1. Let $G$ be a connected 1-dimensional algebraic group. Show, using what we have proved in class, that $G$ is commutative. (It turns out that $G$ is isomorphic to $\mathbb{G}_m$ or $\mathbb{G}_a$, but you are not allowed to use this.)

\textit{Solution:} This is in the standard texts (Humphreys 20.1, and Springer 3.1.3). Here is the argument given by Springer: fix $g \in G$ and consider the morphism $\phi(x) = xgx^{-1}$. If $G$ is not commutative, then we may suppose $g$ is non-central, and then $\phi$ is non-constant. In that case the closed irreducible subset $\phi(G)$ is not a point, so must be all of $G$. Since the image $\phi(G)$ contains an open dense subset of $\phi(G) = G$, and since $\dim(G) = 1$, it follows that the complement $G - \phi(G)$ is finite. Now choose a faithful representation of $G$ in some $\text{GL}_n$. There are only finitely many possibilities for the characteristic polynomials $\det(T-x)$ of $x \in G$. Since the coefficients of $\det(T-x)$ are regular functions on $G$ and $G$ is connected, they must be constant on $G$. But then every element of $G$ satisfies $(T-1)^n = 0$, i.e., $G$ is unipotent, and thus also nilpotent. From this we see that $(G,G) = 1$, since it is closed, connected, and of dimension strictly less than $\dim(G) = 1$. Thus, $G$ is commutative.

2. Let $G$ be a connected algebraic group whose elements are semisimple. Show that $G$ is a torus. (Hint: consider a Borel subgroup.)

\textit{Solution:} Let $B$ denote a Borel subgroup. As we proved in class, we may write $B = T \ltimes U$, where $U := B_u$, the set of unipotent elements in $B$, and $T$ is a maximal torus in $B$ and in $G$. By hypothesis we have $U = 1$, so that $B = T$.

Let $t \in T$ and consider the automorphism $\sigma(x) = txt^{-1}$, and the associated map $\chi(x) = \sigma(x)x^{-1}$. Clearly $\chi(B) = 1$ since $B = T$ is commutative, and so $\chi$ induces a morphism $\bar{\chi} : G/B \to G$ defined by $\bar{\chi}(xB) = \chi(x)$. Since $G/B$ is complete, the image of $\bar{\chi}$ is closed and complete in the affine variety $G$, and is thus finite. Being connected, that image must be trivial. This shows that $\chi(x) = 1$ for all $x$, that is, $t \in Z(G)$. Since $t$ was arbitrary, we see $T \subset Z(G)$.

Since $T = B$ is normal in $G$, the space $G/B = G/T$ is simultaneously connected, complete, and affine. Hence $G/T$ is a point, and $G = T$, as desired.

3. (a) Assume $\text{char}(k) = 2$ and let $G = \text{SL}_2$. Denote by $\sigma$ the inner automorphism defined by

$$u = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$ 

Show that $L(G_{\sigma}) \neq g_{\sigma}$.

(b) Let $G = \text{GL}_n$ and let $\sigma$ be any inner automorphism. Show that $L(G_{\sigma}) = g_{\sigma}$.

\textit{Solution:} (a) Fix a $2 \times 2$ matrix $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then the condition that $X$ commutes with $u$ is that $c = 0$ and $a = d$. 


Now $G_\sigma$ consists of such matrices which have determinant 1, that is

$$G_\sigma = \{ \begin{bmatrix} \pm 1 & b \\ 0 & \pm 1 \end{bmatrix} \},$$

from which we see $\dim(G_\sigma) = 1$.

Recall that $\mathfrak{sl}_2$ consists of the trace zero $2 \times 2$ matrices. So, $g_\sigma$ consists of the trace zero matrices $X$ with $c = 0$ and $a = d$. Thus

$$g = \{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \},$$

the trace zero condition holding automatically because $\text{char}(k) = 2$. We see that $\dim(g_\sigma) = 2$, and hence $L(G_\sigma) \neq g_\sigma$.

(b) Let $A$ denote an invertible $n \times n$ matrix, and let $X$ denote any $n \times n$ matrix. Let $\sigma$ be the automorphism $X \mapsto AXA^{-1}$. To prove $L(G_\sigma) = g_\sigma$, it is enough to show $\dim(G_\sigma) = \dim(g_\sigma)$, because we always have $L(G_\sigma) \subset g_\sigma$.

Note that the centralizer $G_\sigma = C_{\text{GL}_n}(A)$ is the principal open subset given by $\det \neq 0$ in $g_\sigma = C_G(A)$, the Lie subalgebra of $g = \mathfrak{gl}_n$ consisting of all $n \times n$ matrices which commute with $A$. Now $g_\sigma$ is itself an algebraic variety as well as a $k$-vector space, and the $k$-vector space operations are given by morphisms of algebraic varieties over $k$. It follows that $g_\sigma$ is an irreducible algebraic variety (choosing a $k$-basis for $g_\sigma$ allows us to define a bijective morphism $A_k^l \to g_\sigma$ for some $l \geq 1$, proving that $g_\sigma$ is irreducible). Finally, this implies that $G_\sigma$ is irreducible, since non-empty principal open subsets in irreducible affine varieties are open and dense (it is obvious that $G_\sigma \neq \emptyset$).

4. Let $G$ be a connected closed subgroup of $\text{GL}_n$. Assume that the subspace $g \subset \mathfrak{gl}_n$ has a complement that is stable under $\text{Ad}(G)$.

(a) If $\sigma$ is an inner automorphism of $G$, then $L(G_\sigma) = g_\sigma$. (Hint: proceed as in the beginning of the proof of the analogous statement we proved in class under the assumption that $\sigma$ is a semisimple automorphism.)

(b) Let $x \in G$ and let $C$ be the conjugacy class of $x$ in $\text{GL}_n$. Show that $C \cap G$ consists of finitely many conjugacy classes in $G$. (Hint: let $X$ be a component of $C \cap G$ containing $x$ and let $D$ be the conjugacy class of $x$ in $G$; using (a) show that $T_x X = T_x D$ and $X = D$.)

(c) Show that the number of unipotent conjugacy classes in $\text{GL}_n$ is finite (Use Jordan normal forms.)

Solution:

(a): Let $\sigma$ denote the inner automorphism $g \mapsto hgh^{-1}$ of $G$, which of course extends to an inner automorphism of $\text{GL}_n$. Consider the morphism $\chi(x) = \sigma(x)x^{-1}$, thought of as a map $G \to G$ and also a map $\text{GL}_n \to \text{GL}_n$. We also have $\psi : G \to \chi(G)$ (and its $\text{GL}_n$-variant), sending $x \mapsto \chi(x)$. Note that $d\psi_e = \text{Ad}(h) - 1$ (similarly for $\text{GL}_n$).

We discussed in class the equivalences

$$L(G_\sigma) = g_\sigma \iff d\psi_e \text{ is surjective} \iff \psi \text{ is separable}.$$

By the part #3(b) we know part (a) holds for $\text{GL}_n$, and so in that case we know from (1) that the $\text{GL}_n$-variant of $d\psi_e$ is surjective. Write $\mathfrak{gl}_n = g \oplus m$, where $m$ is an $\text{Ad}(G)$-stable complement. Given $X \in g$ in $T_e(\chi(G)) \subset T_e(\chi(\text{GL}_n))$, there is $Y \in T_e(\text{GL}_n) = \mathfrak{gl}_n$ such that $d\psi^{\text{GL}_n}_e(Y) = (\text{Ad}(h) - 1)(Y) = X$. Since the complement $m$ is stable under $\text{Ad}(h) - 1$, we can assume $Y \in g$, showing that $d\psi_e : T_e(G) \to T_e(\chi(G))$ is surjective. Thus, using (1) again, we deduce that $L(G_\sigma) = g_\sigma$. 


(b) Fix a GL$_n$-conjugacy class $C$, and consider the variety $C \cap G$. This is stable under the $G$-conjugacy action, hence is a union of $G$-conjugacy classes. Also, being a variety it has only finitely many irreducible components $X$. It is ETS that each such $X$ is itself a $G$-conjugacy class. We will prove a weaker statement, which suffices for our purposes.

We will establish the following two statements, which imply the result we want.

(i): Every $G$-conjugacy class $D \subset C \cap G$ is contained in some component $X$ of $C \cap G$.

(ii): Each component $X$ of $C \cap G$ contains at most one $G$-conjugacy class $D$.

It follows that the number of $D$’s is at most the number of $X$’s, and is thus finite.

**Proof of (i):** Since $C \cap G = \cup_{X}X$, intersecting this with $D$ gives $D = \cup_{X}X \cap D$, a finite union of closed subsets of $D$. Since $D$ is irreducible, for some $X$ we have $D = X \cap D$, that is, $D \subset X$.

**Proof of (ii):** Suppose $D \subset X$ and choose $x \in D$. We want to prove that $\dim(D) = \dim(X)$, for then it follows that $D$ contains an open dense subset of $X$. This would apply to any other $G$-conjugacy class $D' \subset X$ as well, and so $D$ would meet $D'$, hence they would coincide, proving that $X$ can contain at most one $G$-conjugacy class.

To prove $\dim(D) = \dim(X)$, it is enough to prove that $\dim(T_{x}D) = \dim(T_xX)$, for then we’d have

$$
\dim(D) = \dim T_x D = \dim T_x X \geq \dim(X),
$$

the first equality holding because $D$ is smooth (the inequality always holds, even if $X$ is singular at $x$). But since $D \subset X$ this would force the equality $\dim(D) = \dim(X)$ to hold.

Finally, to prove $\dim(T_{x}D) = \dim(T_x X)$, we need to prove $\dim(T_{x}Dx^{-1}) = \dim(T_{x}Xx^{-1})$. Consider the morphism $\psi(g) = gxg^{-1}x^{-1}$ and its GL$_n$-analogue $\psi_{GL_n}$. Note that $Dx^{-1} = \text{im}(\psi)$, and $Xx^{-1} \subset \text{im}(\psi_{GL_n})$. This $\psi$ is the map $g \mapsto g\sigma(g^{-1})$, where $\sigma(g) := xgx^{-1}$. On the level of tangent spaces, we know (by using the equality $L(G_{\sigma}) = G_{\sigma}$ proved in part (a) along with (1)) that $d\psi_{e}$ maps onto $T_e(Dx^{-1})$ (and a similar statement holds for $d\psi_{GL_n}$).

Also, note that $d\psi_{e} = 1 - \text{Ad}(x)$. So, writing $\mathfrak{g}l_{n} = \mathfrak{g} \oplus \mathfrak{m}$ for an $\text{Ad}(G)$-stable complement $\mathfrak{m}$ as before, we get the following chain of inclusions:

$$
T_e(Xx^{-1}) \subset \mathfrak{g} \cap \text{im}(d\psi_{GL_n})
= \mathfrak{g} \cap (1 - \text{Ad}(x))(\mathfrak{g} \oplus \mathfrak{m})
= (1 - \text{Ad}(x))(\mathfrak{g})
= T_e(Dx^{-1})
\subset T_e(Xx^{-1}).
$$

It follows that $T_e(Dx^{-1}) = T_e(Xx^{-1})$, and this completes the proof.

**Remark:** Note that we did not establish the hint that was given. The hint turns out to be true, but the proof seems to be more involved. This whole problem (as well as the proof of the finer information contained in the hint) is due to Richardson: R.W. Richardson, *Conjugacy classes in Lie algebras and algebraic groups*, Annals of Math. 86 (1967), 1-15.
(c) This easily follows from Jordan normal form. In fact the number of unipotent conjugacy classes in $\text{GL}_n$ is $\mathcal{P}(n)$, the number of partitions of $n$. For example, $\mathcal{P}(5) = 7$:

1+1+1+1+1
1+1+1+2
1+2+2
1+1+3
1+4
2+3
5