Practice problems for Exam #1

1. Consider the matrix \( A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 2 & 2 & 1 \\ 3 & 4 & 2 \end{bmatrix} \). You must use the result of (a) to answer (b)–(g).

I recommend that you check your result of (a) with the solution before continuing.

(a) Perform Gaussian elimination to find the row echelon form \( U \), the matrix \( L \) of multipliers, and the permutation vector \( p \). Always use the first available pivot candidate.

(b) Find a basis for \( \text{range} \ A \).

(c) Find a basis for \( \text{null} \ A \).

(d) Find a basis for \( \text{range} \ A^\top \).

(e) Find a basis for \( \text{null} \ A^\top \).

(f) Consider a linear system \( Ax = b \) where \( b \in \mathbb{R}^4 \) is given. How many conditions does the vector \( b \) have to satisfy so that a solution exists? State these conditions (using your earlier results).

(g) Consider the linear system \( Ax = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \). Find the general solution.

2. Assume we have a matrix \( A \in \mathbb{R}^{2 \times 4} \)

(a) What are the possible values for \( r = \text{rank} \ A \)? Give an example matrix \( A \) for each case!

(b) Assume \( \text{rank} \ A = 1 \). What is the dimension of the spaces \( \text{range} \ A, \text{null} \ A, \text{range} \ A^\top, A^\top ? \)

3. For a matrix \( A \in \mathbb{R}^{3 \times 5} \) we obtain the row echelon form \( U = \begin{bmatrix} 2 & 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \) and the permutation vector \( p = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \).

DO NOT find the matrix \( A \! \! \! \! \! \)!

(a) Which of the columns of the matrix \( A \) form a basis for the column space?

(b) Which of the rows of the matrix \( A \) form a basis for the row space?

(c) Find a basis for the orthogonal complement of the row space.