Solutions of HW6 Problems in Stat 700

(1). (a) As we saw in class, \( n\bar{X} \sim \text{Poisson}(n\lambda) \) and \( E[(n\bar{X})^2 - n\bar{X}] = n(n\lambda)^2 + n\lambda - n\lambda = n^2\lambda^2 \), so that \( T = \bar{X}^2 - n^{-1}\bar{X} \) is unbiased for \( \lambda^2 \). Since \( T \) is obviously a function of the complete sufficient statistic \( X \), the Lehmann-Scheffé Theorem shows that \( T \) is the UMVUE.

(b) Writing the Poisson(\( \lambda \)) probability mass function in terms of the changed parameter \( \theta = \lambda^2 \), the Cramer-Rao bound is found directly as

\[
\left\{ n E\left( \frac{\partial^2}{\partial \theta^2} \log(e^{-\sqrt{n}\frac{\theta X/2}{X!}}) \right) \right\} = \left\{ n(-\frac{\theta^{-3/2}}{4} - E(-\frac{X}{2\theta^{1/2}})) \right\}^{-1} = \frac{4\theta^{3/2}}{n} = \frac{4\lambda^3}{n}
\]

(c) We can calculate \( \text{Var}(T) = ET^2 - (ET)^2 \) in terms of \( n\bar{X} = X_1 + \cdots + X_n \) as

\[
= n^{-4} E\left[n\bar{X}(n\bar{X}-1)(n\bar{X}-2)(n\bar{X}-3) + 4n\bar{X}(n\bar{X}-1)(n\bar{X}-2) + 2n\bar{X}(n\bar{X}-1) - \lambda^4 \right]
\]

= \( 4\lambda^3 + 2\lambda^2/n^2 \). The bound is not achieved (note that this variance of \( T \) is strictly larger than the C-R lower bound) because the parametric distribution in terms of \( \theta \) is not a natural exponential family.

(2). (a) For \( x \leq y < \vartheta \), we can calculate the probability (up to top-order terms in \( dx \) and \( dy \)) as

\[
P(V_1 \in [x, x+dx), V_{(n)} \in [y, y+dy)] = \frac{2x}{y^2} \left( \frac{x}{y} \right)^{2n-2} I_{[x=y]} + \frac{2x}{y^2} dx \frac{(2n-2) y^{2n-3}}{y^{2n-2}} I_{[x<y]}
\]

After dividing by the probability \( P(V_{(n)} \in [y, y+dy]) = 2my^{2n-1} dy / \vartheta^{2n} \), we find that the conditional probability distribution of \( V_1 \) at \( x \) given \( V_{(n)} \in [y, y+dy] \) is the mixture with weights \( 1/n \) and \( (n-1)/n \) respectively of a mass-point 1 at \( y \) with a density \( 2(x/y^2) I_{[0<x<y]} \), and the conditional distribution function for \( 0 < y < \vartheta \) is

\[
F_{V_1|V_{(n)}}(t|y) = \frac{1}{n} I_{[x=y]} + \frac{n-1}{n} \frac{(\min(y,t))^2}{y^2} I_{[t\geq 0]}
\]

Since \( V_{(n)} \) is a sufficient statistic by the factorization theorem and is easily checked to be complete, and since \( (3/2)V_1 \) is obviously unbiased for \( \vartheta \), the Lehmann-Scheffé Theorem implies that the UMVUE is

\[
\frac{3}{2} E(V_1 | V_{(n)}) = \frac{3}{2} \left\{ \frac{1}{n} V_{(n)} + \frac{n-1}{n} \int_0^{V_{(n)}} \frac{2y^2}{V_{(n)}^2} dy \right\} = \frac{2n+1}{2n} V_{(n)}
\]

This final result could have been obtained directly by noticing that \( E V_{(n)} = 2n\vartheta/(2n+1) \).

(3). Let \( \mu_j = 32, 36, 40 \) for \( j = 1, 2, 3 \), and denote \( \theta = (\mu, 1/\sigma^2) \), with prior probability the product of the probability mass function \( 0.3\delta_{\mu_1} + 0.4\delta_{\mu_2} + 0.3\delta_{\mu_3} \)
(in μ) and the density \(100t, e^{-10t}\) (which is Gamma(2, 10), in \(1/\sigma^2\) at argument \(t > 0\)). Then the marginal density for \(Y = (Y_1, \ldots, Y_n)\), which we will need for the denominator of our Bayes’-rule posterior density calculation, is

\[
f_Y(y) = \frac{(2\pi)^{-n/2} (0.3)^n}{\prod_{j=1}^{n} \Gamma(\frac{t}{j})^{\frac{j}{2}}} \frac{100 \Gamma((n/2) + 1)}{(10 + 0.5 [(n-1)S_y^2 + n(y - \mu_j)^2])^{n/2+2}}
\]

where \(S_y^2 = (n-1)^{-1} \sum_{i=1}^{n} (y_i - \bar{y})^2\), \(\bar{y} = n^{-1} \sum_{i=1}^{n} y_i\). In getting this expression, we integrated

\[
f_{Y|\theta}(y \mid m, t) = (2\pi)^{-n/2} t^{n/2} \exp \left( -\frac{(t/2)\sum_{i=1}^{n} y_i^2 - 2nm \bar{y} + nm^2}{(10 + 0.5 [(n-1)S_y^2 + n(\bar{y} - m)^2])} \right)
\]

Then the posterior mass-function/density for \(\theta\) at \((\mu_j, t)\) (discrete in the 1st coordinate, continuous in the 2nd) is

\[
\begin{align*}
\left(\frac{4}{3}\right)^{l_{i=1} n} t^{n/2+1} \exp \left( -\frac{(10 + 0.5 [(n-1)S_y^2 + n(\bar{y} - \mu_j)^2]) t}{(10 + 0.5 [(n-1)S_y^2 + n(\bar{y} - \mu_j)^2])} \right) \\
\sum_{k=1}^{n} \left(\frac{4}{3}\right)^{l_{k=1} n} \exp \left( -\frac{(10 + 0.5 [(n-1)S_y^2 + n(\bar{y} - \mu_k)^2]) t}{(10 + 0.5 [(n-1)S_y^2 + n(\bar{y} - \mu_k)^2])} \right)
\end{align*}
\]

With the given numbers \(S_y^2 = 6.25, \bar{y} = 37.2\) in this problem,

\[
(n - 1)S_y^2 + n(\bar{y} - \mu_j)^2 = \begin{cases} 99(6.25) + 100(5.2)^2 = 3322.75 \\ 99(6.25) + 100(2.2)^2 = 1102.75 \\ 99(6.25) + 100(2.8)^2 = 1402.75 \end{cases}
\]

and the ratio \((10 + 0.5(1102.75))/(10 + 0.5(1402.75)) = 1 - 0.21086\), which raised to the power 52 is really negligible. Therefore the posterior is approximately denoted at \(\mu_2 = 36\) in the \(\mu\) component, and in \(1/(\sigma^2) = t\) is Gamma(100/2 + 2, 10 + 0.5(1102.75)) = Gamma(52, 561.38).

(4). Recall throughout that \(X\) and \(Y\) are discrete random variables, with \(p_X(x, \theta) = \sum_{y : h(y) = x} p_Y(y, \theta)\). Then we compute directly from the definition, freely passing the derivative with respect to \(\theta\) across the possibly infinite summation over all \(y\) such that \(h(y) = x\):

\[
I_X(\theta) = E \left[ \left( \frac{\partial}{\partial \theta} \log p_X(X, \theta) \right)^2 \right] = \sum_x \frac{1}{p_X(x, \theta)} \left( \frac{\partial}{\partial \theta} p_X(x, \theta) \right)^2
\]

\[
= \sum_x \frac{1}{p_X(x, \theta)} \left( \sum_{y : h(y) = x} \frac{\partial}{\partial \theta} p_Y(y, \theta) \right)^2
\]

\[
= \sum_x \frac{1}{p_X(x, \theta)} \left( \sum_{y : h(y) = x} \sqrt{p_Y(y, \theta)} \left( \frac{1}{\sqrt{p_Y(y, \theta)}} \frac{\partial}{\partial \theta} p_Y(y, \theta) \right) \right)^2
\]

2
Applying the Cauchy-Schwarz inequality to the final $(\cdot)^2$, we find

$$I_X(\vartheta) \leq \sum_x \frac{1}{p_X(x, \vartheta)} \sum_{y: h(y) = x} p_Y(y, \vartheta) \cdot \sum_{y: h(y) = x} \left( \frac{\partial}{\partial \vartheta} p_Y(y, \vartheta) \right)^2 / p_Y(y, \vartheta)$$

$$= \sum_x \sum_{y: h(y) = x} \left( \frac{\partial}{\partial \vartheta} p_Y(y, \vartheta) \right)^2 / p_Y(y, \vartheta) = I_Y(\vartheta)$$

(5) (a) First we note that by independence of sample elements, $E(X_1 X_2) = p^2$ is unbiased for $p^2$ in this setting where the complete sufficient statistic is $S_n = n\bar{X}$. So Lehmann-Scheffé implies that the unique UMVUE is the Rao-Blackwellized estimator (when evaluated at $m = S_n$)

$$E(X_1 X_2 \mid n\bar{X} = m) = \sum_{j=0}^1 \sum_{k=0}^1 j k P(X_1 = j, X_2 = k \mid S_n = m)$$

$$= \sum_{j=0}^1 \sum_{k=0}^1 jk p^2 \left( \frac{n-2}{m-j-k} \right) p^{m-j-k} (1-p)^{n-2-(m-j-k)} \left[ \binom{n}{m} p^m (1-p)^{n-m} \right]$$

and this last expression is immediately checked to be equal to $\binom{n-2}{m-2} / \binom{n}{m} = (m(m-1))/(n(n-1))$. Thus, substituting $S_n = n\bar{X}$ for $m$, we find the UMVUE $X(n\bar{X} - 1)/(n-1)$.

(6) The posterior is proportional (as a function of the parameter $\lambda$) to $\gamma e^{-\gamma \lambda} e^{-n\lambda} \lambda^{n\bar{Y}}/(Y_1! \cdots Y_n!)$, and therefore is Gamma($n\bar{Y} + 1, n + \gamma$). In general, for $Z \sim \text{Gamma}(\alpha, \beta)$, we minimize the convex function $E((a - Z)^2/Z) = E(Z) - 2a + E(a^2/Z)$ of the real variable $a$ by setting the derivative equal to 0, i.e.,

$$a = 1/ E(1/Z) = 1/ \int_0^{\infty} \frac{\beta^\alpha}{\Gamma(\alpha)} t^{\alpha-1} e^{-\beta t} dt = 1/ \left( \frac{\beta}{\alpha-1} \right) = \frac{\alpha - 1}{\beta}$$

Substituting $\alpha = n\bar{Y} + 1, \beta = n + \gamma$ for our posterior density, we find that the Bayes optimal estimator for this strange loss function is $n\bar{Y} / (n + \gamma)$. Note that these last lines correct an error in my previous posted solution: the answer is different from the optimal estimator $(n\bar{Y} + 1)/(n + \gamma)$ under the squared-error loss function.