Solution: Practice Problem for Numerical Integration

1. Consider the integral \( I = \int_0^2 x^3 \, dx \).

   (a) Find the value of the (i) midpoint rule, (ii) trapezoid rule, (iii) Simpson rule (on the whole interval).
   Find an upper bound for the error \(|Q - I| \leq \cdots\) using the error formulas in each case.
   Here \( a = 0, b = 2 \) and \( f(x) = x^3 \), so we obtain
   \[
   Q_{\text{Midpt}} = 2 \cdot f(1) = 2, \quad Q_{\text{Trap}} = 2 \cdot \frac{f(0) + f(2)}{2} = 8, \quad Q_{\text{Simpson}} = 2 \cdot \frac{f(0) + 4 \cdot f(1) + f(2)}{6} = 4
   \]
   For the error estimates we have \( f''(x) = 6x \) and \( \max_{[0,2]} |f''(x)| = 12 \) yielding
   \[
   |Q_{\text{Midpt}} - I| \leq \frac{(b-a)^3}{24} \max_{[0,2]} |f''(x)| = \frac{2^3}{24} \cdot 12 = 4
   \]
   \[
   |Q_{\text{Trap}} - I| \leq \frac{(b-a)^3}{12} \max_{[0,2]} |f''(x)| = \frac{2^3}{12} \cdot 12 = 8
   \]
   \[
   |Q_{\text{Simpson}} - I| \leq \frac{(b-a)^5}{90 \cdot 32} \max_{[0,2]} |f^{(4)}(x)| = 0
   \]
   since we have \( f^{(4)}(x) = 0 \) in this case. Recall that the Simpson rule is actually exact if \( f(x) \) is a polynomial of degree \( \leq 3 \).

(b) Find the value of the composite trapezoid rule \( Q_{\text{Trap}}^2 \) with 2 subintervals of equal size.
   Here \( N = 2 \) and \( h = (b-a)/N = 1 \) yielding
   \[
   Q_{\text{Trap}}^2 = h \cdot \left[ f(0) + f(1) \right] + \left[ f(1) + f(2) \right] = 1 \cdot \frac{0 + 2 \cdot 1 + 8}{2} = 5
   \]

(c) Find a value \( N \) such that we can guarantee \( |Q_{\text{Trap}}^N - I| \leq 10^{-10} \) for the composite trapezoid rule with \( N \) intervals of equal size.
   Using \( \max_{[0,2]} |f''(x)| = 12 \) we obtain
   \[
   |Q_{\text{Trap}}^N - I| \leq \frac{1}{12} \frac{(b-a)^3}{N^2} \max_{x \in [a,b]} |f''(x)| = \frac{1}{12} \frac{2^3}{N^2} \cdot 12 = \frac{8}{N^2}
   \]
   We need to choose \( N \) such that
   \[
   \frac{8}{N^2} \leq 10^{-10} \iff \sqrt{\frac{8}{10^{-10}}} \leq N;
   \]
   i.e., \( N \geq 10^5 \cdot \sqrt{8} \approx 282842.7 \). Therefore we need \( N \geq 282843 \) to guarantee \( |Q_{\text{Trap}}^N - I| \leq 10^{-10} \).