Problem 1 (30 pts). A scientist took the following measurements of the concentration of a certain component of a chemical process:

<table>
<thead>
<tr>
<th>time (t)</th>
<th>1</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>10</th>
<th>12</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>concentration (c)</td>
<td>9.7</td>
<td>8.3</td>
<td>7.7</td>
<td>6.7</td>
<td>4.8</td>
<td>3.8</td>
<td>2.9</td>
</tr>
</tbody>
</table>

(a) Formulate the least squares problem for a straight line that best fits the data.
(b) Solve using the normal equations (use the command \ to solve the resulting linear system).
(c) Solve using the QR decomposition (use the MATLAB command [Q,R]=qr(A) to compute the QR decomposition of A; type help qr for an explanation of the output).
(d) What are the heights of the straight line at the given instants of time?
(e) What is the error at each time and its 2-norm?
(f) At what time do you think that the concentration will be zero?

The next three problems describe how eigenvalues can be calculated by iterative methods that employ the QR factorization, and provide some theoretical justification. There is no simple way to compute eigenvalues of matrices larger than $2 \times 2$. Calculating the roots of the characteristic polynomial does not work well numerically. The most successful algorithm for computing eigenvalues is based on the iterative use of the QR factorization. The MATLAB command eig implements this method. Consider the matrices

$$M_1 = \begin{bmatrix} 1 & -2 & 8 \\ 7 & -7 & 6 \\ 5 & 7 & -8 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 4 & -2 & 3 & -7 \\ 1 & 2 & 6 & 8 \\ 8 & 5 & 1 & -5 \\ -5 & 8 & -5 & 3 \end{bmatrix}.$$ 

Problem 2 (20 pts). Compute the eigenvalues and eigenvectors of $M_1$ and $M_2$ using the command eig. Type [V,D]=eig(A) to perform this task; help eig explains the meaning of both V and D.

Problem 3 (20 pts). Let $A \in \mathbb{R}^{n \times n}$. Let $A = Q_0R_0$ be a QR factorization of $A$ and create $A_1 = R_0Q_0$, that is change the order of multiplication. Let $A_1 = Q_1R_1$ be a QR factorization of $A_1$ and create $A_2 = R_1Q_1$. Explain why the following statements are true:
(a) $A = Q_0A_1Q_0^T$;
(b) $A = (Q_0Q_1)A_2(Q_0Q_1)^T$;
(c) $Q_0Q_1$ is orthogonal;
(d) $A, A_1,$ and $A_2$ all have the same eigenvalues.

The basic QR algorithm for eigenvalues computes iteratively the QR factorization of the $m$-th matrix $A_m = Q_mR_m$, and creates the next matrix $m + 1$ as $A_{m+1} = R_mQ_m$, and continues until the entries below the diagonal of $A_m$ are sufficiently small.

Problem 4 (30 pts). For both matrices $M_1$ and $M_2$, perform enough steps of the basic QR algorithm to make every entry below the diagonal smaller than 0.1. Record the number of steps for each matrix. Write a for or while loop to implement the iteration. The basic MATLAB commands for $M_1$ read as follows:

```matlab
A = M1;
[Q,R] = qr(A);
A = R*Q;
```