Calculus 141, section 9.5 Integral Test and Comparison Tests

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Consider series such as \( \sum_{n=1}^{\infty} \frac{1}{n} \), \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \), \( \sum_{n=1}^{\infty} \frac{1}{n^2} \), and \( \sum_{n=1}^{\infty} \frac{1}{2^n} \). You may notice that these resemble improper integrals \( \int_1^{\infty} \frac{dx}{x} \), \( \int_1^{\infty} \frac{dx}{\sqrt{x}} \), \( \int_1^{\infty} \frac{dx}{x^2} \), and \( \int_1^{\infty} \frac{dx}{2^x} \). Indeed, both \( \sum_{n=1}^{\infty} \frac{1}{n} \) (harmonic series, section 9.4) and \( \int_1^{\infty} \frac{dx}{x} \) (section 8.7) diverge. Might we suspect a similar relationship between the other infinite series and corresponding improper integrals?

By observing the relationships between the terms \( a_n \) of a positive decreasing sequence, and the values of the function \( f \) for which \( f(n) = a_n \), the text proves Theorem 9.12, the Integral Test: For a positive decreasing sequence \( \{a_n\}_{n=1}^{\infty} \), and \( f \) a continuous function on \([1, \infty)\) such that \( f(n) = a_n \), then the series \( \sum_{n=1}^{\infty} a_n \) converges if and only if \( \int_1^{\infty} f(x) \, dx \) converges.

Examples A: Determine whether \( \sum_{n=1}^{\infty} \frac{1}{n} \), \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \), \( \sum_{n=1}^{\infty} \frac{1}{n^2} \), and \( \int_1^{\infty} \frac{dx}{2^x} \) converge or diverge.

\textit{Answers}: diverges, converges, converges

Example B: Determine values of \( p \) for which \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) converges and diverges. \textit{Answers}: \( p > 1 \); \( 0 < p \leq 1 \)
Example C: Does \( \sum_{n=1}^{\infty} \frac{1}{n^{1/n}} \) converge?  \textit{Answer: yes}

Example D: Does the infinite series \( \sum_{n=2}^{\infty} \frac{1}{n^{\sqrt{\ln n}}} \) converge?  \textit{Answer: no}

Example A revisited: Determine the value to which \( \sum_{n=1}^{\infty} \frac{1}{n^{2}} \) converges.  \textit{Answer:} \( \frac{\pi^{2}}{6} \) (trick question)

The question now becomes: How quickly and how closely do the partial sums generated by a series approach the actual value of the series? The \( j \)th truncation error is defined to be

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E_{j} = \left| \sum_{n=1}^{\infty} a_{n} - \sum_{n=1}^{j} a_{n} \right| = \left| \sum_{n=j+1}^{\infty} a_{n} \right| , \text{ i.e. the rest of the sequence beyond the } j \text{th partial sum. The text applies the Integral Test to show that } \int_{j+1}^{\infty} f(x) \, dx \leq E_{j} \leq \int_{j}^{\infty} f(x) \, dx . \text{ In Example 2 the text estimates } E_{100} \text{ for } \sum_{n=1}^{\infty} \frac{1}{n^{2}} .
Example C revisited: Estimate $E_{10000}$ for the series $\sum_{n=1}^{\infty} \frac{1}{n \sqrt{n}}$. Answer: within 0.02

Recall section 8.7, where we developed the comparison test for improper integrals. A similar test exists for infinite sums: Theorem 9.13 gives us the Direct Comparison Test for infinite sums.

a. If $\sum_{n=1}^{\infty} b_n$ converges and $0 < a_n \leq b_n$ for all $n \geq 1$, then $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$.

b. If $\sum_{n=1}^{\infty} b_n$ diverges and $0 < b_n \leq a_n$ for all $n \geq 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Example E: Does $\sum_{n=1}^{\infty} \frac{1}{n!}$ converge? Answer: yes
Direct comparison is sometimes awkward, or at least inconvenient. The Limit Comparison Test provides an alternative (Theorem 9.14): If \( \lim_{{n \to \infty}} \frac{a_n}{b_n} \) exists and is a positive number, then positive series \( \sum_{{n=1}}^\infty a_n \) and \( \sum_{{n=1}}^\infty b_n \) either both converge or both diverge. To apply the Limit Comparison Test, we’ll look for a sequence \( \{b_n\} \) which has known properties and whose \( n \)th term behaves in a fashion similar to the \( n \)th term of the sequence \( \{a_n\} \) for large values of \( n \). If we can determine that \( \lim_{{n \to \infty}} \frac{a_n}{b_n} \) is a positive number, we’ll be able to draw a conclusion about the series \( \sum_{{n=1}}^\infty a_n \).

Example F: Does \( \sum_{{n=1}}^\infty \frac{1}{2^n - 5} \) converge? \textit{Answer: yes}

Example G: Does \( \sum_{{n=1}}^\infty \frac{3n + 2}{\sqrt{n}(3n - 5)} \) converge? \textit{Answer: no}