Selected Solutions to Stat 650 HW8

(I). We know from the discussion in class of nonhomogeneous Poisson processes as time-changed homogeneous Poisson processes that if \( N_0(t) \) is a homogeneous rate-1 Poisson process, then \( N_0(2t^2) \) is a nonhomogeneous Poisson process with exactly the same distribution as \( N(t) \). Equivalently, \( N_0(x) = N(\sqrt{x}/2) \) is a homogeneous unit-rate Poisson process. It follows from the notation for \( Z(t) \) in the problem that

\[
Z(t) = \int_0^t a(s) \, dN(s) = \int_0^{2t^2} a(x) \, dN_0(x) = \sum_{k=1}^{N_0(2t^2)} a(\tau_k)
\]

where \( \tau_k \) denotes the time of the \( k \)'th jump of the Poisson process \( N_0(t) \). We know about homogeneous Poisson processes that conditionally given that they have \( n \) jumps in a fixed time-interval \([0,T]\) the locations of those jumps \( \tau_1, \ldots, \tau_n \) are distributed the same as the order statistics \( U_{(1)} \), \ldots, \( U_{(n)} \) from an iid Uniform(0, \( T \)) sample \( \{U_i\}_{i=1}^n \). Thus, taking \( T = 2t^2 \), we find that \( Z(t) \) has the same distribution as

\[
\sum_{k=1}^{N_0(2t^2)} a(U_k), \quad U_k \overset{\text{iid}}{\sim} \text{Unif}[0, 2t^2]
\]

Thus, \( Z(t) \) is a compound-Poisson distributed random variable which is the sum of a Poisson(\( 2t^2 \))-distributed number of random variables \( a(U_k) \) each of which has the distribution function

\[
P(a(U_k) \leq x) = P(U_k \in a^{-1}([0, x]))
\]

which, for strictly increasing function \( a(u) \), is equal to \( a^{-1}(x)/(2t^2) \) for all \( x \in \{a(u) : 0 \leq u \leq 2t^2\} = [a(0), a(2t^2)] \).

(II). It is easy to check that \( M_1(t) \) and \( M_2(t) \) each have independent Poisson-distributed increments because, for \( 0 < s < t \),

\[
M_1(t) - M_1(s) = N(\{(t_1, t_2) : 0 \leq t_1 \leq t_2 \in (s, t]\})
\]

\[
M_2(t) - M_2(s) = N(\{(t_1, t_2) : t_1 \in (s, t] , \ 0 \leq t_2 \leq 1\})
\]
are planar-Poisson-process counts of event-occurrences within sets disjoint from the sets on which event-occurrences the increments up to time $s < t$ for the respective processes $M_1(\cdot), M_2(\cdot)$. The resulting process $M_1(t)$ is nonhomogeneous-Poisson because its cumulative rate-function is (its expectation, equal to) $\lambda t^2/2$, while $M_2(t)$ with mean $\lambda t$ is evidently a (homogeneous) Poisson process with (instantaneous) rate $\lambda$. 