MACDONALD’S FORMULA IMPLIES THE MIRKOVIC-VILONEN DIMENSION FORMULA

Mirkovic-Vilonen: The dimension of $N_{\pi^\lambda K/K} \cap K_{\pi^\mu K/K}$ is $\langle \rho, \mu + \lambda \rangle$, and the number of irreducible components of top dimension is $m_{\mu}(\lambda)$, the multiplicity of the weight $\lambda$ in the character $E_\mu$.

We will deduce this using Macdonald’s formula. It is enough (by, e.g., the Weil conjectures) to show that

$$\lim_{q \to \infty} \frac{\#(N_{\pi^\lambda K/K} \cap K_{\pi^\mu K/K})(\overline{\mathbb{F}_q})}{q^{\langle \rho, \mu + \lambda \rangle}} = m_{\mu}(\lambda).$$

The numerator in the left hand side is

$$\int_G 1_{A_\rho NK}(\pi^{-\lambda} y) 1_{K_{\pi^\mu K}}(y^{-1}) \, dy = (1_{A_\rho NK} * 1_{K_{\pi^\mu K}})(\pi^{-\lambda})$$

$$= (1_{K_{\pi^{-\omega_0 \mu} K} \cdot 1_{A_\rho NK}})(\pi^{-\lambda})$$

$$= 1_{K_{\pi^{-\omega_0 \mu} K}}(\pi^{-\lambda}) \delta_B^{1/2}(\pi^{-\lambda})$$

$$= 1_{K_{\pi^{-\omega_0 \mu} K}}(\pi^{-\omega_0 \lambda}) q^{\langle \rho, \lambda \rangle}.$$

By Macdonald’s formula, this is the coefficient of $\pi^{-\omega_0 \lambda}$ in

$$\frac{q^{\langle \rho, \mu + \lambda \rangle}}{W_{-\omega_0 \mu}(q^{-1})} \sum_{w \in W} w\left( \prod_{\alpha > 0} \frac{1 - q^{-1} \pi^{-\alpha \gamma}}{1 - \pi^{-\alpha \gamma}} \right) \cdot \pi^{-w \omega_0 \mu}.$$

Divide this by $q^{\langle \rho, \mu + \lambda \rangle}$ and take the limit as $q \to \infty$. The Weyl character formula implies that we get $m_{-\omega_0 \mu}(-w_0 \lambda) = m_{\mu}(\lambda)$. This completes the proof.