Some problems related to Stoke’s and Divergence theorems

Math 241H

1. (a) Show by direct calculation that the divergence theorem does not hold for $\mathbf{F}(r, \theta, \psi) = \frac{r}{r^2}$, where $\mathbf{r}$ denotes the unit radial vector. Why does the theorem fail?
(b) Verify by direct calculation that the divergence theorem does hold for the $\mathbf{F}$ from part (a) when $S$ is the surface $S_1$ of a sphere of radius $R_1$ plus the surface $S_2$ of a sphere of radius $R_2$, both centered at the origin, and $D$ is the region between the two surfaces?
(c) In general, what restriction must be placed on a surface $S$ so that the divergence theorem will hold for the function of part (a)?

2. Use the divergence theorem to show that
$$\int \int_S \mathbf{n} \, dS = 0$$
where $\mathbf{n}$ is the unit vector normal to the surface $S$.

3. Let $S$ be the surface of the sphere $x^2 + y^2 + z^2 = 9$. Evaluate
$$\int_S x^2 \, dy \, dz + y^2 \, dz \, dx + z^2 \, dx \, dy$$
using the divergence theorem.