \square \]

3 The upper-tailed t-test

We wish to give a test to decide between:

\[ H_0 : \mu = \mu_0 \]
\[ H_a : \mu > \mu_0 \]

The upper-tailed t-test is the decision rule:

reject \( H_0 \) if \( \bar{x} \geq \mu_0 + t_{\alpha,n-1} \frac{S}{\sqrt{n}} \).

We will now prove that the two-sided t-test has significance level (i.e. Type I error probability) equal to \( \alpha \). Once again we will need Theorem 1.

We now prove

Theorem 3. The upper-tailed t-test has significance level \( \alpha \).
Proof.

\[ P(\text{Type I error}) = P(\text{Reject } H_0 \text{ when } H_0 \text{ is correct}) \]

\[ = P(\bar{x} \geq \mu_0 + t_{\alpha,n-1} \left( \frac{S}{\sqrt{n}} \right) \text{ when } \mu = \mu_0) \]

\[ P(\bar{X} - \mu_0 \geq t_{\alpha,n-1} \left( \frac{S}{\sqrt{n}} \right) \text{ when } \mu = \mu_0) \]

\[ = P(\frac{\bar{X} - \mu_0}{\frac{S}{\sqrt{n}}} \geq t_{\alpha,n-1} \text{ when } \mu = \mu_0). \]

Now we use the assumption that \( \mu = \mu_0 \) to replace \( \mu_0 \) by \( \mu \) in the ratio \( (\bar{X} - \mu_0)/(\frac{S}{\sqrt{n}}) \).

Then we apply Theorem 1 above to deduce that the rewritten ratio \( T = (\bar{X} - \mu)/(\frac{S}{\sqrt{n}}) \) has t-distribution with \( n-1 \) degrees of freedom. Thus we obtain the new equation

\[ P(\text{Type I error}) = P(T \geq t_{\alpha,n-1}). \]

This last probability is equal to \( \alpha \) by definition (draw a picture). \qed