Calculus 141, section 9.1 Taylor polynomial approximation ~ Introduction

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In the previous section, we were able to approximate the value of an integral using first rectangles (midpoint sum), then trapezoids, then quadratics (Simpson’s Rule). In chapter 9 we turn to a similar process for approximating any curve, but extend it to using higher degree polynomials and higher-order derivatives.

We’ll need the definition of a factorial to make the notation of our process a little easier: “n factorial” is written \( n! \) and defined as

\[
10! = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1
\]

for integers \( n \geq 1 \), and \( 0! = 1 \). For example:

\[
6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720
\]

Now, recall how the definition of the first derivative was developed, i.e. as a linear expression of the tangent line at a given value of \( x \). In slope-intercept form \( y = b + mx \) for \( x = 0 \), we’d have \( l(x) = f(0) + f'(0)x \).

Can we develop a similar quadratic expression? If so, we’d want it to have the following properties:

\[
q(x) = c + bx + ax^2 \quad \text{with} \quad q(0) = c \quad \text{such that} \quad q(0) = f(0)
\]

\[
q'(x) = b + 2ax \quad \text{with} \quad q'(0) = b \quad \text{such that} \quad q'(0) = f'(0)
\]

\[
q''(x) = 2a \quad \text{with} \quad \frac{q''(0)}{2} = a \quad \text{such that} \quad q''(0) = f''(0)
\]

Thus, our quadratic approximation would be \( q(x) = f(0) + f'(0)x + \frac{f''(0)}{2} x^2 \). (See section 3.8 of the text.)

Let’s get bold and daring, and go for a fifth-degree approximation, with the following properties:

\[
p_5(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 \quad \text{with} \quad p_5(0) = a_0 \quad \text{such that} \quad p_5(0) = f(0)
\]

\[
p_5^{(1)}(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 \quad \text{with} \quad p_5^{(1)}(0) = a_1 \quad \text{such that} \quad p_5^{(1)}(0) = f^{(1)}(0)
\]

\[
p_5^{(2)}(x) = 2a_2 + 3 \cdot 2a_3 x + 4 \cdot 3a_4 x^2 + 5 \cdot 4a_5 x^3 \quad \text{with} \quad \frac{p_5^{(2)}(0)}{2} = a_2 \quad \text{such that} \quad p_5^{(2)}(0) = f^{(2)}(0)
\]

\[
p_5^{(3)}(x) = 3 \cdot 2a_3 + 4 \cdot 3 \cdot 2a_4 x + 5 \cdot 4 \cdot 3a_5 x^2 \quad \text{with} \quad \frac{p_5^{(3)}(0)}{3 \cdot 2} = a_3 \quad \text{such that} \quad p_5^{(3)}(0) = f^{(3)}(0)
\]

\[
p_5^{(4)}(x) = 4 \cdot 3 \cdot 2a_4 + 5 \cdot 4 \cdot 3 \cdot 2a_5 x \quad \text{with} \quad \frac{p_5^{(4)}(0)}{4 \cdot 3 \cdot 2} = a_4 \quad \text{such that} \quad p_5^{(4)}(0) = f^{(4)}(0)
\]

\[
p_5^{(5)}(x) = 5 \cdot 4 \cdot 3 \cdot 2a_5 \quad \text{with} \quad \frac{p_5^{(5)}(0)}{5 \cdot 4 \cdot 3 \cdot 2} = a_5 \quad \text{such that} \quad p_5^{(5)}(0) = f^{(5)}(0)
\]

Thus, our fifth-degree approximation is

\[
p_5(x) = f(0) + f^{(1)}(0)x + \frac{f^{(2)}(0)}{2!} x^2 + \frac{f^{(3)}(0)}{3!} x^3 + \frac{f^{(4)}(0)}{4!} x^4 + \frac{f^{(5)}(0)}{5!} x^5
\]

If we continue this process for higher-order polynomials, we would get the \( n \)th Taylor polynomial of \( f \) about 0:

\[
p_n(x) = f(0) + f^{(1)}(0)x + \frac{f^{(2)}(0)}{2!} x^2 + \frac{f^{(3)}(0)}{3!} x^3 + \ldots + \frac{f^{(n)}(0)}{n!} x^n
\]

(When we get to section 9.9, we’ll develop a more general formula for a Taylor expansion about values other than 0, and include a formula for estimating the error involved in such an approximation.)

Example A: Given a function \( f \) such that \( f(0) = 1, f^{(1)}(0) = -3, f^{(2)}(0) = 5, f^{(3)}(0) = -7 \) find the third-degree Taylor polynomial approximation. Answer: \( p_3(x) = 1 - 3x + \frac{5}{2} x^2 - \frac{7}{6} x^3 \)
The text develops the Taylor approximations for $f(x) = e^x$ for $n = 10$ and $f(x) = \ln(1 + x)$ for $n = 10$ and $n = 20$ in a very straightforward manner, so we won’t recreate those. Instead, let’s consider a trigonometric function.

Example B: Find $p_2(x)$, $p_3(x)$, $p_4(x)$, $p_6(x)$, and $p_8(x)$ for $f(x) = \cos x$.

*Answers:* $1 - \frac{1}{2}x^2$; $1 - \frac{1}{2}x^2$; $1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$; $1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6$; $1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{1}{40320}x^8$

Graphs of $f$ and our Taylor approximations are given below:

Note that the polynomials each veer away from $\cos x$ as $x$ gets larger, but that the higher-degree approximations “follow” $\cos x$ longer than the lower-degree polynomials, i.e. as the oscillations of the polynomial more closely match the periodic nature of $\cos x$. The graph of $\cos x$ versus $p_{20}(x)$ is given below.

In later topics we will address questions of whether our approximations converge or diverge for given values of $x$, how we can know the difference, and (in cases where the approximations converge) the degree of the approximation needed to contain errors within a given margin. In particular for $\cos x$, we’ll be able to show that the Taylor approximations as derived above do converge to $\cos x$ as $n$ approaches $\infty$. 