Calculus 131, section 11.2  1st Order Linear Differential Equations
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Recall the clue telling you that you have a separable differential equation (DE): Separable DEs will generally have one of two forms.

\[
\frac{dy}{dx} = f(x)g(y) \iff \int \frac{1}{g(y)} \, dy = \int f(x) \, dx \quad \text{or} \quad \frac{dy}{dx} = \frac{f(x)}{g(y)} \iff \int g(y) \, dy = \int f(x) \, dx
\]

When a DE is not separable, we’ll need another method. Today we’ll develop one for linear first-order differential equations. These are of the form \( \frac{dy}{dx} + P(x) \cdot y = Q(x) \). “First-order” means \( y’ \) not \( y’’ \), \( y’’’ \), etc. “Linear” means \( y \), not \( y^2 \), \( y^3 \), etc.

Before we get back to this model, we need some preliminaries.

Example A: Solve \((\cos t)y’ – (\sin t)y = t + 20\).  \textit{answer:} \( y = \frac{t^2}{2 \cos t} + \frac{20t}{\cos t} + \frac{C}{\cos t} = \frac{t^2 + 20t + C}{2 \cos t} \)

\textit{Note the position of ‘‘+C’’ in this and other answers!}

Example B: Solve \( e^{10t} \cdot y’ + 10e^{10t} \cdot y = t + 20\).  \textit{answer:} \( y = e^{-10t} \left( \frac{1}{2} t^2 + 20t + C \right) \)

\textit{This example will be particularly helpful in learning the process for solving linear first-order DEs.}

Theory: Consider the DE \( ty’ = t^2 + 3y, \quad t > 0 \). As it is, we cannot rearrange to look like \( f \cdot y’ + f’ \cdot y = \ldots \)

The coefficient of \( y \) is not the derivative of the coefficient of \( y’ \). However, all is not lost. Recall the general form first-order linear DE: \( \frac{dy}{dx} + P(x) \cdot y = Q(x) \). To solve this type of DE we’ll do some clever things. First we’ll find \( \int P(x) \, dx \), i.e. an anti-derivative of \( P(x) \), the coefficient of \( y \) in the first order linear DE.

Next, we’ll form the integrating factor \( e^{\int P(x) \, dx} \) and multiply both sides of the DE as follows.
\[
\left( e^{\int P(x) \, dx} \right) y' + \left( e^{\int P(x) \, dx} \right) a(t) y = \left( e^{\int P(x) \, dx} \right) Q(x)
\]

\[
f \quad g' + f' \quad g = \frac{d}{dx} \left[ f \ast g \right] = \frac{d}{dx} \left[ e^{\int P(x) \, dx} \, y \right] = \left( e^{\int P(x) \, dx} \right) Q(x)
\]

\[
\int \frac{d}{dt} \left[ e^{\int P(x) \, dx} \, y \right] \, dx = \int \left( e^{\int P(x) \, dx} \right) Q(x) \, dx
\]

\[
e^{\int P(x) \, dx} \, y = \int \left( e^{\int P(x) \, dx} \right) Q(x) \, dx
\]

The right-hand side may simplify nicely so we can integrate, or we can use substitution or integration by parts.

Example C: Solve \( x \frac{dy}{dx} = x^2 + 3y, \quad x > 0 \). \textit{answer}: \( y = -x^2 + Cx^3, \quad x > 0 \)

1) Rewrite in standard form.

2a) Find \( \int P(x) \, dx \).

2b) Find the integrating factor.

3) Multiply by the integrating factor and integrate both sides.

Example C extended: Given the initial condition \( y(1) = 2 \), find the particular solution. \textit{answer}: \( y = 3x^3 - x^2 \)

Note that since \( \int P(x) \, dx \) can be \textit{any} anti-derivative, we might as well pick the simplest one.

In using this process, it will be important to correctly identify \( P(x) \), and also to correctly construct the \textit{integrating factor} \( I(x) = e^{\int P(x) \, dx} \). It is called an integrating factor because it transforms the left-hand side into a form that can be integrated because it is the result of differentiating by the Product Rule.
Example D: Solve \( \frac{dy}{dx} + 2xy = 6x^3 \). \( \text{answer: } y = 3x^2 - 3 + Ce^{-x^2} \)

Examples E: \( x \frac{dy}{dt} = x^2 + y^2 \) is not linear (because of the \( y^2 \)), and \( x \frac{d^2y}{dx^2} = x^2 + y \) is not first-order (because it has \( \frac{d^2y}{dx^2} \)!

Example F: An anti-coagulant is introduced intravenously at a rate of 0.5 mg per hour. On a continuous basis 2% of the drug in the bloodstream is absorbed into the body. a) Set up and solve a differential equation that is satisfied by \( y(t) \), the amount of anti-coagulant the bloodstream of the patient. b) Determine the equilibrium amount of the anti-coagulant in the bloodstream of the patient. \( \text{answer: } y = 25 + Ce^{-0.02t}, 25 \text{ mg} \)