Partial Solutions to Stat 700 HW2, Fall 2014

(1). The easiest solution is to take $F^{-1}(U)$ by the probability integral theorem, where $U \sim \text{Uniform}[0,1]$ and for $0 \leq r \leq 1$ the right-continuous inverse $F^{-1}(r)$ of $F$ is see from the definition to be

$$0 \cdot I_{[0 \leq r \leq 1/4]} + \{12r-3\} \cdot I_{[1/4 < r \leq 1/3]} + \{(3r-1)^2+1\} \cdot I_{[1/3 < r \leq 2/3]} + 2 \cdot I_{[2/3 < r \leq 1]}$$

(2). \(m_X(s) = E((\exp(s + 2s^2))^M).\)

(3). \(F(t) = pP(W \leq t) + (1-p)P(X \leq t) = p(1-0.9^t) + (1-p)(1-e^{-2t}).\)

(4). Since $X$ and $Y$ are jointly normal, we know from class that $X - Y \text{Cov}(X,Y)/\text{Var}(Y) = X - Y/4$ is normaly distributed and independent of $Y$. Then after calculating $\text{Var}(X - Y/4) = 3/4$, we have $X$ conditionally given $Y$ has $\mathcal{N}(-1/2 + Y/4, 3/4)$ distributed.

(5). (a) Conditionally given $Q$, $M \sim \text{NegBin}(2,Q)$. So unconditionally, $P(M = k) = E(E(I_{[M=k]} \mid Q)) = \int_0^1 \binom{k-1}{1} q^2(1-q)^{k-2} \frac{\Gamma(8)}{\Gamma^2(4)} q^3 (1-q)^3 dq = (k-1) \frac{\Gamma(8) \Gamma(6) \Gamma(k+2)}{\Gamma^2(4) \Gamma(k+8)}$

(b) Same idea using $L_1 \sim \text{NegBin}(20,Q)$ given $Q$. Variances are found from the formula $\text{Var}(L_1) = E(\text{Var}(L_1 \mid Q)) + \text{Var}(E(L_1 \mid Q)) = 10 E(\text{Var}(M) \mid Q) + 100 \text{Var}(E(M \mid Q))$.

(c) Here the pairs $(M_i, Q_i)$ are iid across $i = 1, \ldots, 10$, so that

$$\text{Var}(L_2) = 10 \text{Var}(M_1) = 10 E(\text{Var}(M \mid Q)) + 10 \text{Var}(E(M \mid Q)) < \text{Var}(L_1)$$

(6). By taking covariances, all 4 of the jointly normal variables whose squares we are summing are independent. Two of them are $2 \cdot \chi^2_1$ random variables and two are $4 \cdot \chi^2_1$ random variables. Since $\chi^2 = \Gamma(1/2, 1/2)$, we have $2 \cdot \chi^2_1 = \Gamma(1/2, 1/4)$ and $4 \cdot \chi^2_1 = \Gamma(1/2, 1/8)$, and sums of pairs of independent variables of these two types are respectively $\Gamma(1, 1/4)$ and $\Gamma(1, 1/8)$, or equivalently Expon(1/4) and Expon(1/8). Finally, the sum of two such independent random variables $X \sim \text{Expon}(\lambda)$, $Y \sim \text{Expon}(\mu)$ are, with $\lambda = 1/4$, $\mu = 1/8$:

$$f_{X+Y}(t) = \frac{d}{dt} E(I_{[X+Y \leq t]} \mid X) = \frac{d}{dt} \int_0^t \lambda e^{-\lambda x} (1 - e^{-\mu(t-x)}) dx = \frac{d}{dt} [1 - e^{-\lambda t} - \frac{\lambda e^{-\mu t}}{\mu - \lambda} (e^{(\mu-\lambda)t} - 1)] = \frac{\lambda \mu}{\mu - \lambda} (e^{-\lambda t} - e^{-\mu t})$$

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