Homework 3 – due 05/11/06
Math 608G

1. Let $G$ be a connected algebraic group, and $H$ a closed connected normal subgroup. Let $T \subset G$ be a maximal torus. Show that $H \cap T$ is a maximal torus in $H$. (Important consequence: $G_{\text{der}} \cap T$ is a maximal torus in $G_{\text{der}} := (G, G)$.)

2. Let $G$ be a connected reductive group, and fix a Borel subgroup and a maximal torus $G \supset B \supset T$. Write $B = TU = T \prod_{\alpha > 0} U_\alpha$.

(I) Let $s \in T$ be an element, with centralizer $C_G(s)$. Prove that $C_G(s)$ is reductive, using the following steps:

(a) Let $x \in C_G(s)$, and use the Bruhat decomposition to write $x$ uniquely in the form $x = un_wtv$, where $n_w \in N_G(T)$ represents an element $w$ in the Weyl group, $u \in U \cap n_w U^{-1} n_w^{-1}, t \in T$, and $v \in U$.

(b) Deduce from the uniqueness of the expression that $s$ commutes with $u, v$ and $n_w$. Deduce that $x$ lies in the group generated by $T$ along with the $U_\alpha$ such that $\alpha(s) = 1$ and the elements $n_w$ which commute with $s$.

(c) Let $H$ denote the subgroup of $G$ generated by $T$ and the groups $U_\alpha$ such that $\alpha(s) = 1$. Show that $H$ is a closed connected subgroup of $G$ which is normal in $C_G(s)$. Show that $C_G(s)/H$ is a subgroup of the Weyl group $N_G(T)/T$, hence is finite. Deduce that $H = C_G(s)^0$.

(d) Show that $C_G(s)$ is reductive: assume not, and let $V$ denote its non-trivial unipotent radical. Show that since $V$ is normalized by $T$, it must contain some $U_\alpha$. Show that since $V$ is also normalized by $U_{-\alpha}$, the commutation relations for the groups $U_\alpha$ and $U_{-\alpha}$ (seen from computations with $SL_2$) yield a semi-simple element in $V$, a contradiction.

(II) Deduce that if $S$ is any torus in $G$, its centralizer $C_G(S)$ is reductive.

(III) Show that if $T \subset G$ is a maximal torus, then $C_G(T) = T$ (important consequence: $Z(G) \subset T$). Hint: Note that $C_G(T)$ is both nilpotent (shown in class), and reductive by part (II).
3. The following exercises are taken from Springer, 7.4.7.

(I) Let $G = \text{SL}_n$ and $T$ the diagonal matrices in $G$. Show $T$ is a maximal torus. Show that the root datum of $G, T$ is $(X, F, X^\vee, R^\vee)$, where $X = \mathbb{Z}^n/\mathbb{Z}(e_1 + \cdots + e_n)$, $X^\vee = \{ (x_1, \ldots, x_n) \in \mathbb{Z}^n \mid \sum_i x_i = 0 \}$, with the obvious pairing. If $\pi : \mathbb{Z}^n \to X$ is the canonical projection, then show that $R = \pi\{e_i - e_j \mid 1 \leq i, j \leq n \}$, and $R^\vee = \{ e_i - e_j \mid 1 \leq i, j \leq n \}$.

(II) Assume $\text{char}(k) \neq 2$. Let $G = \text{SO}(2n+1)$ be identified with the group of matrices which preserve the symmetric bilinear pairing $(x, y)$ on $k^{2n+1}$, where

$$(v, w) = Q(v + w) - Q(v) - Q(w)$$

and where

$$Q((\xi_0, \ldots, \xi_{2n+1})) = \xi_0^2 + \sum_{i=1}^{n} \xi_i \xi_{n+i}.$$

Identify a maximal torus $T \subset G$ (look for “diagonal” elements in $G$). Show that the root data for $G, T$ is $(X, R, X^\vee, R^\vee)$, where $X = X^\vee = \mathbb{Z}^n$ (standard pairing), $R = \{ \pm e_i, \pm e_i \pm e_j \mid i \neq j \}$, and $R^\vee = \{ \pm 2e_i, \pm e_i \pm e_j \mid i \neq j \}$. 