Dot Product, Cross Product, Determinants

We considered vectors in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \). We will write \( \mathbb{R}^d \) for statements which work for \( d = 2, 3 \) (and actually also for \( d = 4, 5, \ldots \) although this is not needed in this course).

Our goal is to measure lengths, angles, areas and volumes. This will be used later for lengths of curves, surface areas, volumes with curved surfaces etc.

Review of Dot Product

For vectors \( \vec{a}, \vec{b} \in \mathbb{R}^d \) we define the dot product by

\[
\vec{a} \cdot \vec{b} = a_1 b_1 + \cdots + a_d b_d.
\]

The length or norm of a vector \( \vec{a} \in \mathbb{R}^d \) is defined by

\[
\|\vec{a}\| := (\vec{a} \cdot \vec{a})^{1/2} = \sqrt{a_1^2 + \cdots + a_d^2}.
\]

There holds

\[
|\vec{a} \cdot \vec{b}| \leq \|\vec{a}\| \|\vec{b}\| \quad \text{Cauchy-Schwarz inequality}
\]

\[
\|\vec{a} + \vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\| \quad \text{triangle inequality}
\]

If we denote the angle between nonzero vectors \( \vec{a}, \vec{b} \) by \( \theta \) there holds

\[
\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos \theta
\]

This can be used to compute the angle \( \theta \) between two vectors \( \vec{a}, \vec{b} \) by

\[
Q := \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \|\vec{b}\|}, \quad \theta = \cos^{-1} Q.
\]

By the Cauchy-Schwarz inequality we have \(-1 \leq Q \leq 1\), so applying the inverse cosine gives a value \( \theta \in [0, \pi] \).

Two vectors \( \vec{a}, \vec{b} \) are parallel if there exists a scalar \( q \in \mathbb{R} \) such that \( \vec{b} = q \vec{a} \). If \( q > 0 \) we have \( Q = 1 \) and \( \theta = 0 \). If \( q < 0 \) we have \( Q = -1 \) and \( \theta = \pi \).

Two vectors \( \vec{a}, \vec{b} \) are orthogonal if \( \vec{a} \cdot \vec{b} = 0 \). In this case we have \( Q = 0 \) and \( \theta = \pi/2 \).

Let \( \vec{a}, \vec{b} \in \mathbb{R}^d \) with \( \vec{a} \neq \vec{0} \). We want to decompose the vector \( \vec{b} \) into a vector parallel to \( \vec{a} \) and a vector orthogonal to \( \vec{a} \):

\[
\vec{b} = \vec{c} + \vec{d}, \quad \vec{c} \parallel \vec{a}, \quad \vec{d} \perp \vec{a}
\]

(1)

We must have \( \vec{c} = q \vec{a} \) where \( q \in \mathbb{R} \) is chosen such that \( \vec{d} = \vec{b} - q \vec{a} \perp \vec{a} \), i.e.,

\[
(\vec{b} - q \vec{a}) \cdot \vec{a} = 0 \iff \vec{b} \cdot \vec{a} - q \vec{a} \cdot \vec{a} = 0 \iff q = \frac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}}
\]

We define the projection of the vector \( \vec{b} \) onto the vector \( \vec{a} \) as this vector \( \vec{c} \), i.e.,

\[
\text{pr}_{\vec{a}} \vec{b} := \frac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}} \vec{a}
\]

Note that \( \|\text{pr}_{\vec{a}} \vec{b}\| = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|} \).
Area of a Parallelogram

For vectors $\bar{a}, \bar{b} \in \mathbb{R}^d$ we now consider the parallelogram with the four vertices $0, \bar{a}, \bar{b}, \bar{a} + \bar{b}$. The area $A$ of this parallelogram can be computed as base times height.

If the angle between the vectors $\bar{a}, \bar{b}$ is denoted by $\theta$ we obtain the area as

$$A = \|\bar{a}\| \|\bar{b}\| \sin \theta.$$  \hfill (2)

If we decompose $\bar{b} = \bar{c} + \bar{d}$ as in (1) we can write the area as

$$A = \|\bar{a}\| \|\bar{d}\| = \|\bar{a}\| \|\bar{b} - \text{pr}_a \bar{b}\|$$

Since $\bar{c} \perp \bar{d}$ we have $\|\bar{d}\|^2 = \|\bar{b}\|^2 - \|\bar{c}\|^2$ and obtain

$$A^2 = \|\bar{a}\|^2 \left(\|\bar{b}\|^2 - \|\text{pr}_a \bar{b}\|^2\right) = \|\bar{a}\|^2 \left(\|\bar{b}\|^2 - \frac{(\bar{a} \cdot \bar{b})^2}{\|\bar{a}\|^2}\right)$$

$$A^2 = \|\bar{a}\|^2 \|\bar{b}\|^2 - (\bar{a} \cdot \bar{b})^2$$  \hfill (3)

In order to write $A$ in a different way we consider

$$\sum_{i=1}^{d} \sum_{j=1}^{d} (a_i b_j - a_j b_i)^2 = \left(\sum_{i=1}^{d} a_i^2\right) \left(\sum_{j=1}^{d} b_j^2\right) - 2 \left(\sum_{i=1}^{d} a_i b_i\right) \left(\sum_{j=1}^{d} a_j b_j\right) + \left(\sum_{i=1}^{d} a_i^2\right) \left(\sum_{j=1}^{d} b_i^2\right)$$

$$= \|\bar{a}\|^2 \|\bar{b}\|^2 - 2(\bar{a} \cdot \bar{b})^2 + \|\bar{a}\|^2 \|\bar{b}\|^2 = 2A^2$$

Note that $(a_i b_j - a_j b_i)^2$ is zero for $i = j$, and that its value is the same if we interchange $i$ and $j$. Hence we obtain

$$A^2 = \sum_{i,j=1,\ldots,d \atop i < j} (a_i b_j - a_j b_i)^2$$  \hfill (4)

For $d = 2$ this sum has one term, for $d = 3$ this sum has 3 terms (for $d = 4$ it has 6 terms). We will now look at this formula for $d = 2$ and $d = 3$.

Parallelogram in $\mathbb{R}^2$

For $d = 2$ the sum in (4) contains only one term with $i = 1$, $j = 2$ and we have

$$A^2 = (a_1 b_2 - a_2 b_1)^2$$

$$A = |a_1 b_2 - a_2 b_1|.$$  \hfill (5)

Note that the expression $a_1 b_2 - a_2 b_1$ is the so-called determinant of the 2 by 2 matrix consisting of the vectors $\bar{a}, \bar{b}$:

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} := a_1 b_2 - a_2 b_1$$

The determinant may be negative. Its absolute value is the area of the parallelogram:

$$A = \left|\text{det} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\right|$$

The sign of the determinant specifies the orientation of two vectors $\bar{a}, \bar{b} \in \mathbb{R}^2$: A positive determinant means that the vector $\bar{b}$ is to the left of vector $\bar{a}$, a negative determinant means that the vector $\bar{b}$ is to the right of vector $\bar{a}$. 

Note that

- The sign of the determinant specifies the orientation of two vectors $\bar{a}, \bar{b} \in \mathbb{R}^2$: A positive determinant means that the vector $\bar{b}$ is to the left of vector $\bar{a}$, a negative determinant means that the vector $\bar{b}$ is to the right of vector $\bar{a}$. 

- The determinant may be negative. Its absolute value is the area of the parallelogram:

  $$A = \left|\text{det} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\right|$$
Parallelogram in $\mathbb{R}^3$

For $d = 3$ the sum in (4) contains three terms and we have

$$A^2 = (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2.$$ 

Hence $A = \|\vec{x}\|$ with the vector $\vec{x}$ defined by

$$\vec{x} := (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1).$$

Note that $x_j = a_kb_j - a_kb_k$ where

1. $j, k, \ell$ are different
2. $k, \ell$ are in “positive order” if we arrange 1, 2, 3 on a circle.

This vector $\vec{x}$ is the so-called cross product of the vectors $\vec{a}, \vec{b} \in \mathbb{R}^3$

$$\vec{a} \times \vec{b} := (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1).$$

Note that $\vec{x} := \vec{a} \times \vec{b}$ satisfies

$$\vec{a} \cdot \vec{x} = a_1(a_2b_3 - a_3b_2) + a_2(a_3b_1 - a_1b_3) + a_3(a_1b_2 - a_2b_1) = 0$$

$$\vec{b} \cdot \vec{x} = b_1(a_2b_3 - a_3b_2) + b_2(a_3b_1 - a_1b_3) + b_3(a_1b_2 - a_2b_1) = 0.$$ 

The cross product $\vec{a} \times \vec{b}$ therefore has the following properties:

1. $\|\vec{x}\|$ is the area $A$ of the parallelogram defined by $\vec{a}, \vec{b}$, i.e., $\|\vec{x}\| = \|\vec{a}\| \|\vec{b}\| \sin \theta$.
2. $\vec{x}$ is orthogonal to $\vec{a}, \vec{b}$.

Note that for two nonzero, non-parallel vectors $\vec{a}, \vec{b}$ there are two vectors $\vec{x}$ which satisfy these two conditions where one is the negative of the other. We will see below that the three vectors $\vec{a}, \vec{b}, \vec{x}$ should have “positive orientation” in the sense of the “right hand rule”: If the thumb of the right hand points in the direction of $\vec{a}$, the index finger in the direction of $\vec{b}$, then $\vec{x}$ points in the direction given by the middle finger. This resolves the ambiguity between the two possible vectors.

Area of Triangle

For vectors $\vec{a}, \vec{b} \in \mathbb{R}^d$ we consider the triangle with the three vertices $\vec{0}, \vec{a}, \vec{b}$. Obviously the area of this triangle is $\frac{1}{2}A$ where $A$ is the area of the parallelogram with vertices $\vec{0}, \vec{a}, \vec{b}, \vec{a} + \vec{b}$.

Volume of Parallelepiped in $\mathbb{R}^3$

Three vectors $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$ define a so-called parallelepiped. This is a polyhedron with 6 sides which are parallelograms. Its 8 vertices are given by the points $r\vec{a} + s\vec{b} + t\vec{c}$ where $r, s, t \in \{0, 1\}$. Its volume $V$ is computed as (area of base) times height.

If we consider the parallelepiped given by the vectors $\vec{a}, \vec{b}$ as the base, its area is $\|\vec{x}\|$ where $\vec{x} := \vec{a} \times \vec{b}$.

Note that the vector $\vec{x}$ is orthogonal on the parallelogram. Therefore the height $h$ is the component of the vector $\vec{c}$ in the direction of $\vec{x}$, i.e.,

$$h = \|\text{pr}_{\vec{x}}\vec{c}\| = \left|\frac{\vec{c} \cdot \vec{x}}{\|\vec{x}\|}\right|$$

yielding for the volume $V$ of the parallelepiped

$$V = \|\vec{x}\| \left|\frac{\vec{c} \cdot \vec{x}}{\|\vec{x}\|}\right| = \left|\vec{c} \cdot \vec{x}\right|$$

$$V = \left|\left(\vec{a} \times \vec{b}\right) \cdot \vec{c}\right|. \quad (6)$$
The expression \((\vec{a} \times \vec{b}) \cdot \vec{c}\) is called the determinant of the 3 by 3 matrix consisting of the vectors \(\vec{a}, \vec{b}, \vec{c}\): 

\[
\det \begin{bmatrix}
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 \\
c_1 & c_2 & c_3
\end{bmatrix} := (\vec{a} \times \vec{b}) \cdot \vec{c}
\]

\[
= a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1 - a_1 b_3 c_2 - a_2 b_1 c_3
\]

Note that this is the sum of terms \(\pm a_j b_k c_\ell\) where

1. \(j, k, \ell\) are different
2. “+” if \(j, k, \ell\) are in “positive order”, “−” if \(j, k, \ell\) are in negative order

if we arrange 1,2,3 on a circle.

The absolute value of the determinant gives the volume \(V\) of the parallelepiped. The sign of the determinant gives the orientation of the three vectors \(\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3\): A positive sign means that the vectors are arranged according to the “right hand rule”, a negative sign means that the vectors are arranged according to the “left hand rule”.

Note that for the three vectors \(\vec{a}, \vec{b}, \vec{x} := \vec{a} \times \vec{b}\) we have

\[
\det \begin{bmatrix}
\vec{a} \\
\vec{b} \\
\vec{x}
\end{bmatrix} = (\vec{a} \times \vec{b}) \cdot \vec{x} = \vec{x} \cdot \vec{x} \geq 0,
\]

i.e., the three vectors \(\vec{a}, \vec{b}, \vec{a} \times \vec{b}\) are positively oriented (if \(\vec{a} \parallel \vec{b}\) we have \(\vec{a} \times \vec{b} = \vec{0}\)).

**Volume of Tetrahedron in \(\mathbb{R}^3\)**

For vectors \(\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3\) we consider the tetrahedron with the four vertices \(\vec{0}, \vec{a}, \vec{b}, \vec{c}\). We can compute the volume of a tetrahedron as \((\text{area of base}) \cdot \text{height}/3\). Compared with the parallelepiped, the area of the base is half as large. The volume \(V\) of the parallelepiped was given as \(V = (\text{area of base}) \cdot \text{height}\). Hence the volume of the tetrahedron is \(\frac{1}{6}V\).