Homework 1 – due 10/24/03
Math 603

Do at least 3 problems to earn a grade of √. Do more to earn a √+.


4. (a) Suppose $A$ is an integrally closed Noetherian domain with field of fractions $K$. Suppose $L/K$ is a finite separable extension, and let $B$ denote the integral closure of $A$ in $L$. Prove that $B$ is a finite $A$-module. Hint: Use Atiyah-Macdonald, Prop. 5.17.

(b) Let $k$ be a field and suppose $B$ is a domain which is also a finitely generated $k$-algebra. Let $L = \text{Frac}(B)$ and let $	ilde{B}$ denote the integral closure of $B$ in $L$. Assume that $\text{char}(k) = 0$ (for simplicity). Prove that $	ilde{B}$ is a finite $B$-module and a finitely generated $k$-algebra. Hint: use the Noether Normalization lemma applied to $B$ together with part (a). (Remark: Spec($\tilde{B}$) is called the normalization of the scheme Spec($B$). This exercise implies, for example, that the normalization of an irreducible variety is still a variety, because it is still finite-type over the coefficient field.)

(c) Suppose $A$ is a domain which is also a finitely generated algebra over a field $k$ of characteristic zero. Let $L$ be a finite extension of $K = \text{Frac}(A)$, and let $B$ be the integral closure of $A$ in $L$. Prove that $B$ is a finite $A$-module and a finitely-generated $k$-algebra. (Remark: This is essentially Theorem 3.9A in Hartshorne’s book Algebraic Geometry, and is stated there – without proof – in the more general situation where $k$ need not have characteristic zero.)

5. Let $k$ be any algebraically closed field, and let $B$ be a domain which is a finitely generated $k$-algebra. Let $x$ be a closed point of Spec($B$), and let $U \subset \text{Spec}(B)$ be a non-empty open set. Show that there is a curve in Spec($B$) joining $x$ to some point in $U$. (Note: for the purposes of this exercise, a curve in Spec($B$) is a closed set of form $V(p)$, $p$ a prime ideal, where $\dim(B/p) = 1$). Hint: do it first in the case $B = k[X_1, \ldots, X_n]$, then reduce to this case by using the Noether Normalization and the Going-Down theorem.