Here is the stuff you really need to remember for the exams.

1 Linear systems $Ax = b$

Problem:

We consider a linear system of $m$ equations for $n$ unknowns $x_1, \ldots, x_n$:
For a given matrix $A \in \mathbb{R}^{m \times n}$ and a vector $b \in \mathbb{R}^m$ we want to find all vectors $x \in \mathbb{R}^n$ such that $Ax = b$.

Algorithms:

Gaussian elimination: This gives the decomposition

\[
\begin{bmatrix}
\text{row } p_1 \text{ of } A \\
\vdots \\
\text{row } p_m \text{ of } A
\end{bmatrix} = LU
\]

- the row echelon form $U \in \mathbb{R}^{m \times n}$ has $r$ nonzero rows and $r$ nonzero pivots in columns $q_1, \ldots, q_r$. The variables $x_{q_1}, \ldots, x_{q_r}$ are called basic variables, the remaining variables are called free variables.
- the matrix $L \in \mathbb{R}^{m \times m}$ is lower triangular: the diagonal elements are 1, below the diagonal are the multipliers used in the elimination
- the permutation vector $p$ contains the numbers $1, \ldots, m$ in a scrambled order

For given $L, U, p$

- find basis for range $A$: Use columns $q_1, \ldots, q_r$ of $A$
- find basis for null $A$: Set one of the free variables to 1 and the others to zero. Then use $Ux = \mathbf{0}$ and back substitution to find the basic variables $x_{q_1}, \ldots, x_{q_r}$. This gives vectors $v^{(1)}, \ldots, v^{(n-r)}$.
- find basis for range $A^\top$: Use rows $1, \ldots, r$ of $U$
- find basis for null $A^\top$: For $j = 1, \ldots, m - r$: Solve $L^\top u = e^{(r+j)}$ by forward substitution. Then let

\[
\begin{cases}
    w_{p_1} := u_1 \\
    \vdots \\
    w_{p_m} := u_m
\end{cases}
\]

gives $m - r$ basis vectors $w^{(1)}, \ldots, w^{(m-r)}$. Here $e^{(k)} := [0, \ldots, 0, 1, 0, \ldots, 0]^\top$.

For given $L, U, p$ and right hand side vector $b \in \mathbb{R}^m$ find general solution of linear system $Ax = b$:

- solve $Ly = \begin{bmatrix} b_{p_1} \\ \vdots \\ b_{p_m} \end{bmatrix}$ by forward substitution
- find particular solution $x_{\text{part}}$ of $Ux = y$: Set all free variables to 0. Then use back substitution to find the basic variables $x_{q_1}, \ldots, x_{q_1}$.
- Find a basis $v^{(1)}, \ldots, v^{(n-r)}$ for null $A$ using $U$ (see above)
- The general solution is $x = x_{\text{part}} + c_1 v^{(1)} + \cdots + c_{n-r} v^{(n-r)}$ where $c_1, \ldots, c_{n-r} \in \mathbb{R}$ are arbitrary
Subspace $V = \text{span}\left\{ v^{(1)}, \ldots, v^{(k)} \right\}$ with given vectors $v^{(j)} \in \mathbb{R}^n$

- find basis for $V$:
  - Let $A$ have the rows $v^{(1)\top}, \ldots, v^{(k)\top}$, then $V = \text{range} A^\top$
  - Find row echelon form $U$ (we don’t need $L, p$)
  - use rows $1, \ldots, r$ of $U$
- find basis for orthogonal complement $V^\perp$:
  - since $V^\perp = \text{null} A$: use row echelon form $U$ to find basis for $\text{null} A$

Important facts to remember:

- range $A$ and range $A^\top$ have both dimension $r$
- $\text{null} A$ and range $A^\top$ are orthogonal complements in $\mathbb{R}^n$, $\dim \text{null} A = n - r$
- null $A^\top$ and range $A$ are orthogonal complements in $\mathbb{R}^m$, $\dim \text{null} A^\top = m - r$

Hence: $Ax = b$ has a solution $\iff b$ is orthogonal on basis vectors of $\text{null} A^\top$ (we have $m - r$ solvability conditions)

Case of square matrix $A \in \mathbb{R}^{n\times n}$

- If $r = \text{rank} A < n$ we say that $A$ is singular. If $r = n$ we say $A$ is nonsingular
- $A$ is singular $\iff \det A = 0$ $\iff$ rows of $A$ are linearly dependent $\iff$ columns of $A$ are linearly dependent
- If $A$ is singular: linear system $Ax = b$ has either no solution or infinitely many solutions
- If $A$ is nonsingular: linear system $Ax = b$ has a unique solution $x \in \mathbb{R}^n$.
Let $u^{(j)}$ denote the solution vector of $Au^{(j)} = e^{(j)}$, then $A^{-1} = [u^{(1)}, \ldots, u^{(n)}] \in \mathbb{R}^{n\times n}$ is called inverse matrix.

Then the solution of $Ax = b$ is given by $x = A^{-1}b$.

2 Least squares problem $\|Ac - b\| = \min$

Problem:

We have a matrix $A \in \mathbb{R}^{m\times n}$ with $m \geq n$ and linearly independent columns. Since the linear system $Ac = b$ does in general not have a solution, we want to find $c \in \mathbb{R}^n$ such that $\|Ac - b\|$ is as small as possible.

Application:

“Curve fitting”: We have measured experimental data $\begin{array}{cccc} t_1 & \cdots & t_m \\ y_1 & \cdots & y_m \end{array}$ and want to fit this with a curve

$$y = c_1 g_1(t) + \cdots + c_n g_n(t)$$

Here $g_1(t), \ldots, g_n(t)$ are given functions and $c_1, \ldots, c_n$ are unknown parameters which we want to find.

We want that the residuals $r_j = c_1 g_1(t_j) + \cdots + c_n g_n(t_j) - y_j$ are “small”:

Least squares fit: find $c_1, \ldots, c_n$ such that $r_1^2 + \cdots + r_m^2$ is minimal.

With $A = \begin{bmatrix} g_1(t_1) & \cdots & g_m(t_1) \\ \vdots & \ddots & \vdots \\ g_1(t_m) & \cdots & g_m(t_m) \end{bmatrix}$ we have $r = Ac - y$. Hence we want to find $c \in \mathbb{R}^n$ such that $\|Ac - y\|$ is minimal.
Algorithms:

- **Method 1: Normal equations:** Solve the $n \times n$ linear system $(A^TA)c = A^Tb$.
- **Method 2: Orthogonalization**
  - Find orthogonal basis $p^{(1)}, \ldots, p^{(n)}$ for span $\{a^{(1)}, \ldots, a^{(n)}\}$ using Gram-Schmidt process:
    
    $$p^{(1)} := a^{(1)}, \quad p^{(2)} := a^{(2)} - \frac{a^{(2)} \cdot p^{(1)}}{s_{12}} p^{(1)}, \quad p^{(3)} := a^{(3)} - \frac{a^{(3)} \cdot p^{(1)}}{s_{13}} p^{(1)} - \frac{a^{(3)} \cdot p^{(2)}}{s_{23}} p^{(2)}, \ldots$$

    yielding a decomposition $A = PS$ where
    
    $$P = \begin{bmatrix} p^{(1)}, \ldots, p^{(n)} \end{bmatrix} \text{ has orthogonal columns, } S = \begin{bmatrix} 1 & s_{12} & \cdots & s_{1n} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & s_{n-1,n} \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

    is upper triangular.

- Let $d_j := \frac{p^{(j)} \cdot b}{p^{(j)} \cdot p^{(j)}}$ for $j = 1, \ldots, n$
- solve $Sc = d$ by back substitution

3 The determinant $\det A$

For a square matrix $A \in \mathbb{R}^{n \times n}$

- $\det A = 0 \iff$ matrix is singular (i.e. columns are linearly dependent)
- expansion formula: Use row 1 or the column 1 to write $\det A$ in terms of $(n-1) \times (n-1)$ determinants:

  $$\det A = a_{11} \det A_{[1]} - a_{12} \det A_{[2]} + \cdots + (-1)^{n+1} a_{1n} \det A_{[n]}$$

  $$= a_{11} \det A_{[1]} - a_{21} \det A_{[2]} + \cdots + (-1)^{n+1} a_{n1} \det A_{[n]}$$

Here $A_{[jk]}$ denotes the matrix where we remove row $j$ and column $k$.
This also works with the row $j$ or column $j$ of the matrix, but with the factor $(-1)^{j-1}$.

- $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc, \quad \det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} = a \cdot \det \begin{bmatrix} e & f \\ h & k \end{bmatrix} - b \cdot \det \begin{bmatrix} d & f \\ g & k \end{bmatrix} + c \cdot \det \begin{bmatrix} d & e \\ g & h \end{bmatrix}$

4 Eigenvalue problem $Av = \lambda v$

4.1 Finding eigenvalues and eigenvectors

Problem:

A square matrix $A \in \mathbb{R}^{n \times n}$ describes a mapping $\mathbb{R}^n \to \mathbb{R}^n$. If we choose a new basis $v^{(1)}, \ldots, v^{(n)}$ of $\mathbb{R}^n$ this mapping is represented by the matrix $B = V^{-1}AV$ where $V = \begin{bmatrix} v^{(1)}, \ldots, v^{(n)} \end{bmatrix}$. We want to find a new basis such that the matrix $B$ becomes “as simple as possible”: if possible, we would like $B$ to be a diagonal matrix.

Hence want to find a vectors $v \neq \vec{0}$ and a numbers $\lambda$ such that $Av = \lambda v$. 
Algorithm:

How to find eigenvalues:

- find characteristic polynomial \( p(\lambda) = \det(A - \lambda I) \)

\[
\begin{bmatrix}
a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} - \lambda & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & \cdots & a_{n,n-1} & a_{nn} - \lambda \\
\end{bmatrix} = (-1)^n \lambda^n + c_{n-1} \lambda^{n-1} + \ldots + c_1 \lambda + c_0 = p(\lambda) \text{ by evaluating the determinant}
\]

- solve \( p(\lambda) = 0 \) to find the eigenvalues \( \lambda_1, \ldots, \lambda_n \).

For \( n = 2 \) you have to solve a quadratic equation. For \( n = 3 \) try to guess one eigenvalue \( \lambda_1 \), then you can write \( p(\lambda) = (\lambda - \lambda_1) q(\lambda) \) with a quadratic polynomial \( q(\lambda) \). I will give you hints how to find the eigenvalues if \( n > 2 \).

How to find eigenvectors: For each eigenvalue \( \lambda_1, \ldots, \lambda_n \) solve the linear system

\[
\begin{bmatrix}
a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} - \lambda & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & \cdots & a_{n,n-1} & a_{nn} - \lambda \\
\end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
\]

i.e., we have to find a basis for \( \ker(A - \lambda I) \). Since \( \det(A - \lambda I) = 0 \) we have \( r := \text{rank}(A - \lambda I) < n \), and we have to find \( n - r \) linearly independent vectors. We call \( \ker(A - \lambda I) \) the eigenspace for the eigenvalue \( \lambda \).

By taking the eigenvectors for all eigenvalues we obtain vectors \( v^{(1)}, \ldots, v^{(m)} \). These vectors are linearly independent, hence

- either \( m = n \) then the matrix is called diagonalizable, i.e., the vectors \( v^{(1)}, \ldots, v^{(n)} \) form a basis for all of \( \mathbb{R}^n \)
- or \( m < n \) then the matrix is called not diagonalizable

Note: eigenvalues and eigenvectors may be complex.

4.2 Case of diagonalizable matrices

If we can find \( n \) linearly independent eigenvectors \( v^{(1)}, \ldots, v^{(n)} \), then \( V = \begin{bmatrix} v^{(1)} \\ \vdots \\ v^{(n)} \end{bmatrix} \) is nonsingular. Then by changing to the new basis \( v^{(1)}, \ldots, v^{(n)} \) the matrix \( A \) becomes the diagonal matrix \( B = \begin{bmatrix} \lambda_1 & \cdot & \cdot \\ \cdot & \ddots & \cdot \\ \cdot & \cdot & \lambda_n \end{bmatrix} \): (Note that blank elements in the matrix are supposed to be zero)

\[
A \begin{bmatrix} v^{(1)} \\ \vdots \\ v^{(n)} \end{bmatrix} = B \begin{bmatrix} \lambda_1 & \cdot & \cdot \\ \cdot & \ddots & \cdot \\ \cdot & \cdot & \lambda_n \end{bmatrix}, \quad A = V B V^{-1}, \quad V^{-1} A V = B
\]

4.3 Case of non-diagonalizable matrices and the Jordan normal form

If all eigenvalues are different the matrix is always diagonalizable. If there are multiple eigenvalues it can happen that there are fewer eigenvectors than the multiplicity of the eigenvalue.

Assume, e.g., that \( \lambda = 2 \) is a triple eigenvalue, but \( \text{dim} \ker(A - 2 I) = 1 \), i.e., there is only one eigenvector \( v^{(1)} \). Then we can find find two generalized eigenvectors \( v^{(1),1}, v^{(1),2} \) satisfying

\[
(A - \lambda I) v^{(1)} = 0, \quad (A - \lambda I) v^{(1),1} = v^{(1)}, \quad (A - \lambda I) v^{(1),2} = v^{(1),1}
\]
We call \( v^{(1)}, v^{(1,1)}, v^{(1,2)} \) a **Jordan chain** of length 3. Note that we have

\[
A \begin{bmatrix} v^{(1)}, v^{(1,1)}, v^{(1,2)} \end{bmatrix} = \begin{bmatrix} v^{(1)}, v^{(1,1)}, v^{(1,2)} \end{bmatrix} \begin{bmatrix} \lambda & 1 \\ \lambda & 1 \\ & & \lambda \end{bmatrix}
\]

where instead of a diagonal matrix \( \begin{bmatrix} \lambda & 1 \\ \lambda & 1 \\ & & \lambda \end{bmatrix} \) we now have a so-called **Jordan box** \( \begin{bmatrix} \lambda & 1 \\ & & \lambda \\ & & & \lambda \end{bmatrix} \).

**Example**: assume that for \( A \in \mathbb{R}^{6 \times 6} \) we have \( \lambda_1 = \lambda_2 = \lambda_3 = 2, \lambda_4 = \lambda_5 = 3, \lambda_6 = 5 \). If there is only one eigenvector \( v^{(1)} \) for \( \lambda = 2 \) we can find a Jordan chain \( v^{(1)}, v^{(1,1)}, v^{(1,2)} \) of length 3. If there is only one eigenvector \( v^{(4)} \) for \( \lambda = 3 \) we can find a Jordan chain \( v^{(4)}, v^{(4,1)} \) of length 2. Then we have \( V = \begin{bmatrix} v^{(1)}, v^{(1,1)}, v^{(1,2)}, v^{(4)}, v^{(4,1)}, v^{(6)} \end{bmatrix} \) and \( AV = VB \) with

\[
B = \begin{bmatrix} 2 & 1 & 2 & 1 & 3 & 1 \\ 2 & & & & 3 & 5 \end{bmatrix}
\]

**Main result (Jordan normal form)**: For any matrix \( A \in \mathbb{R}^{n \times n} \) we can find a basis of Jordan chains for \( \mathbb{R}^n \). If \( V \) is the matrix which has these Jordan chains as columns, then we have \( AV = VB \) where the matrix \( B \) has Jordan boxes along the diagonal.

### 4.4 Symmetric matrices and quadratic forms

A quadratic form is a sum of terms with \( x_i x_j \). We can write it as

\[
q(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j = x^\top Ax
\]

with a **symmetric matrix** \( A \in \mathbb{R}^{n \times n} \). E.g., consider for \( n = 3 \) the quadratic form

\[
q(x) = x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - 2x_1x_3 - 2x_2x_3 = [x_1, x_2, x_3] \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
\]

**Change of basis**: For linearly independent vectors \( v^{(1)}, \ldots, v^{(n)} \) we can write \( x = y_1 v^{(1)} + \cdots + y_n v^{(n)} = Vy \) with the matrix \( V = \begin{bmatrix} v^{(1)}, \ldots, v^{(n)} \end{bmatrix} \). Then the quadratic form becomes

\[
q(x) = x^\top Ax = \underbrace{y^\top V^\top AV}_{B} y = y^\top By
\]

with \( B = V^\top AV \).

**Main result for a symmetric matrix** \( A \in \mathbb{R}^{n \times n} \): We can find an orthonormal basis \( v^{(1)}, \ldots, v^{(n)} \) of eigenvectors (i.e., \( V^\top V = I \) and \( V^{-1} = V^\top \)) such that \( V^\top AV = B = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \) with real **eigenvalues** \( \lambda_1, \ldots, \lambda_n \), hence the quadratic form becomes with the new variables

\[
q(x) = x^\top Ax = y^\top By = \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2
\]
• find characteristic polynomial \( p(\lambda) = \det(A - \lambda I) \)
• solve \( p(\lambda) = 0 \), this gives eigenvalues \( \lambda_1, \ldots, \lambda_n \) which are real
• for each eigenvalue \( \lambda \): find a basis for \( \text{null}(A - \lambda I) \)
  this gives \( n \) linearly independent eigenvectors \( u^{(1)}, \ldots, u^{(n)} \) (i.e., the matrix \( A \) is diagonalizable)
• for multiple eigenvalues: find an orthogonal basis by using the Gram-Schmidt process
  normalize the basis vectors: \( v^{(j)} := u^{(j)}/\|u^{(j)}\| \).

**Example:** For \( A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \) we obtain the eigenvalues \( \lambda_1 = -1, \lambda_2 = 2, \lambda_3 = 2 \).

For \( \lambda = -1 \) we have \( A - \lambda I = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \) which has rank 2, hence we get one eigenvector \( v^{(1)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \).

For \( \lambda = 2 \)

we have \( A - \lambda I = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix} \) which has rank 1, hence we get two eigenvectors \( v^{(2)} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, v^{(3)} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \).

Note that \( v^{(2)} \) and \( v^{(3)} \) are not orthogonal on each other. Therefore we can use Gram-Schmidt to replace \( v^{(3)} \) with

\[
\begin{align*}
  w^{(3)} &= v^{(3)} - \frac{v^{(3)} \cdot v^{(2)}}{v^{(2)} \cdot v^{(2)}} v^{(2)} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}
\end{align*}
\]

Now the three vectors \( \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix} \) are orthogonal on each other. We now divide each vector by its norm to obtain an orthonormal basis: \( \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \sqrt{\frac{2}{3}} \begin{bmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix} \).

Hence we obtain

\[
V = \begin{bmatrix}
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\
\frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{6}} \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & 0 \\
\end{bmatrix}, \quad B = V^T AV = \begin{bmatrix}
-1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 2 \\
\end{bmatrix}, \quad q(x) = y^T By = -y_1^2 + 2y_2^2 + 2y_3^2
\]

**Positive definite matrices, positive semidefinite matrices**

We say a symmetric matrix \( A \in \mathbb{R}^{n \times n} \) is **positive definite** if \( q(x) = x^T Ax > 0 \) for all \( x \neq 0 \).

We say a symmetric matrix \( A \in \mathbb{R}^{n \times n} \) is **positive semidefinite** if \( q(x) = x^T Ax \geq 0 \) for all \( x \neq 0 \).

**Result:** \( A \) is positive definite \( \iff \) all eigenvalues of \( A \) are positive
\( A \) is positive semidefinite \( \iff \) all eigenvalues of \( A \) are \( \geq 0 \).

**Example:** The matrix \( A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \) is not positive definite since it has a negative eigenvalue \( \lambda_1 = -1 \).