(1). Here the likelihood is

\[ L(X, \rho, \lambda) = \lambda^m X e^{-m \lambda} (\rho \lambda)^n Y e^{-n \lambda \rho} \]

and the gradient in \((\rho, \lambda)\) is

\[ \nabla \log L = \begin{pmatrix} n \hat{Y}/\rho - n \lambda \\ (m \hat{X} + n \hat{Y})/\lambda - (m + n \rho) \end{pmatrix} \]

Direct maximization of \(\log L\) both in general and under the restriction \(H_0: \rho = 1\) immediately yields, in terms of the notation \(\pi = m/(m + n)\),

\[ \hat{\rho} = \frac{\hat{Y}}{\hat{X}} , \quad \hat{\lambda} = \hat{X} , \quad \hat{\lambda}^{(r)} = \pi \hat{X} + (1 - \pi) \hat{Y} \]

Next, the per-observation Fisher Information matrix for \(\theta = (\rho, \lambda)\) is given by

\[ \frac{1}{m + n} E \left( \frac{n \hat{Y}/\rho^2}{\pi} \frac{1}{\lambda^2} \right) = \begin{pmatrix} (1 - \pi) \rho \lambda & 1 - \pi \\ 1 - \pi & (\pi + (1 - \pi) \rho)/\lambda \end{pmatrix} \]

Therefore the inverse per-observation information matrix is

\[ I^{-1} = \frac{\rho}{\pi(1 - \pi)} \begin{pmatrix} (\pi + (1 - \pi) \rho)/\lambda & -(1 - \pi) \\ -(1 - \pi) & (1 - \pi) \lambda/\rho \end{pmatrix} \]

From this we find the Wald statistic as \((m + n)(\hat{\rho} - 1)^2/(I^{-1})_{11}\) or

\[ W_{m,n} = (m + n)(\frac{\hat{Y}}{\hat{X}} - 1)^2 \frac{\pi(1 - \pi) \hat{\lambda}^{(r)}}{\rho(\pi + (1 - \pi) \rho)} = \frac{mn}{(m \hat{X} + n \hat{Y})^2} \frac{(\hat{Y} - \hat{X})^2 \hat{X}}{Y} \]

and the Rao Score statistic as \((\nabla \log L)^2 (I^{-1})_{11}/(m + n)\) evaluated at \(\rho = 1\), \(\hat{\lambda}^{(r)} = \pi \hat{X} + (1 - \pi) \hat{Y}\), or

\[ R_{m,n} = \frac{n^2}{\pi(1 - \pi)(m \hat{X} + n \hat{Y})} \left( \frac{\hat{Y} - m \hat{X} + n \hat{Y}}{m + n} \right)^2 = \frac{mn}{m \hat{X} + n \hat{Y}} \frac{(\hat{Y} - \hat{X})^2}{(m \hat{X} + n \hat{Y})} \]

It is easy to see from the previous LRT derivation that the statistic \(T_n\) is

\[ -2 \left[ m \hat{X} \log(1 + \frac{\hat{\lambda}^{(r)} - \hat{X}}{\hat{X}}) + n \hat{Y} \log(1 + \frac{\hat{\lambda}^{(r)} - \hat{Y}}{\hat{Y}}) \right] \]
for which the last inequality fails. For the given data, this says that our

\[ \rho \] for \( N \)

Thus the Likelihood Ratio test with asymptotically valid cutoff for the hypothesis

\( H_0 : p_2 = \rho p_1 \) is to reject when \( N_1 \log(N_1(1+\rho)/(N_1+N_2)) + N_2 \log(N_2(1+\rho)/(\rho(N_1+N_2))) \geq 0.5 \chi^2_{1.05} = 1.921. \)

(b) Inverting the test means defining the confidence set consisting of all \( \rho \) for which the last inequality fails. For the given data, this says that our interval is

\[ \{ \rho : 132 \log(1+\rho) - 85 \log \rho \leq 1.921 - 47 \log \frac{47}{132} - 85 \log \frac{85}{132} \} \]
It is easy to check that the function on the left-hand side of this inequality is strictly decreasing for all values $\rho < 85/47$ and strictly increasing for all $\rho > 85/47$, and the minimum value achieved at $85/47$ is 85.948, which is strictly less than the value 87.869 on the right-hand side. Thus the displayed set is indeed an interval, with the endpoints $(1.2730, 2.6009)$ found as the only roots of the left-hand side minus the right-hand side.

(5). (a) The Neyman-Pearson Lemma says that the Most Powerful test of this simple versus simple hypothesis has the likelihood ratio form, rejecting when
\[
\sum_{i=1}^{n} \log \left( \frac{X_i + 10}{15.5} \right) \geq C
\]
where $C$ giving approximate size 0.05 (in large samples $n$ is found via the Central Limit Theorem as
\[
n\mu + 1.6445 \sqrt{n} \left( 0.1 \sum_{k=1}^{10} \{ \log((k + 10)/15.5) - \mu \}^2 \right)^{1/2}
\]
where $\mu = \sum_{k=1}^{10} (1/10) \log((k + 10)/15.5)$. Thus $C$ would be taken equal to $-0.01772n + 0.31231 \sqrt{n}$.

(b) With the roles of $H_0$ and $H_A$ reversed, we would reject the hypothesis that $q$ is the correct probability mass function, at approximate size $\alpha$, if
\[
\sum_{i=1}^{n} \log \left( \frac{X_i + 10}{15.5} \right) \leq C' = n\tau - 1.96 \sqrt{n} \left( \sum_{k=1}^{10} \frac{k + 10}{155} \log(\frac{k+1}{15.5}) \right)^2
\]
where $\tau = \sum_{k=1}^{10} \frac{k+10}{155} \log(\frac{k+1}{15.5})$. Thus $C' = 0.01735n - 0.30257 \sqrt{n}$.

(c) This means that we would be in the situation of rejecting each hypothesis in favor of the other at significance level 0.05 if the statistic $\sum_{i=1}^{n} \log \left( \frac{X_i + 10}{15.5} \right)$ fell between $-0.01772n + 0.31231 \sqrt{n}$ and $0.01735n - 0.30257 \sqrt{n}$. This can happen if $0.61488 \leq 0.03607 \sqrt{n}$, or $n \geq 291$. For smaller $n$ values, the opposite can happen, namely statistic values for which both of the two hypothesis tests accept. For $n = 100$, under the hypothesis that the discrete-uniform probability mass function $p(\cdot)$ is the correct one, the probability of both hypotheses accepting, expressed in terms of the approximate standard normal random variable $Z = \left| \sum_{i=1}^{n} \log \left( \frac{X_i + 10}{15.5} \right) + 0.01772n \right|/0.18991 \sqrt{n}$, is
\[
P \left( \frac{0.01735n - 0.30257 \sqrt{n} + 0.01772n}{0.18991 \sqrt{n}} \leq Z \leq 1.6445 \right)
\]
which for $n = 100$ is equal to 0.790.